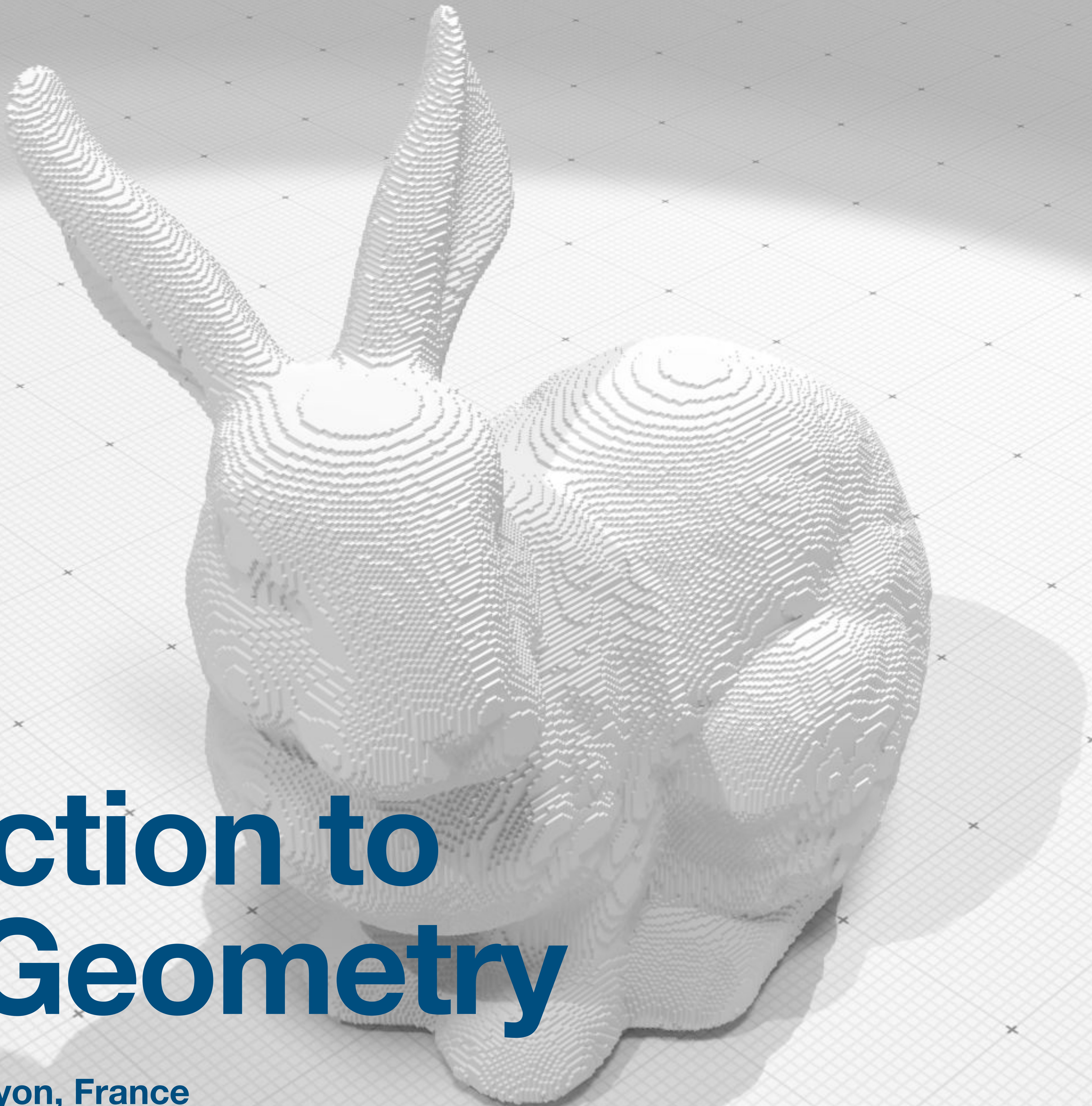


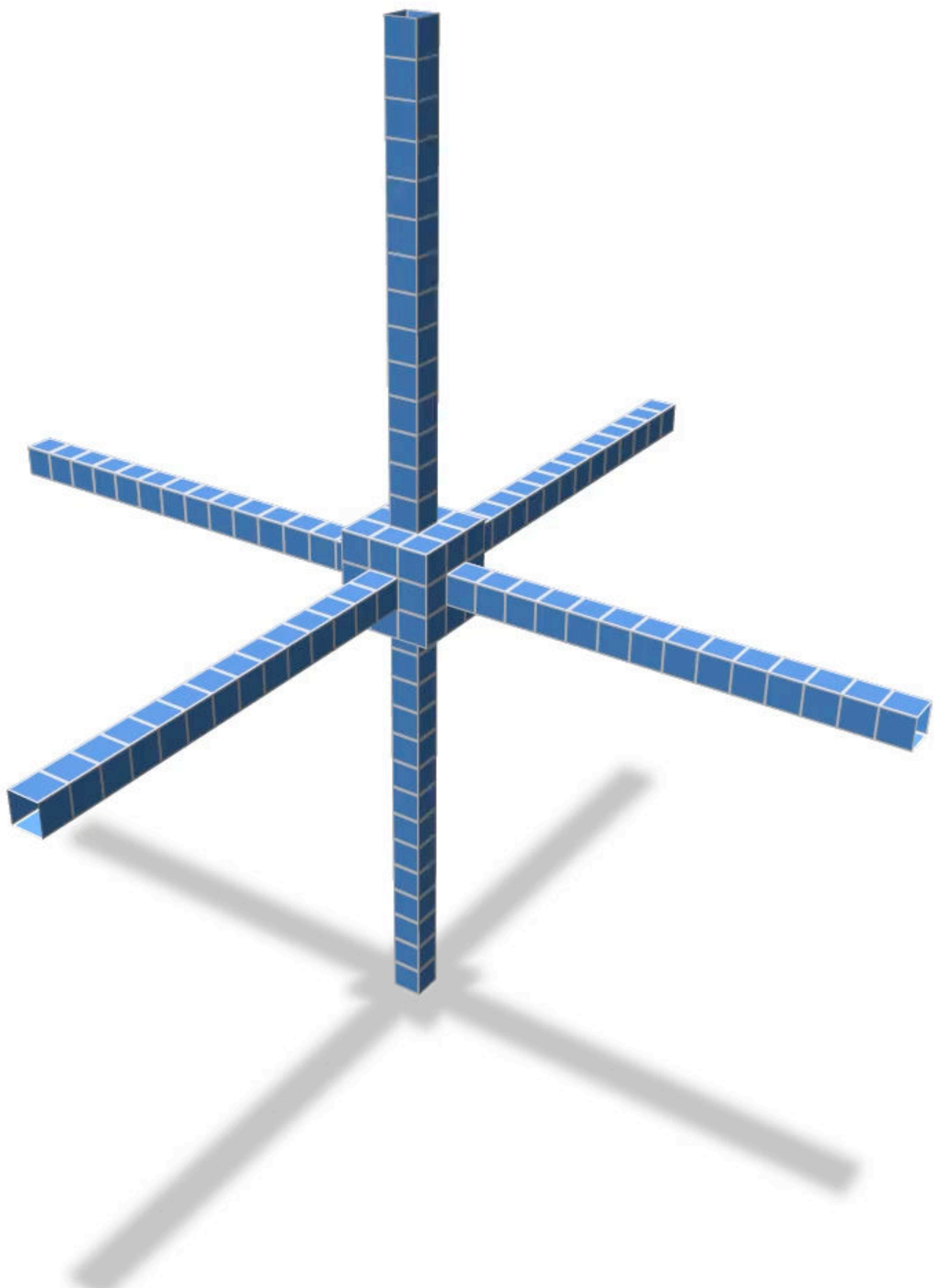
Introduction to Digital Geometry

David Coeurjolly, CNRS, Lyon, France

Jacques-Olivier Lachaud, Université Savoie Mont-Blanc, France

Outline

- context
- dgtal.org
- geometry with integers
- geometry processing on grids
- digital surface processing
- conclusion



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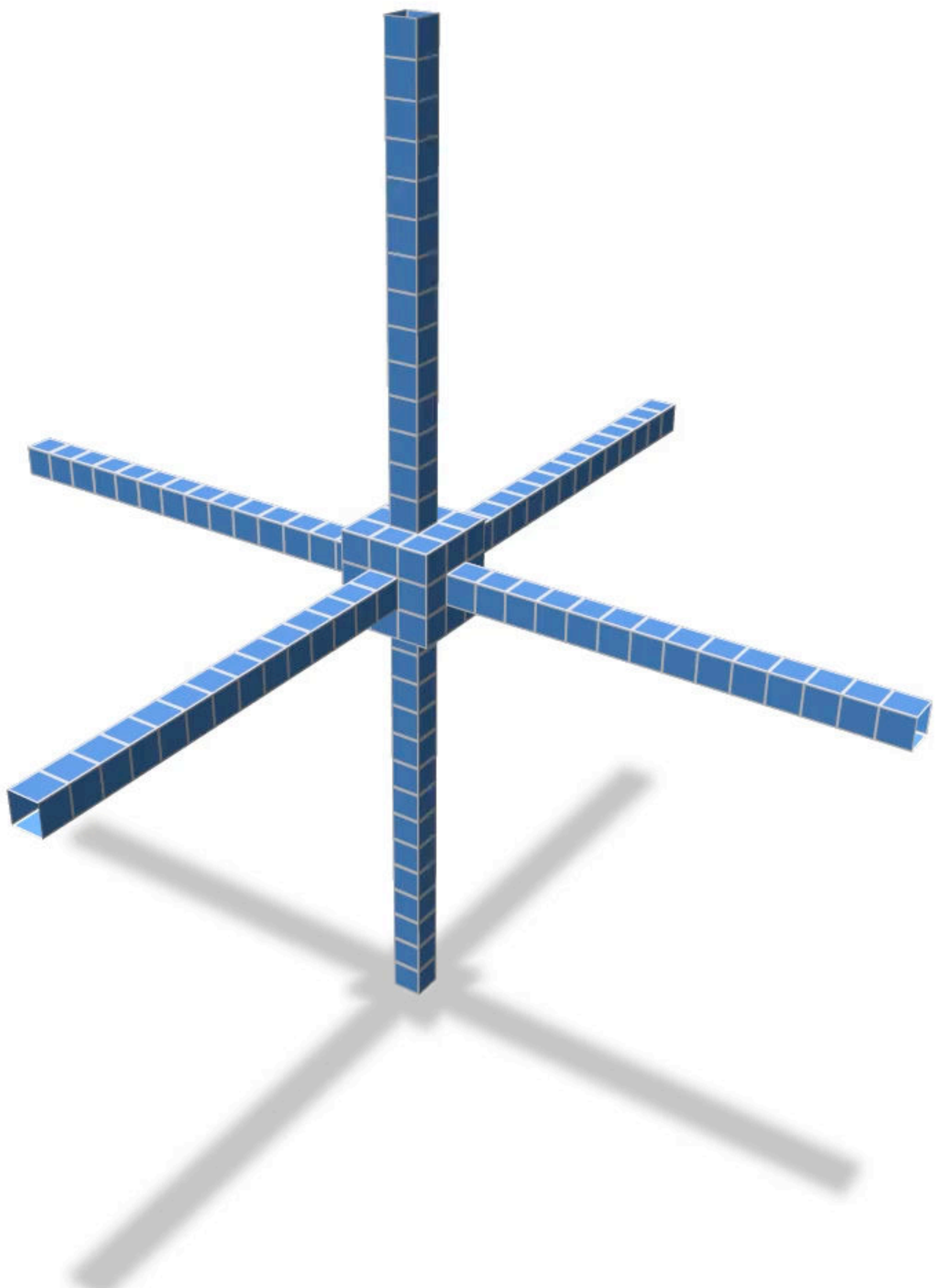
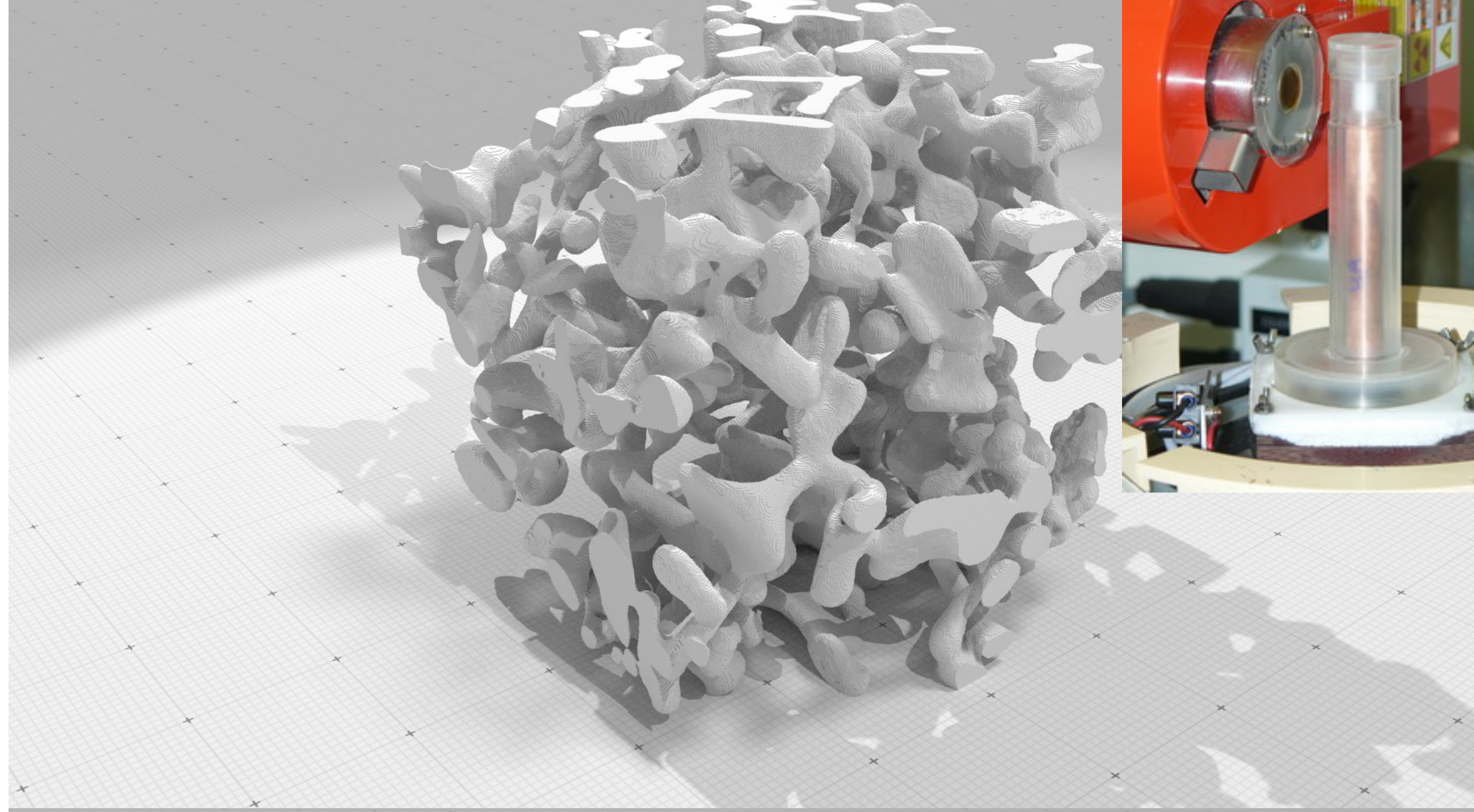
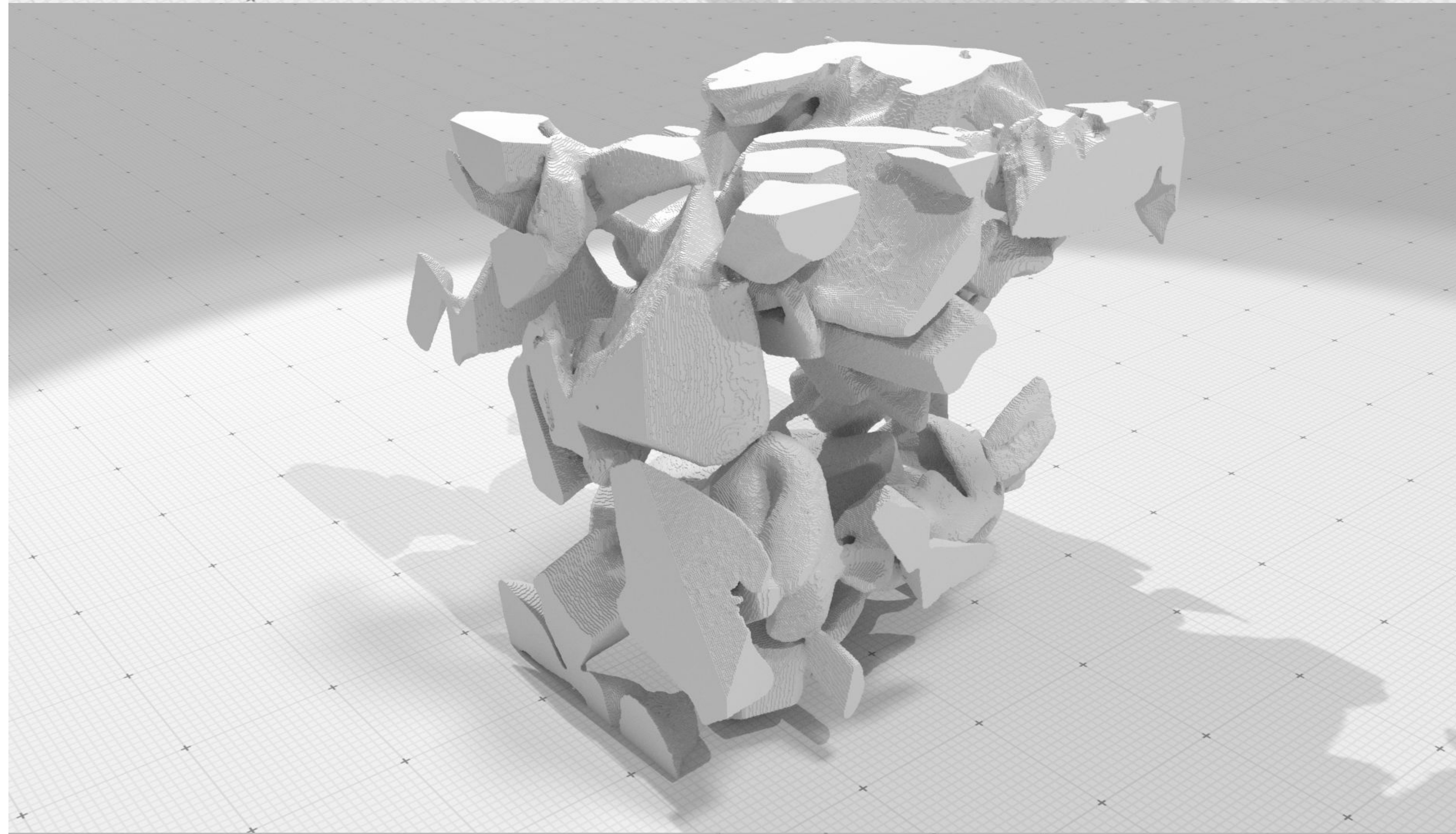
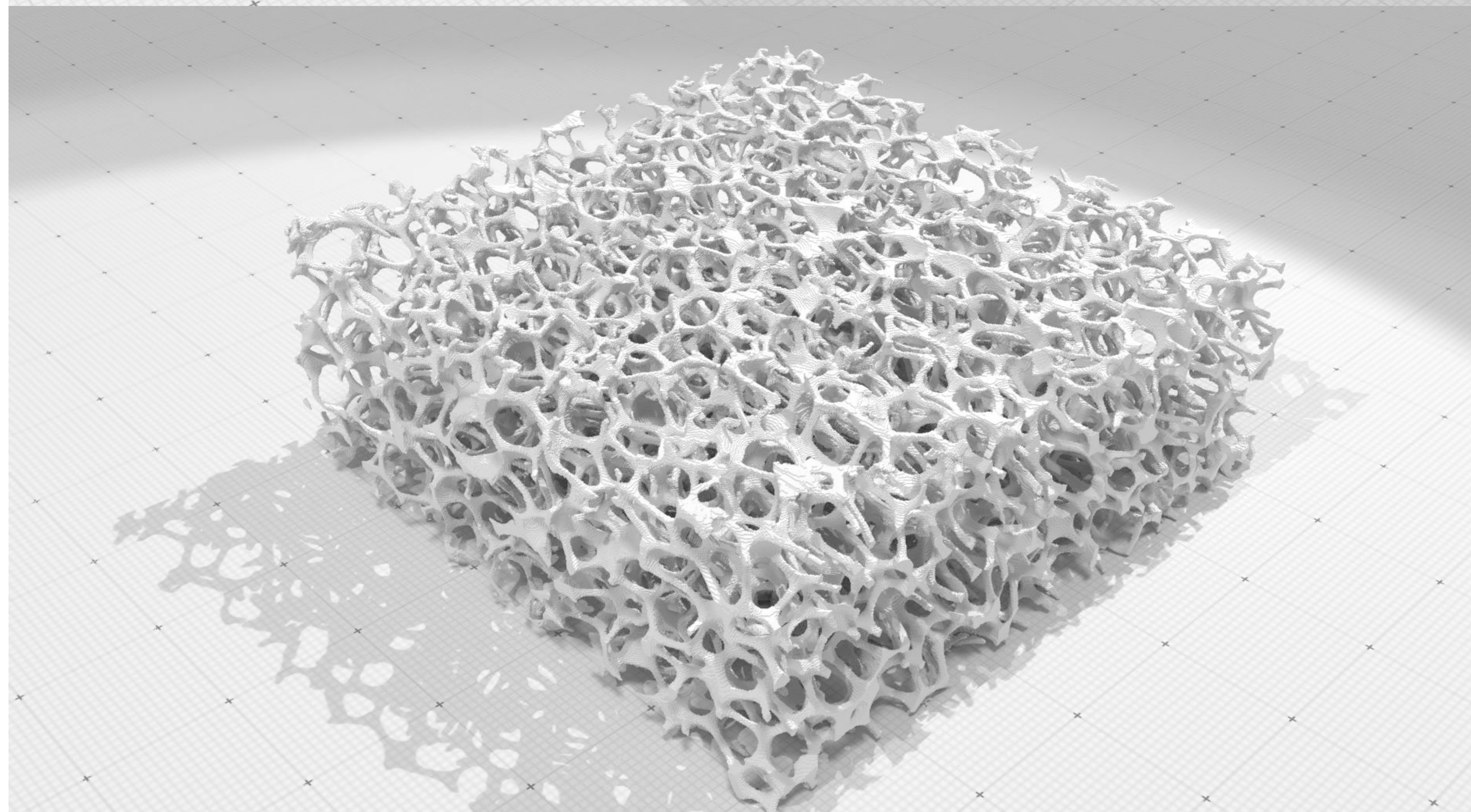


Motivations (1): devices

- Micro-tomographic images
  - material sciences
  - medical images
- Process geometry/topology of images partitions







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- Micro-tomographic images
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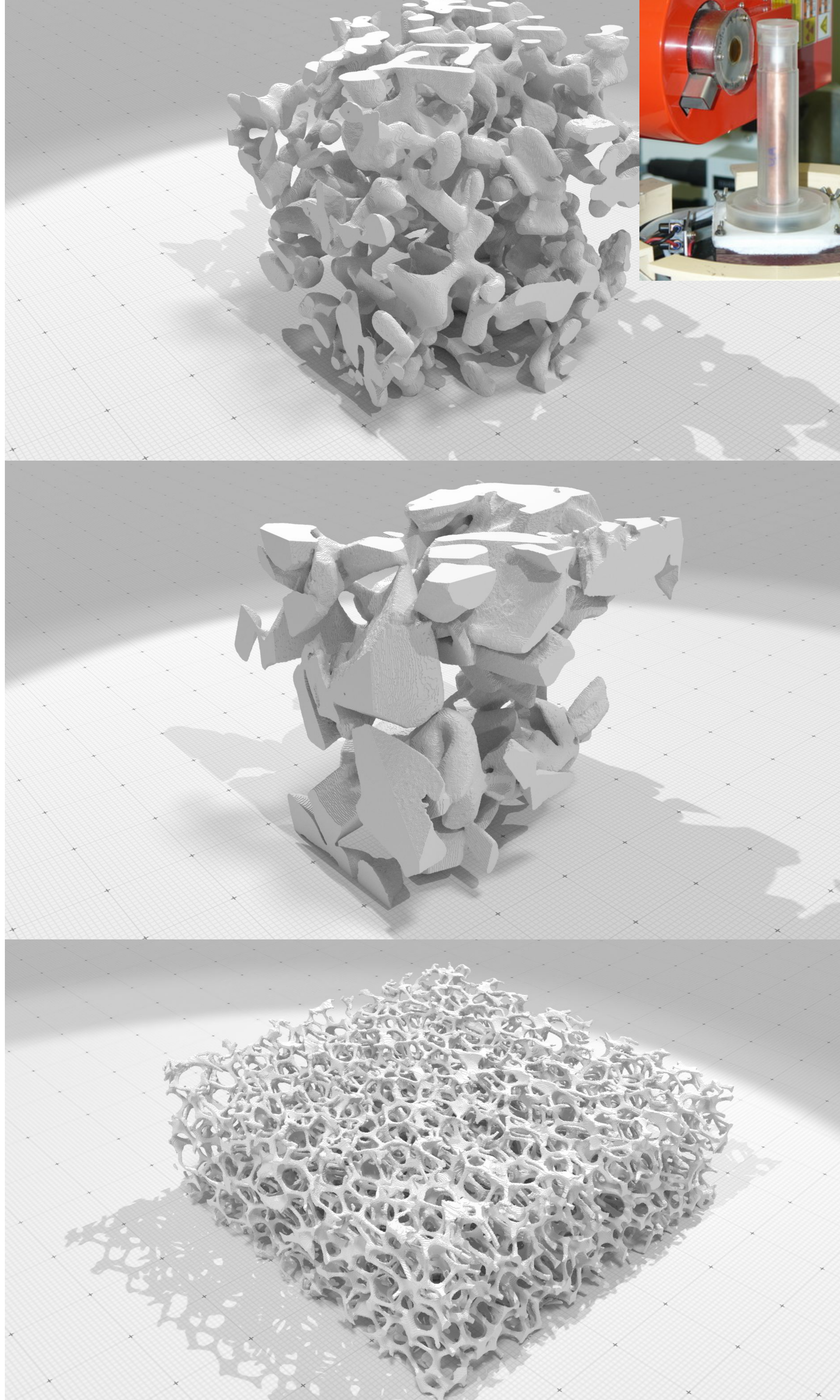
  - medical images

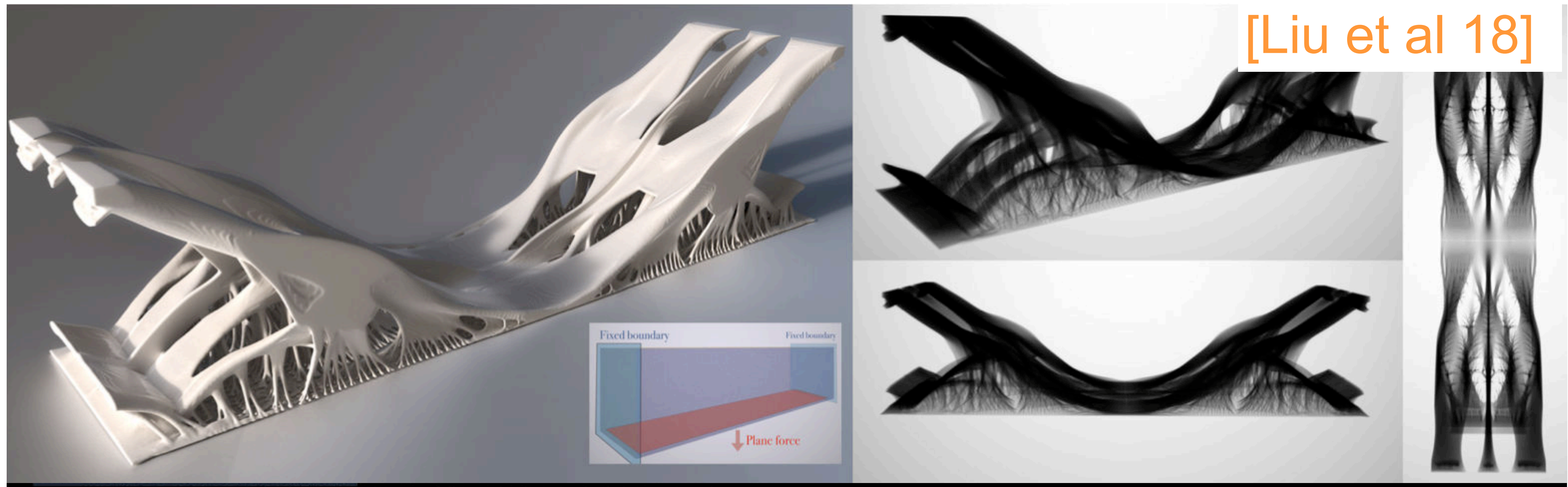
- Process geometry/topology of images partitions

$\Rightarrow X \subset \mathbb{Z}^3$

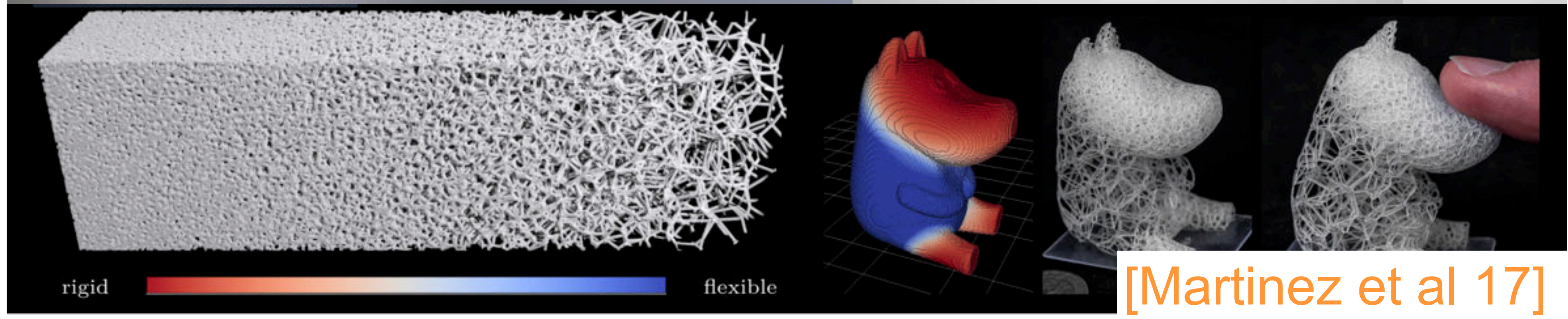


Motivations (2): $\mathbb{Z}^d$ as an efficient modelling space

- Shape optimization / fabrication



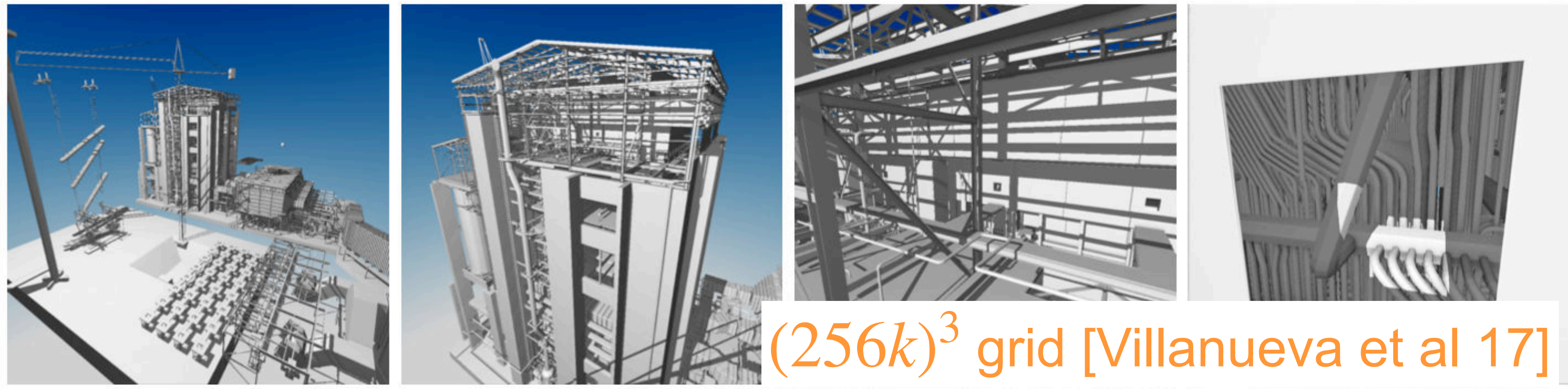
[Liu et al 18]



[Martinez et al 17]

- As a proxy or an intermediate representation

  light transport simulation, booleans, medial axis, distance fields, multiple interfaces/objects tracking in a simulation loop…



$(256k)^3$ grid [Villanueva et al 17]

***Focus**: characteristic functions / labelled images / level sets / …*

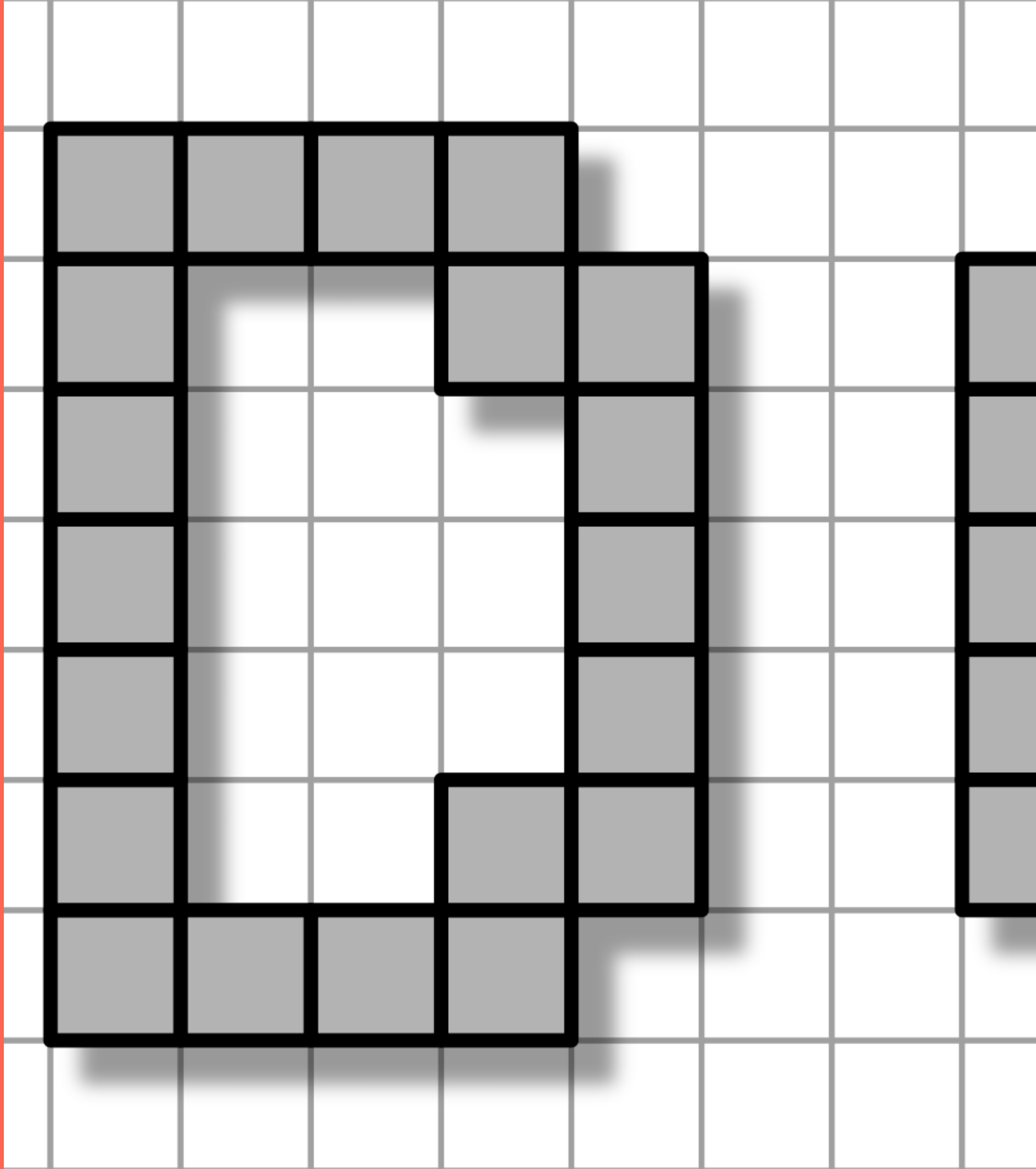
Digital Geometry

Topology and geometry processing on regular data:

- fast algorithms thanks to the regularity of the data
- simple topological structure
- integer based computations
- advanced surface based geometry processing

… in $\mathbb{Z}^d$

dgtal.org



DGtal

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DGtal

Digital Geometry Tools and Algorithms Library

Fork me on GitHub

News

DGtal release 1.3

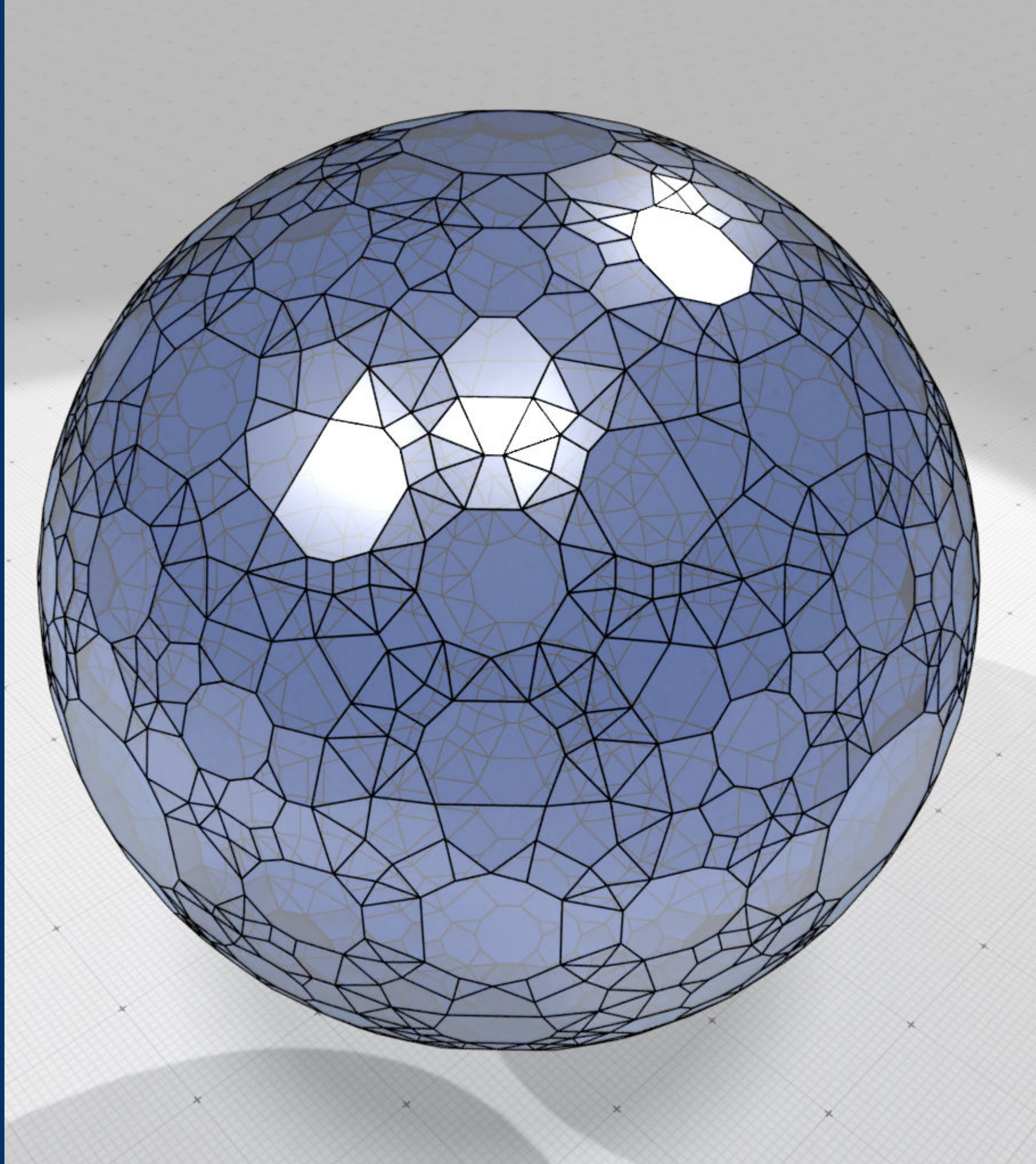
Posted on November 25, 2022

We are thrilled to announce the release 1.3 of DGtal and its tools. Many new features, edits and bugfixes are listed in the Changelog, and we would like to thank all devs involved in this release. In this short review, we would like to only focus on selected new features.... **[Read More]**

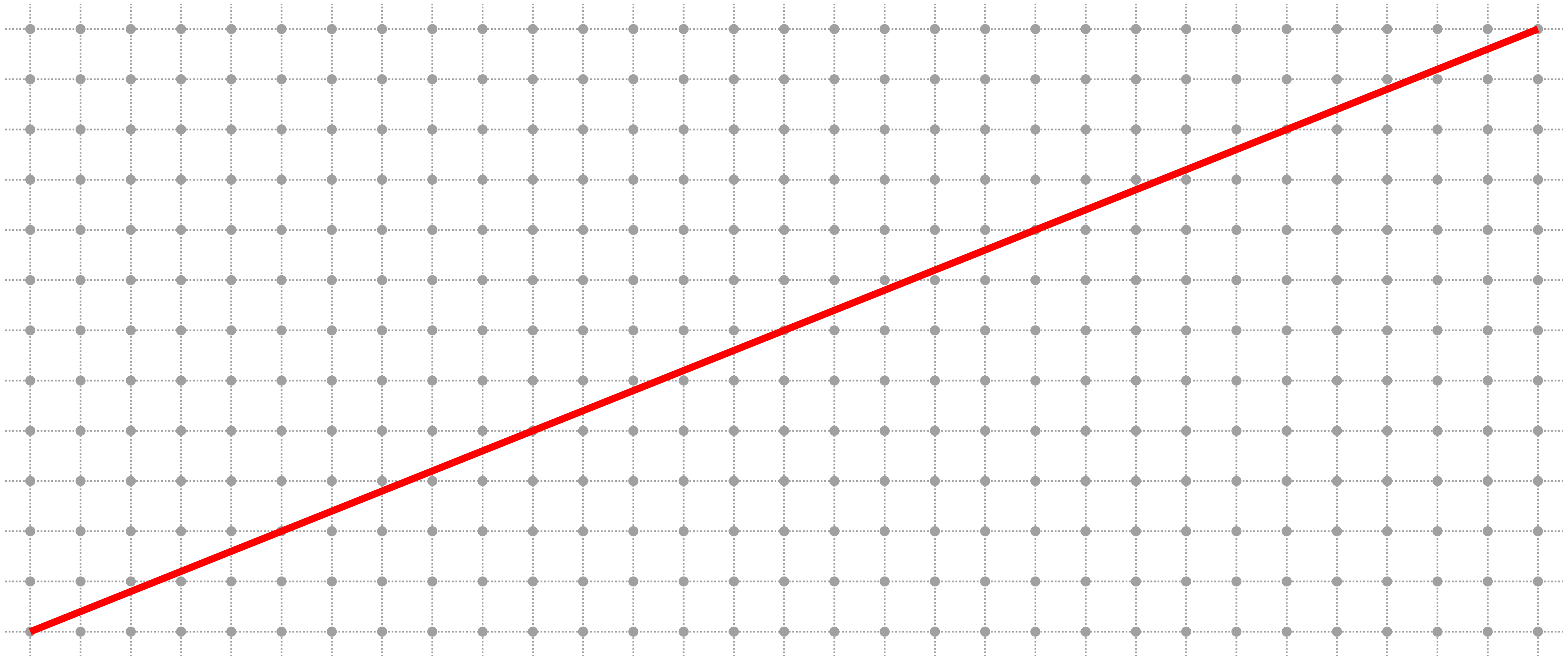
DGtal tutorial at DGMM 2022

https://dgtal.org

python



Quick example



$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}}$$

- Rational slope ⇒ finite set of remainders ⇒ periodic structure ⇒ canonical pattern from **continued fraction**

→ *arithmetization* to speed-up tracing (e.g. fast ray marching on Sparse Voxel Octree)

→ useful to design fast recognition algorithms (pixels/voxels ⇒ digital straight lines, planes, circles…)

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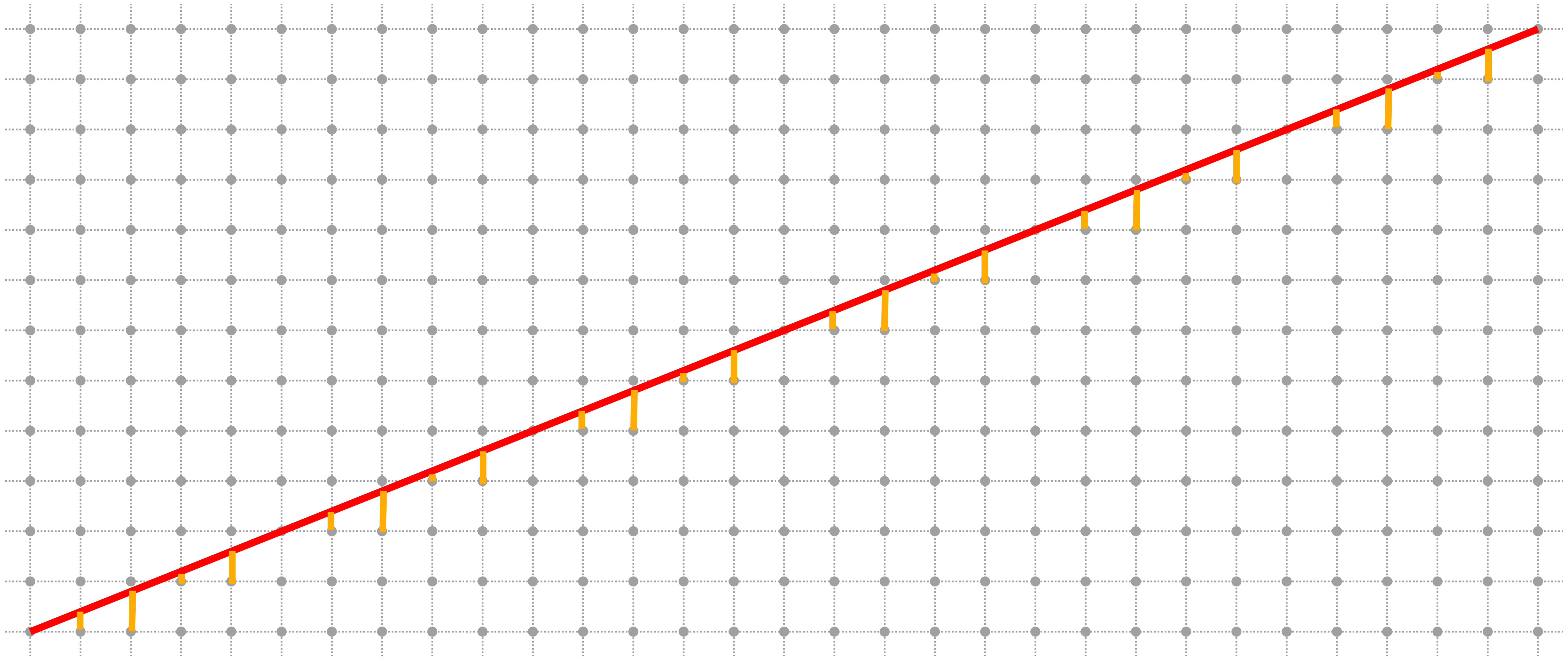

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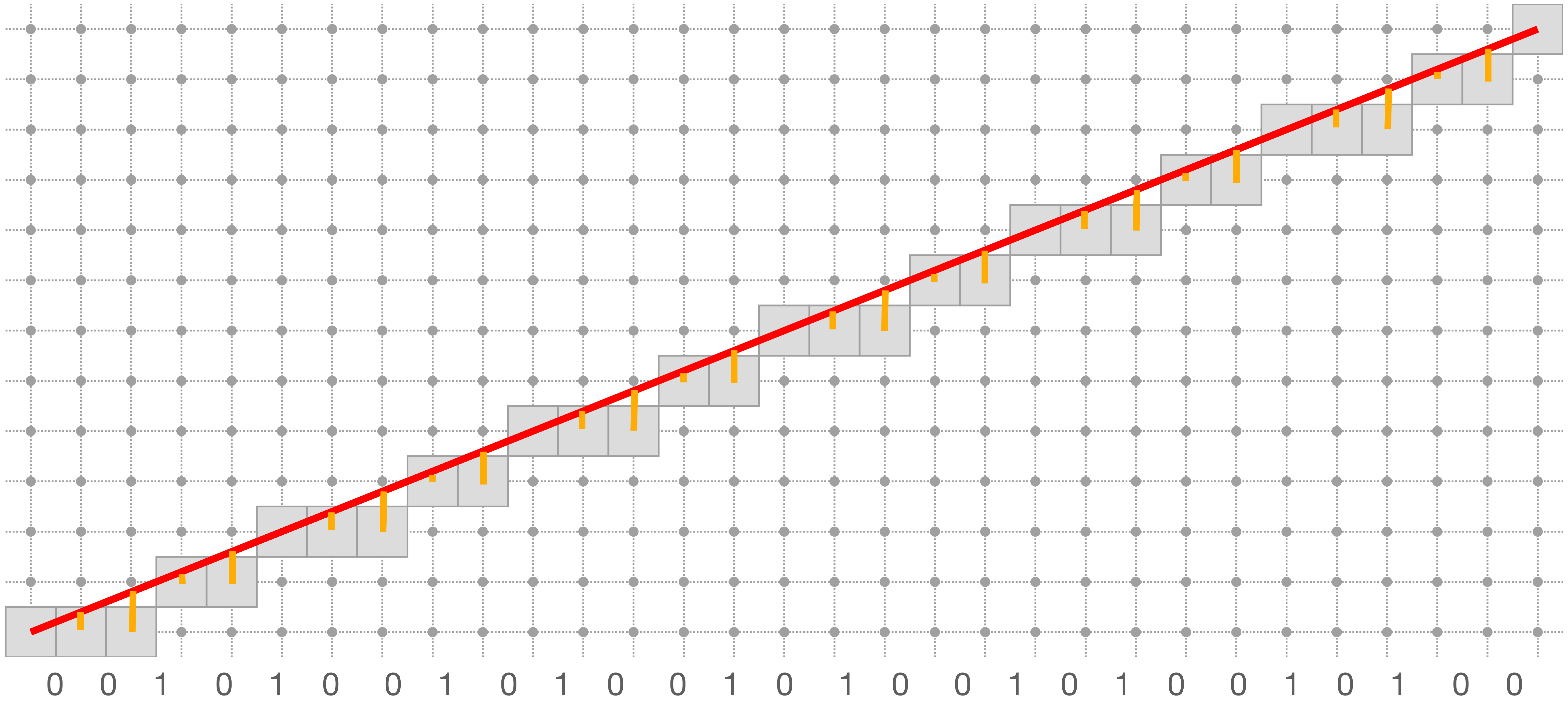

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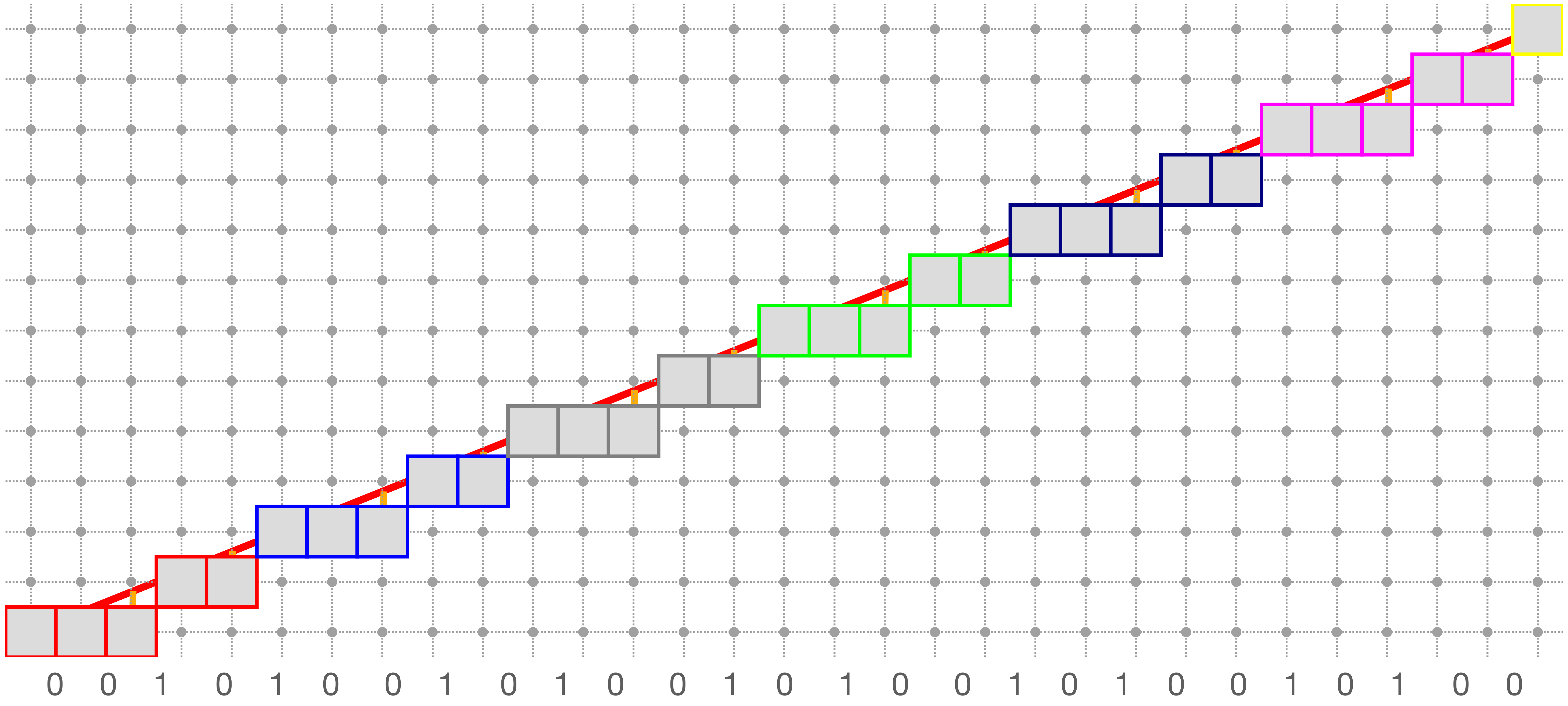

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Further elements

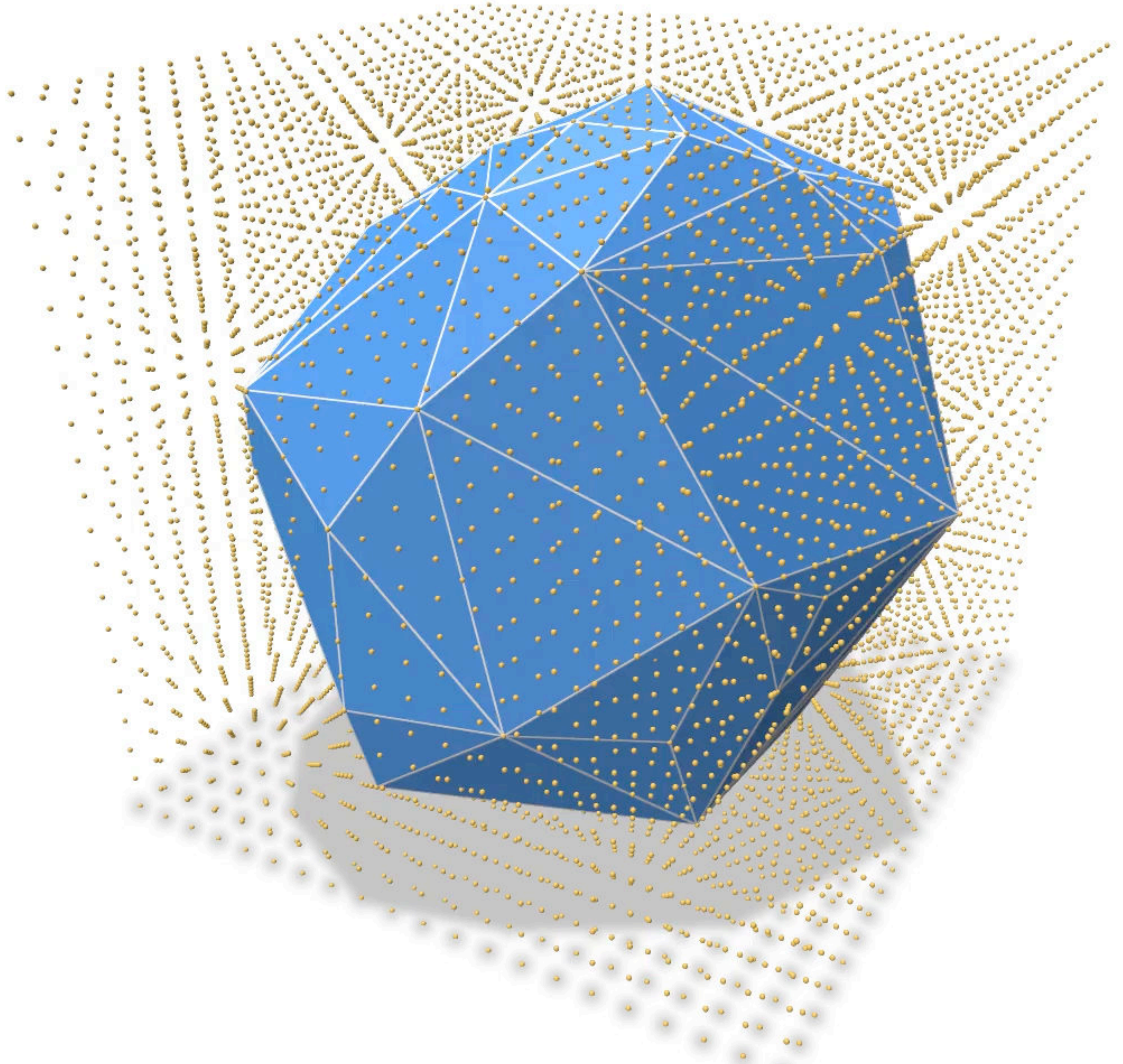
Let $P \subset \mathbb{Z}^d$ a lattice polytope with non-empty interior, then: $f_k \ll c_d (Vol\, P)^{\frac{d-1}{d+1}}$

Convex on the lattice $[1,n]^2$grid has $O(n^{2/3})$ edges

Let $P \subset [1,U]^2$ (with $U \leq 2^m$) and $n := |P|$, the expected time for Voronoi diagram / Delaunay triangulation is:

$$O\left(\min\{n \log n, n\sqrt{U}\}\right)$$

⋮



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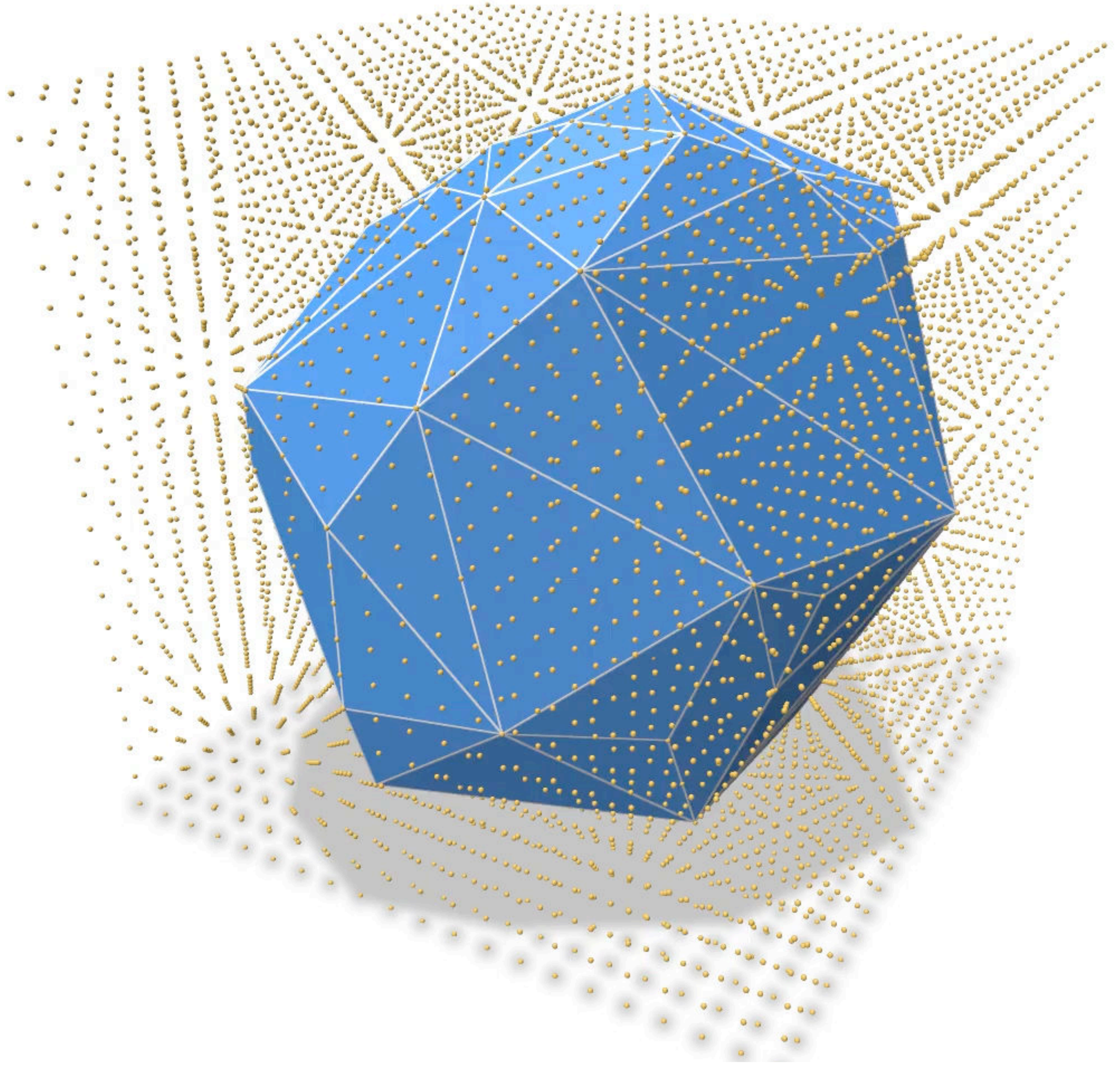


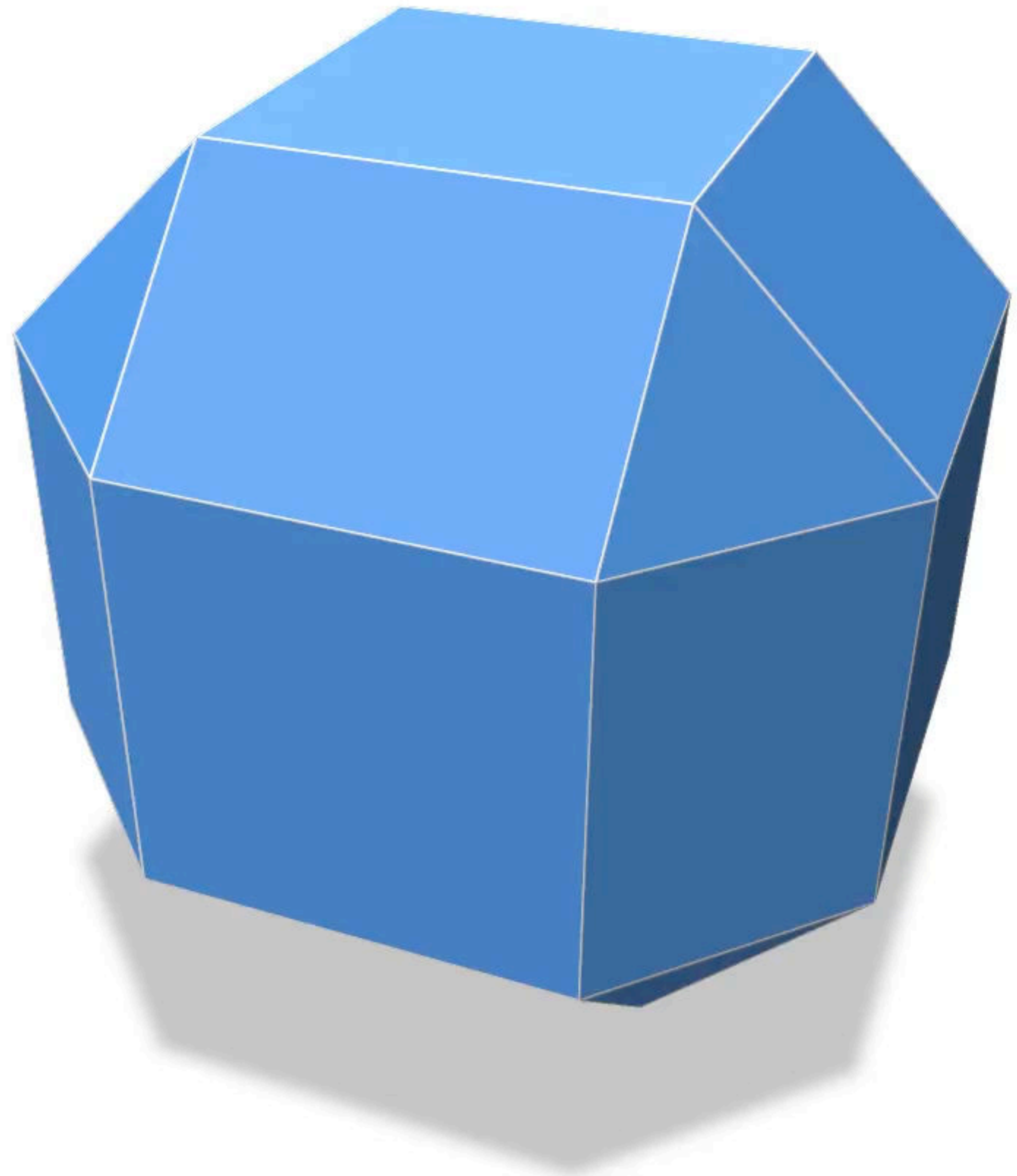
hands on…

```
void oneStep(double myh)
{
  auto params = SH3::defaultParameters();
  params( "polynomial", "sphere1" )( "gridstep", myh )
        ( "minAABB", -1.25 )( "maxAABB", 1.25 );
  auto implicit_shape   = SH3::makeImplicitShape3D  ( params );
  auto digitized_shape = SH3::makeDigitizedImplicitShape3D( implicit_shape, params );

  std::vector<Point> points;
  std::cout << "Digitzing shape" << std::endl;
  auto domain = digitized_shape->getDomain();
  for(auto &p: domain)
    if (digitized_shape->operator()(p))
      points.push_back(p);

  std::cout << "Computing convex hull" << std::endl;
  QuickHull3D hull;
  hull.setInput( points );
  hull.computeConvexHull();
  std::cout << "#points="    << hull.nbPoints()
            << " #vertices=" << hull.nbVertices()
            << " #facets="   << hull.nbFacets() << std::endl;
  std::vector< RealPoint > vertices;
  hull.getVertexPositions( vertices );
  std::vector< std::vector< std::size_t > > facets;
  hull.getFacetVertices( facets );

  polyscope::registerSurfaceMesh("Convex hull", vertices, facets)->rescaleToUnit();
}
```



```cpp
void oneStep(double myh)
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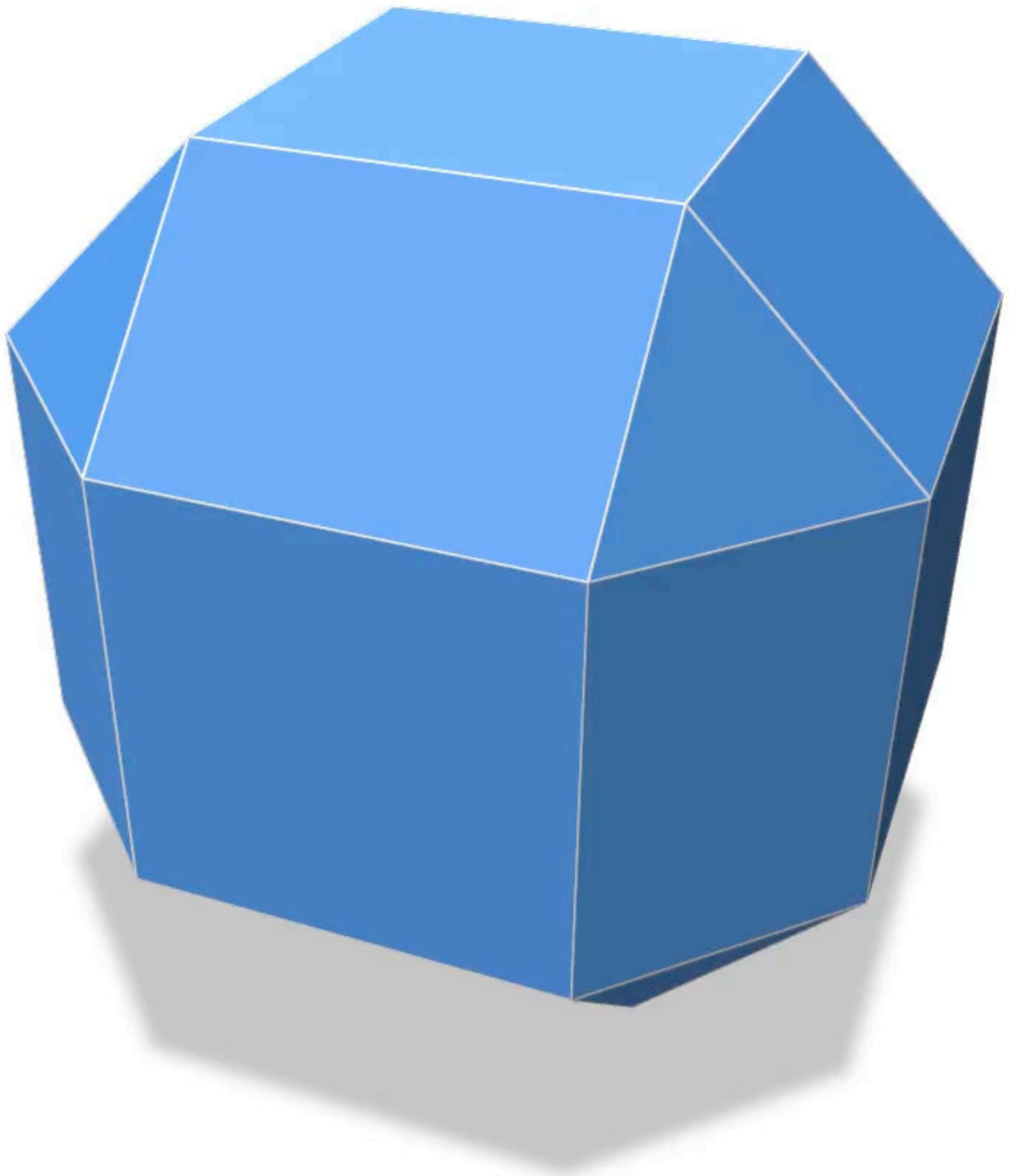
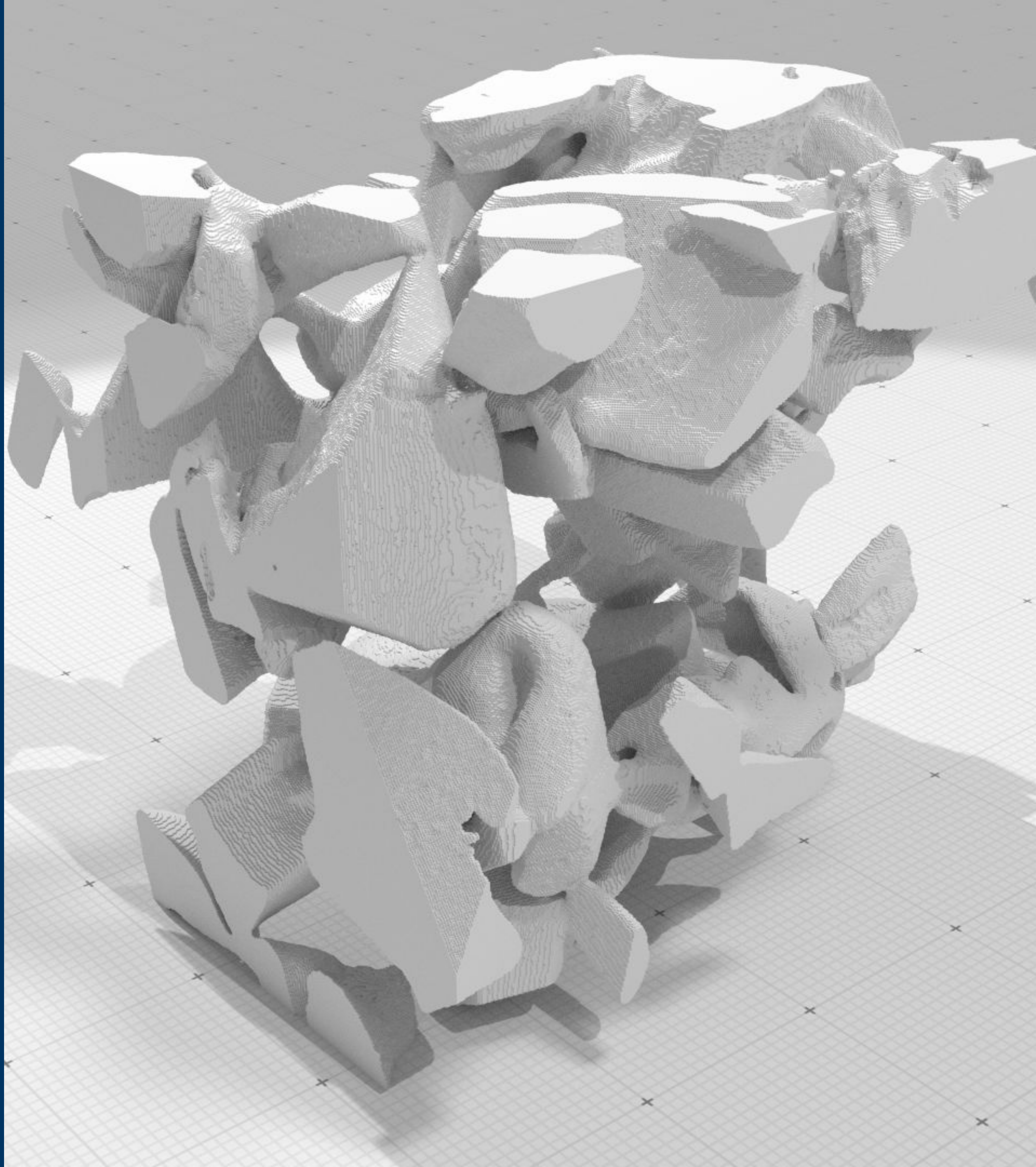
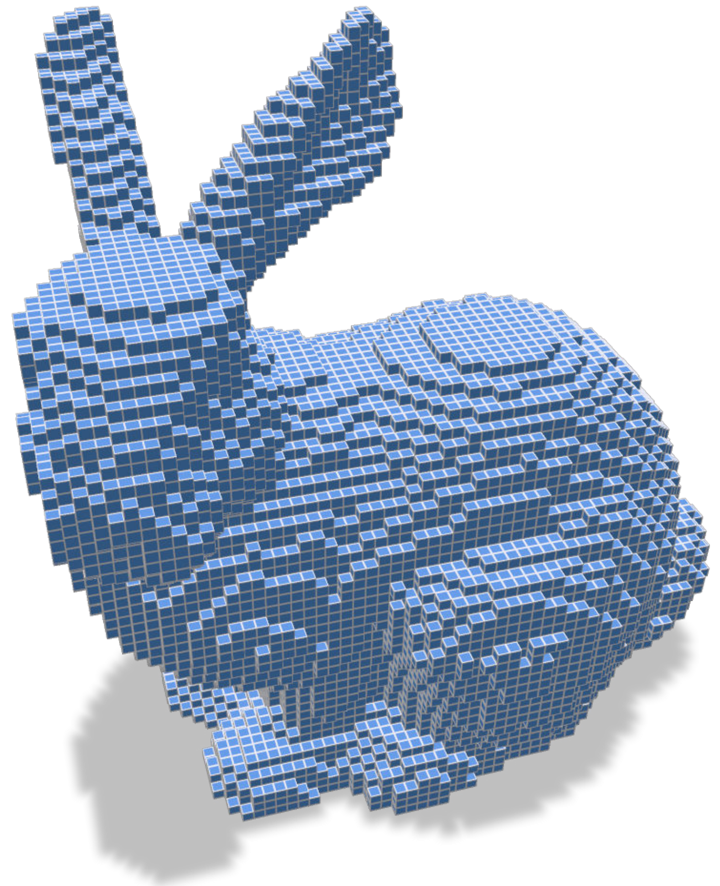


$$\mathbb{Z}^d$$

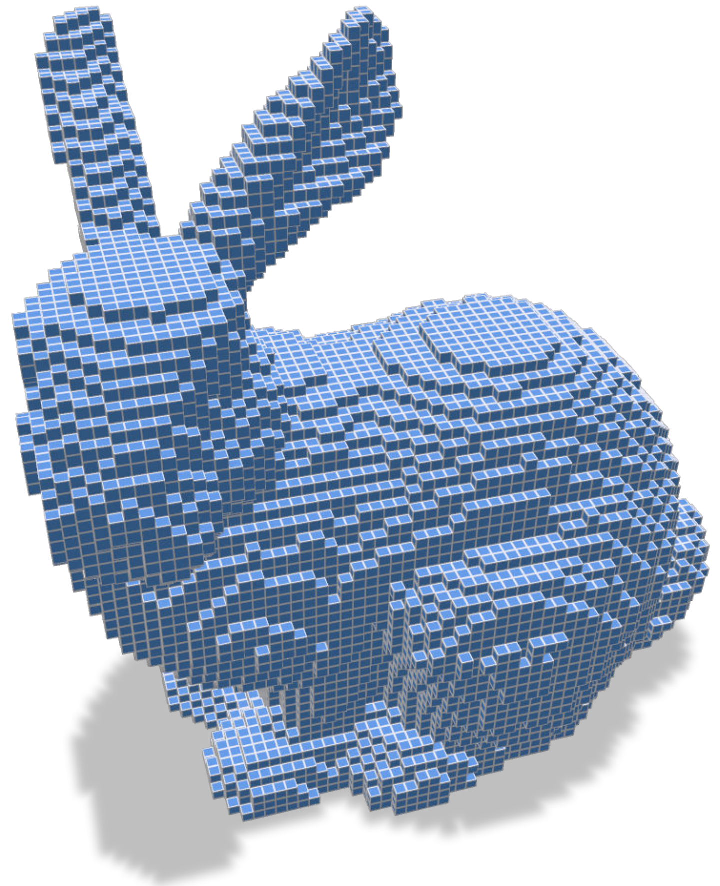


Volumetric analysis



Given $X \subset \mathbb{Z}^d$ ***and a domain*** $[0,n]^d$***, compute:***

Volumetric analysis



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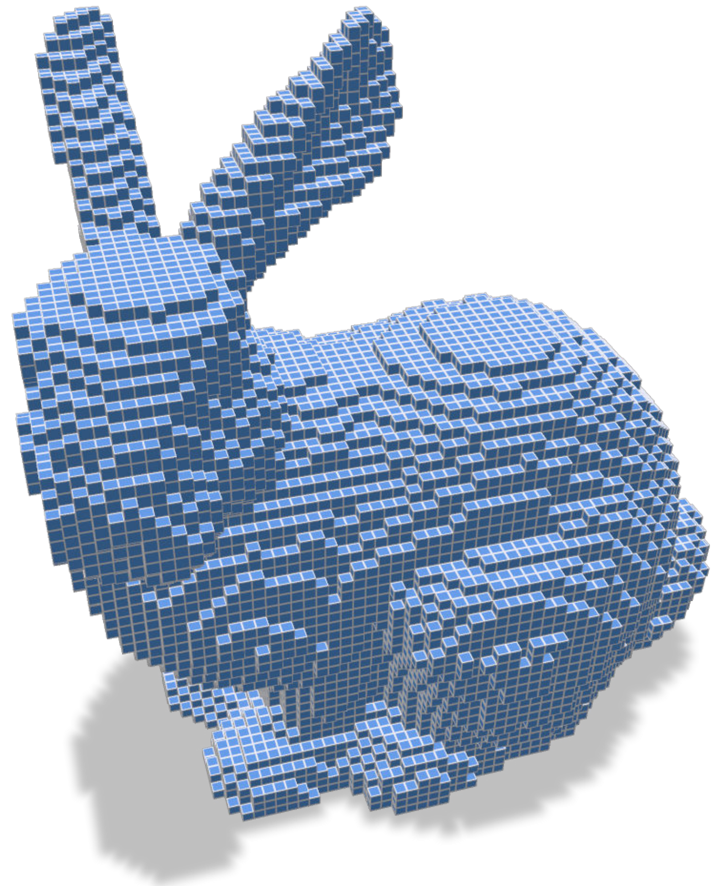

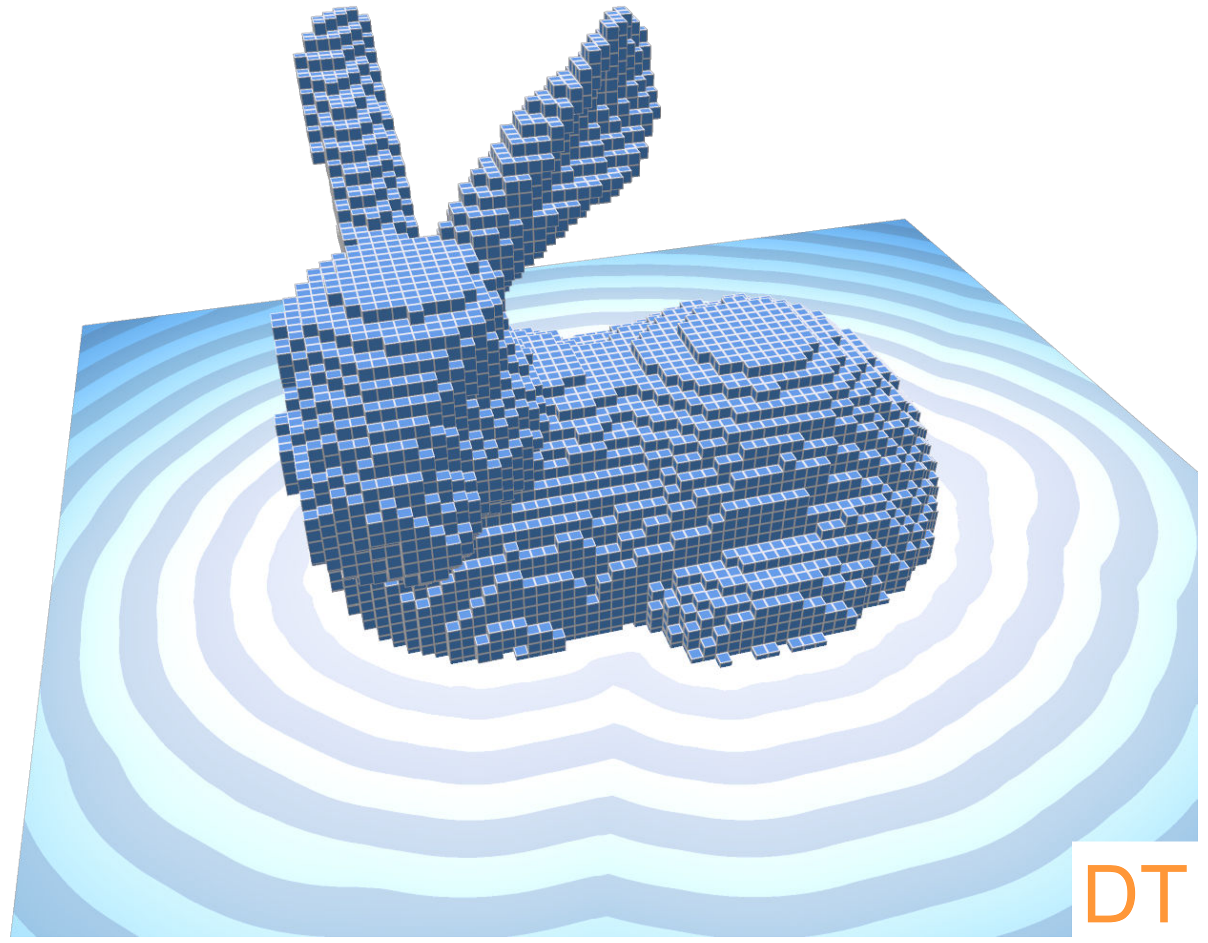

Given $X \subset \mathbb{Z}^d$ ***and a domain*** $[0,n]^d$***, compute:***

$$DT(x) = min_{y \in D \setminus X} \; d(x, y)$$ *(aka distance map)*

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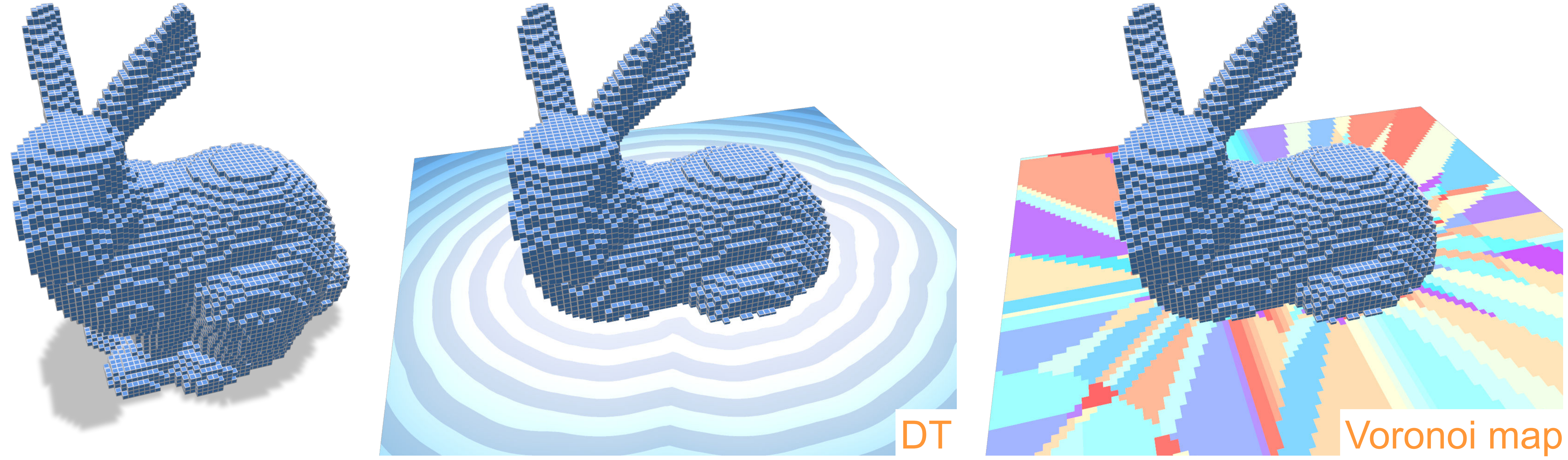

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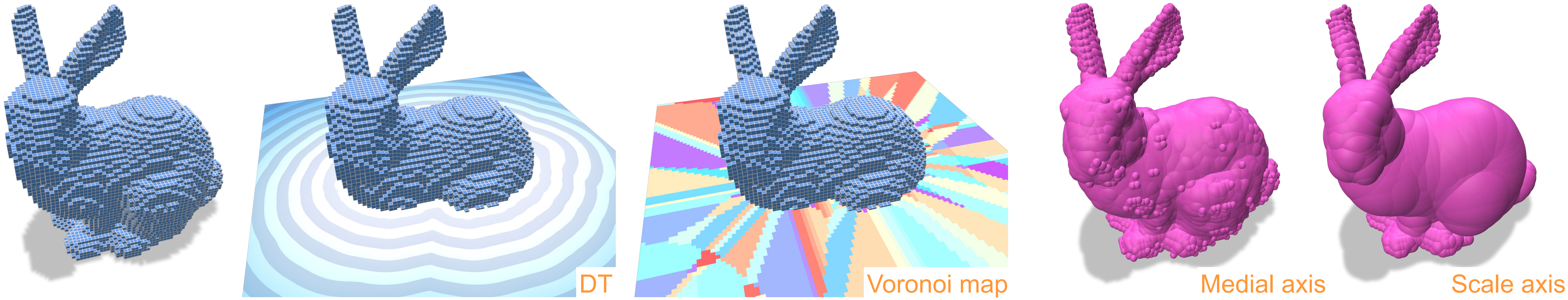
Volumetric analysis



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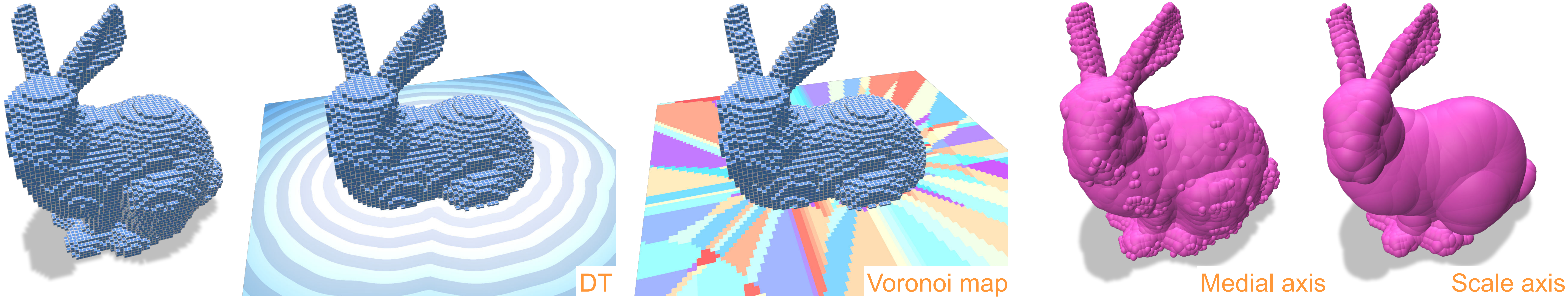
$M = \{(x, r) \in \mathbb{Z}^{d+1} \mid \mathscr{B}(x, r) \cap \mathbb{Z}^d \subset X$, *there is no* (x', r') *s.t.* $\mathscr{B}(x, r) \subset \mathscr{B}(x', r')\}$ *(aka discrete medial axis)*

Volumetric analysis



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$$\pi(x) = \text{argmin}_{(y,r) \in M} \ \|x - y\|_2^2 - r^2$$ *(aka l_2 Power map $\mathcal{P}(M) \cap \mathbb{Z}^d$)*

Volumetric analysis

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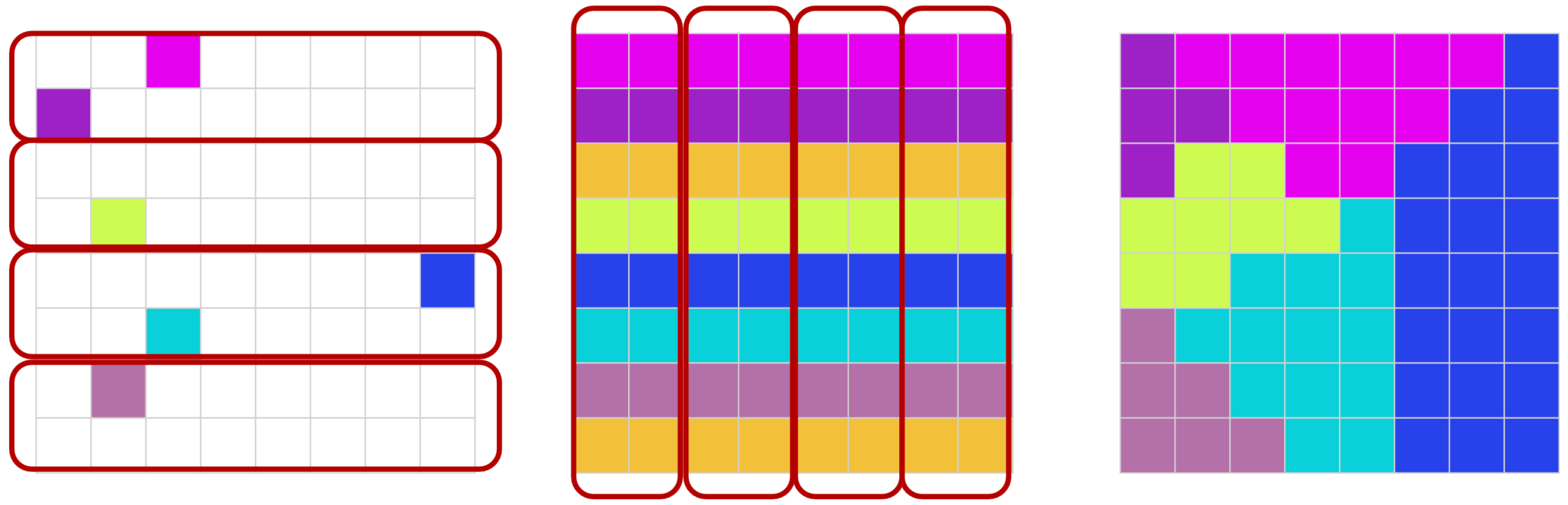
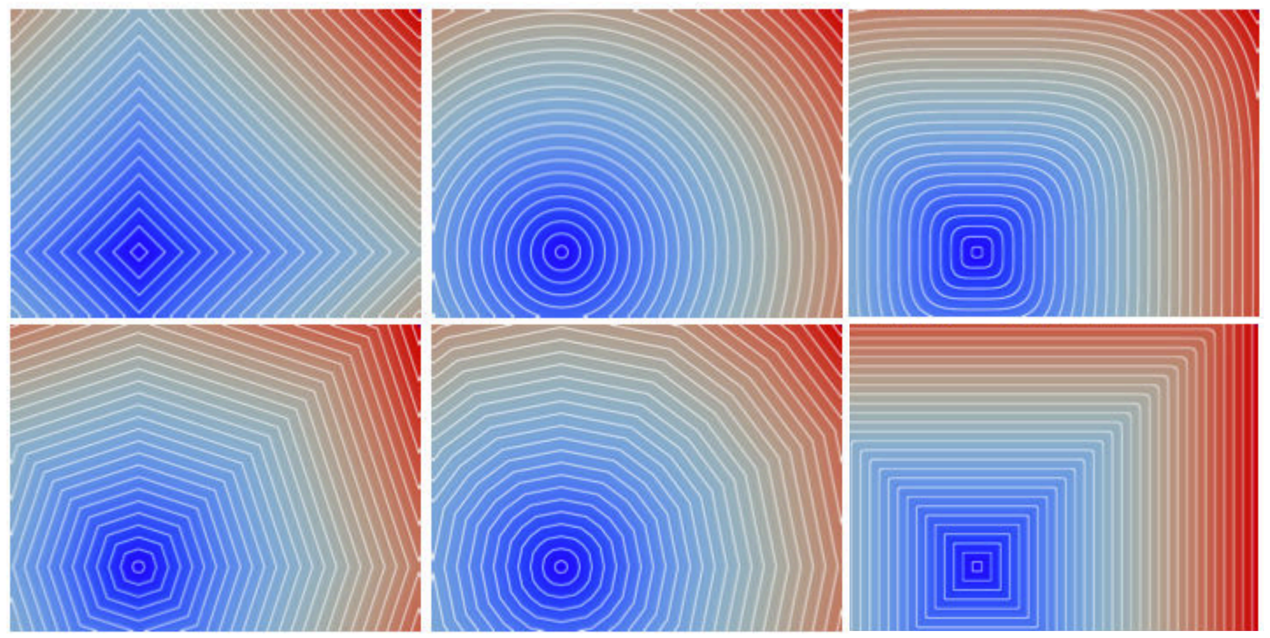
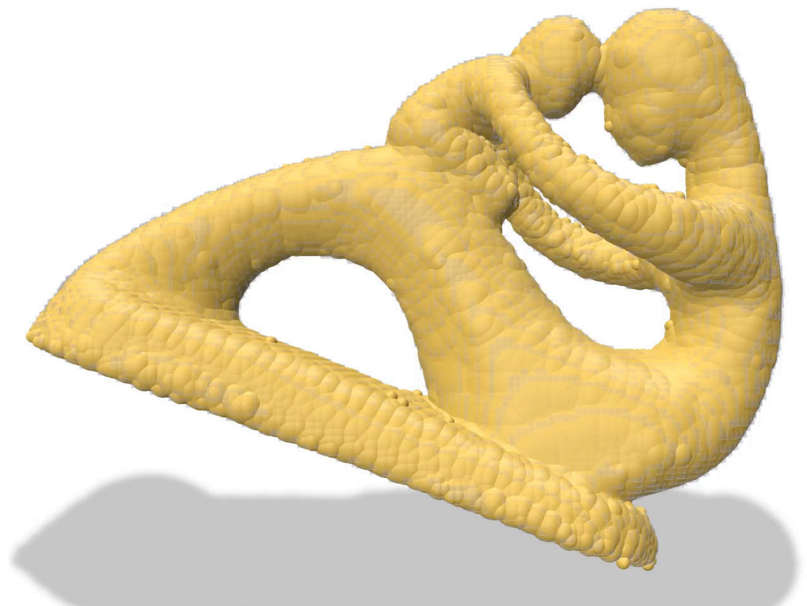
Separable Volumetric approaches







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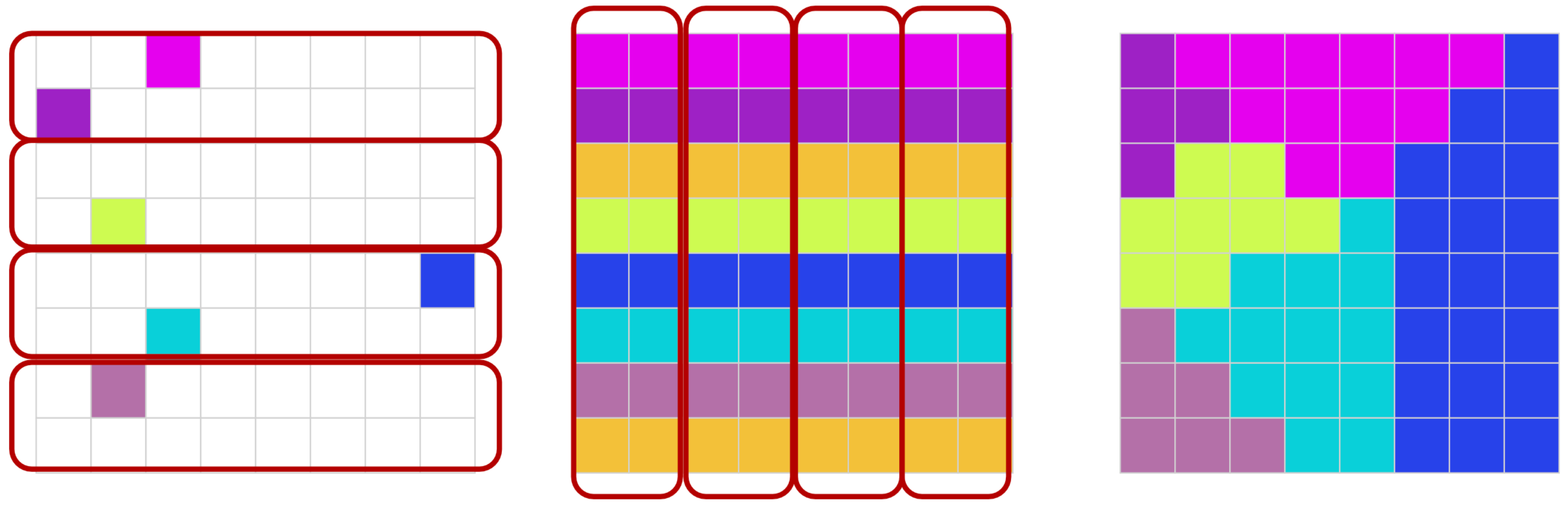
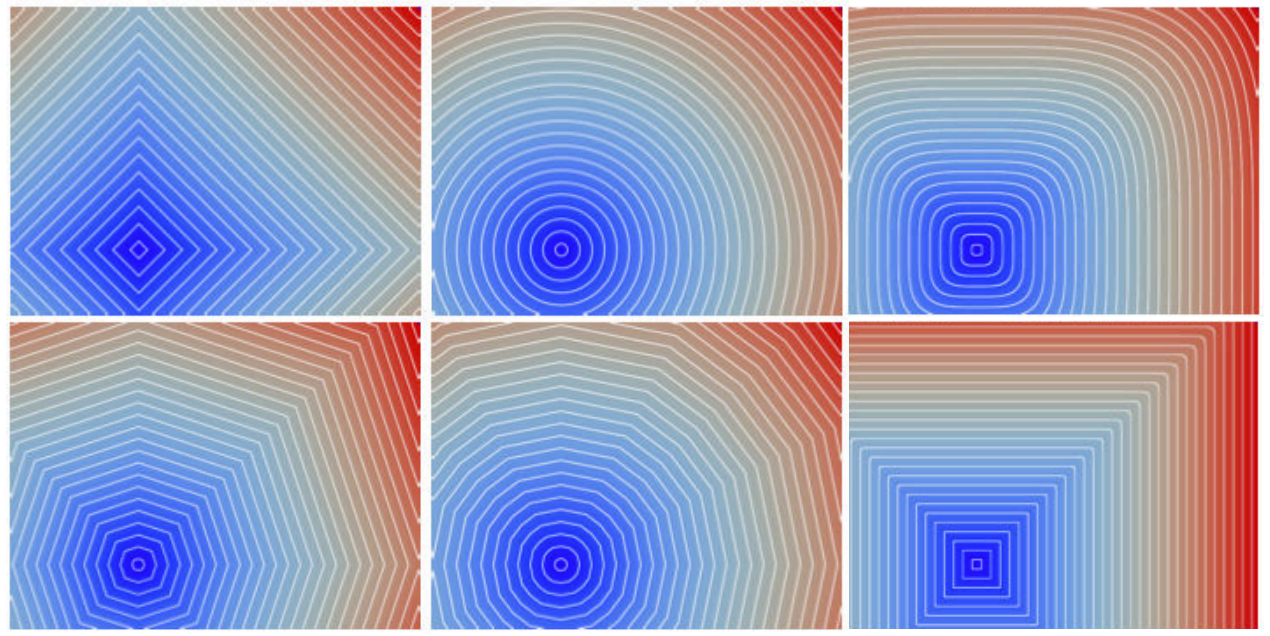
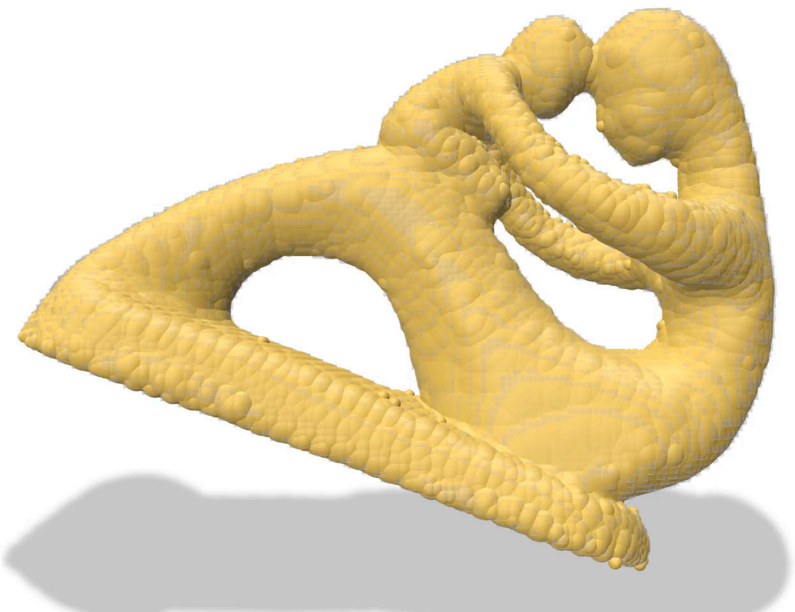




The separable algorithm is correct:



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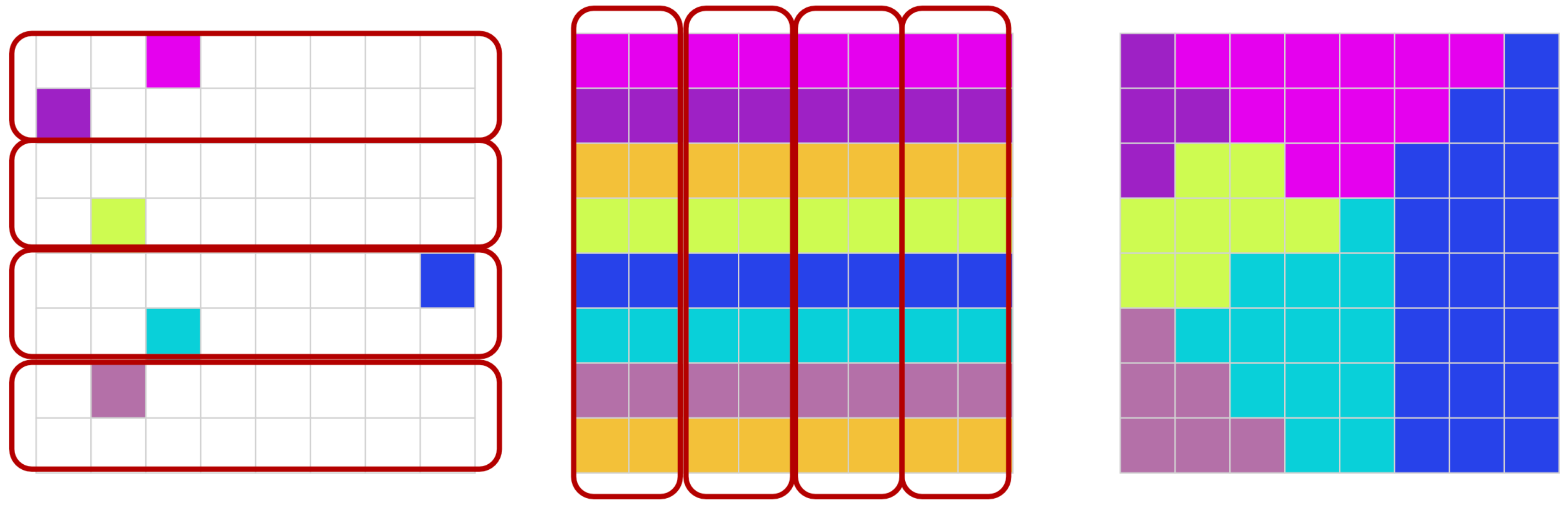

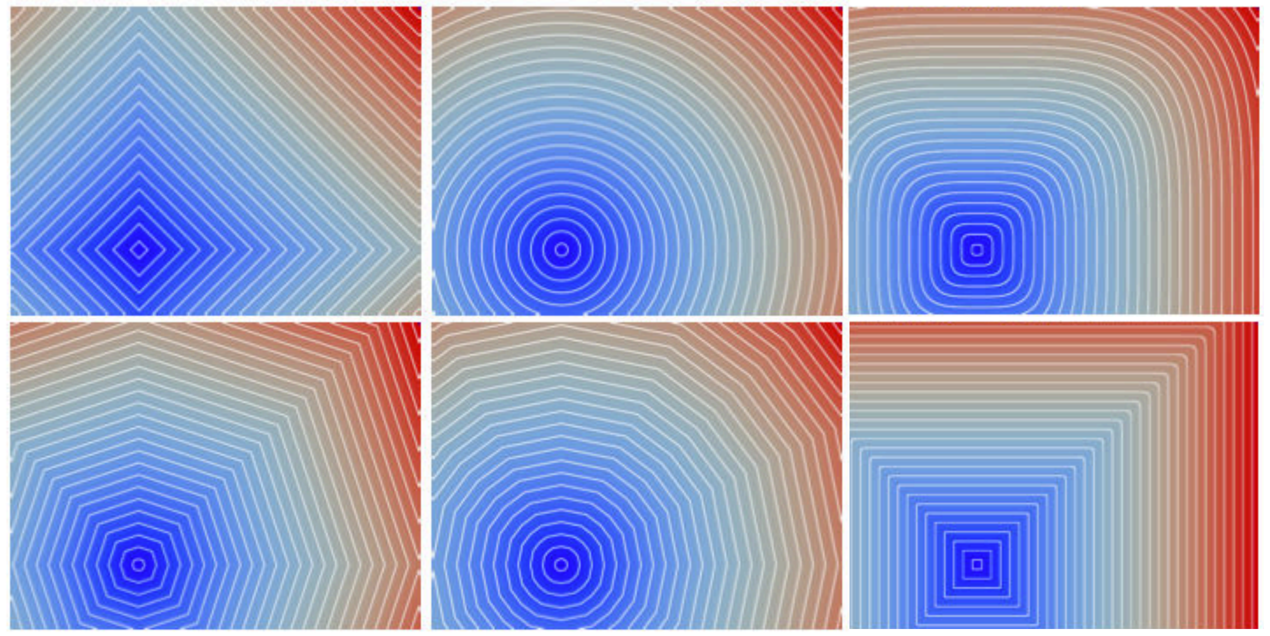

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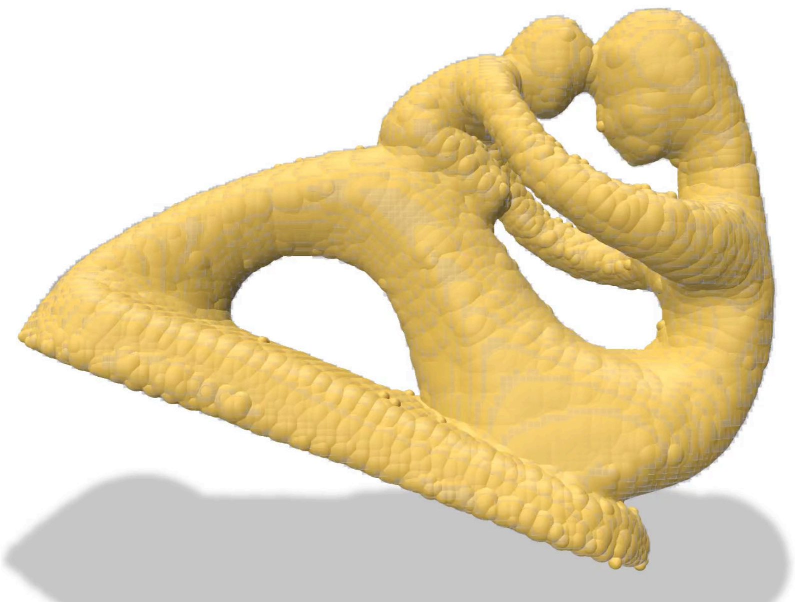
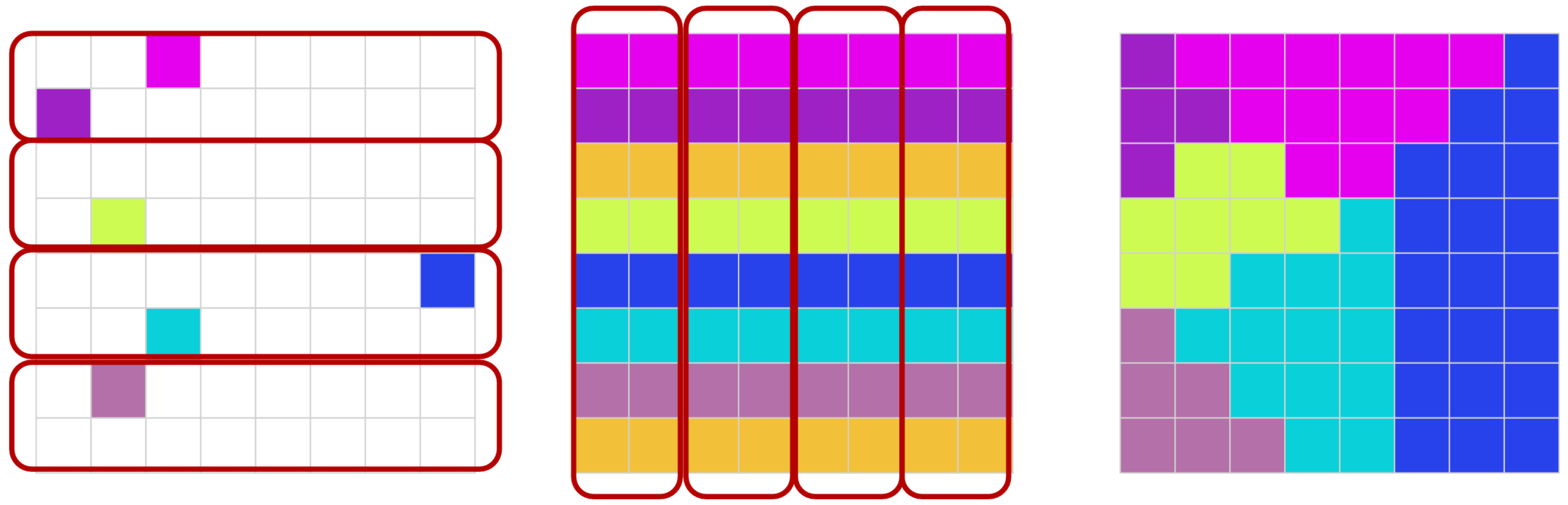
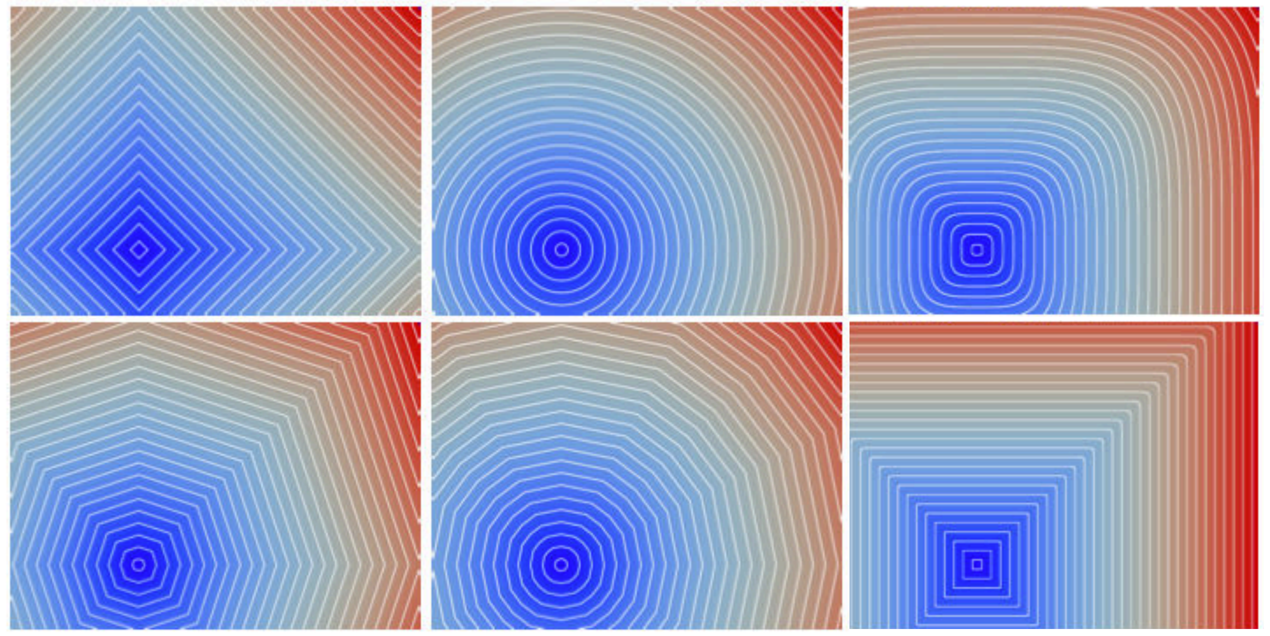


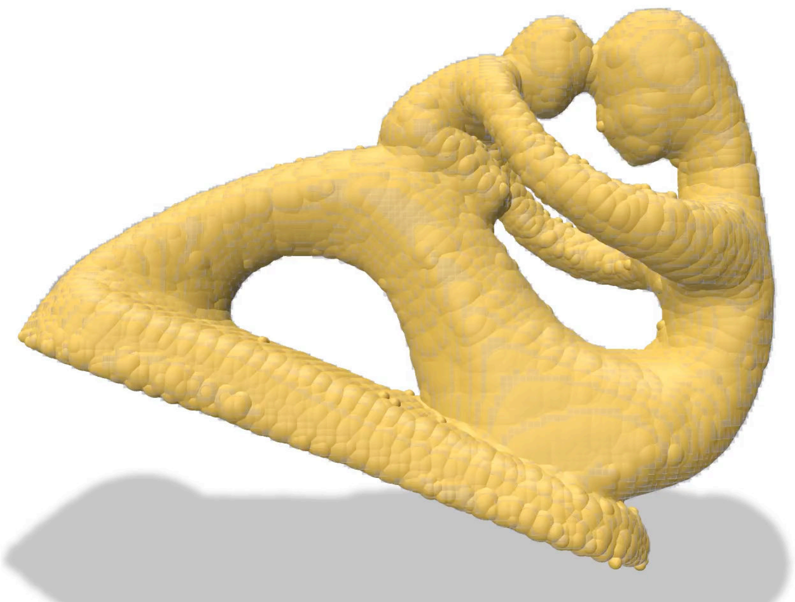
Separable Volumetric approaches



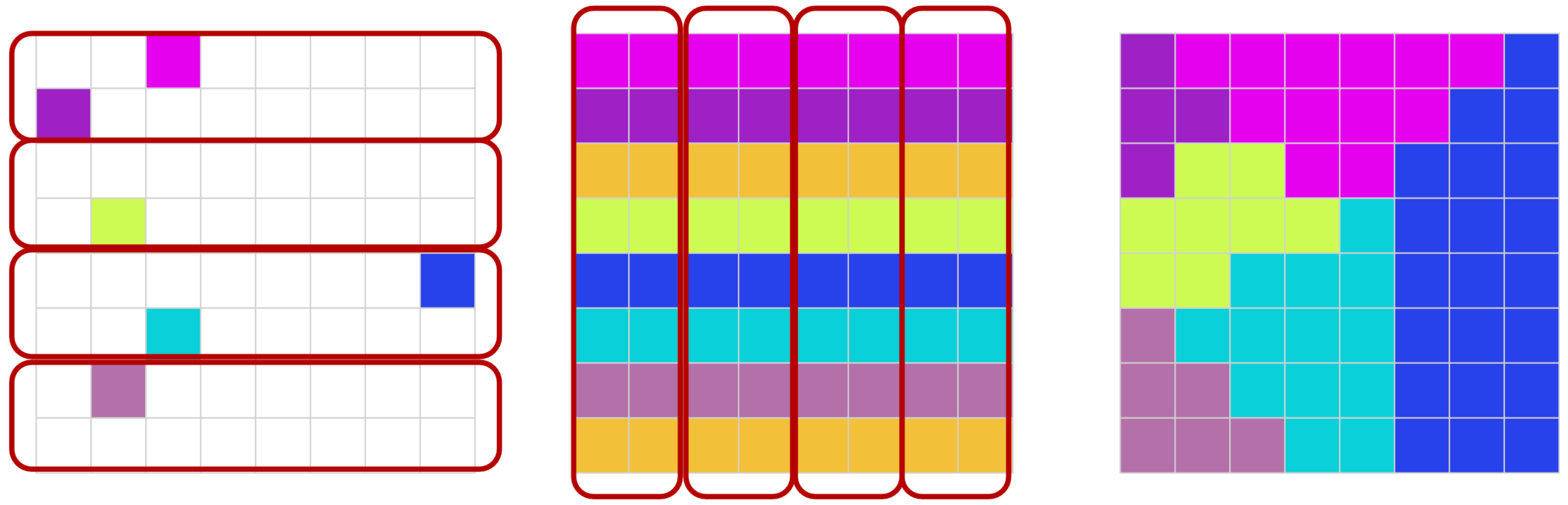
The separable algorithm is correct:

- for any dimension
- for any metric with axis symmetric unit ball (e.g. any l_p, chamfer norms)



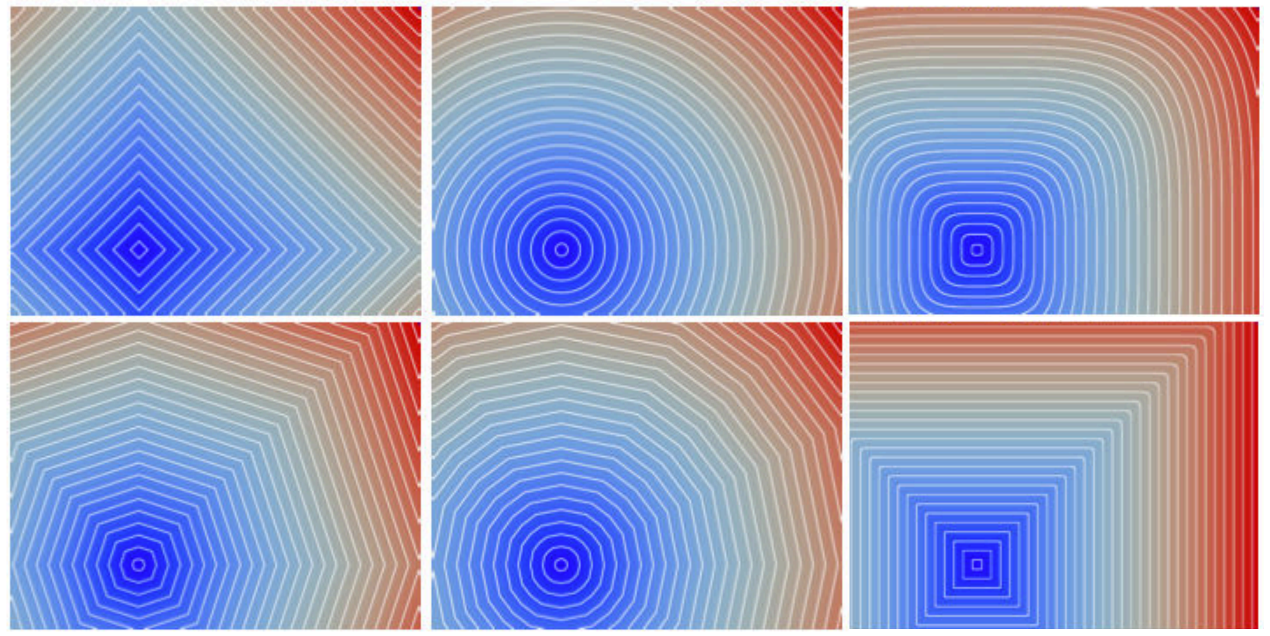


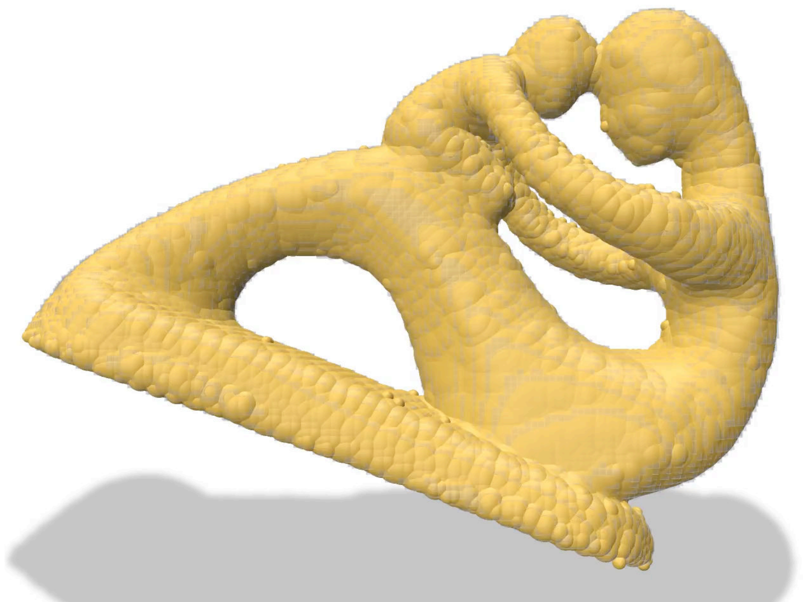
Separable Volumetric approaches



The separable algorithm is correct:

- for any dimension
- for any metric with axis symmetric unit ball (e.g. any l_p, chamfer norms)
- on any toroidal nD domains





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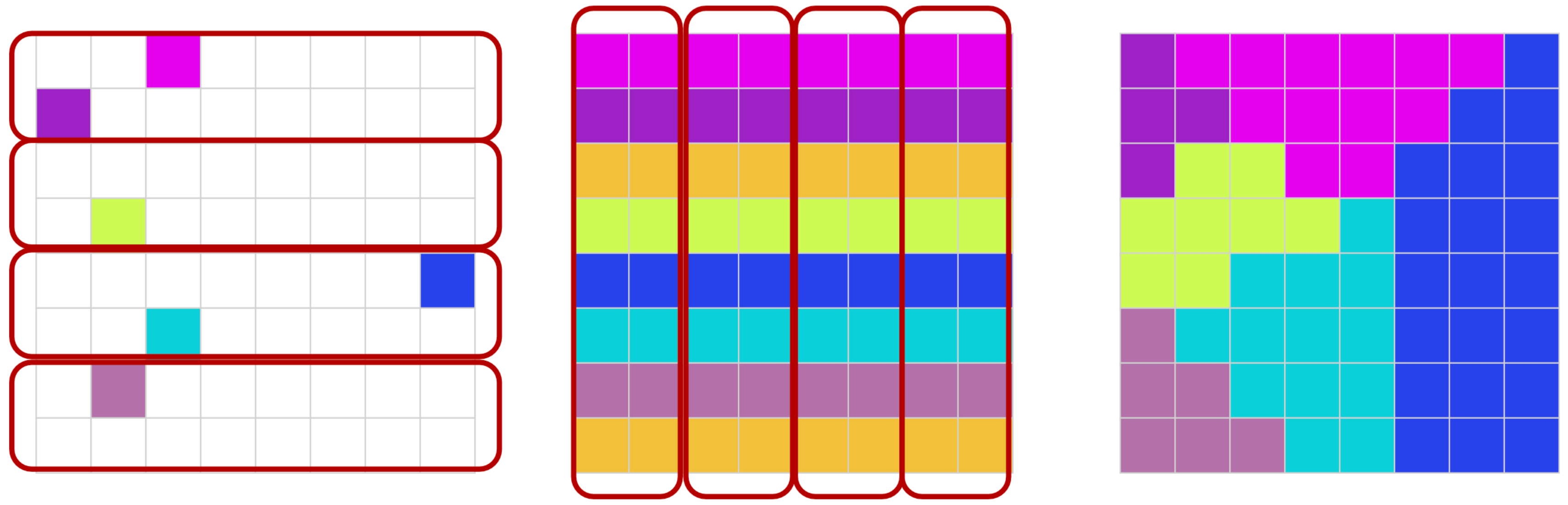

The separable algorithm is correct:

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- for any metric with axis symmetric unit ball (e.g. any l_p, chamfer norms)
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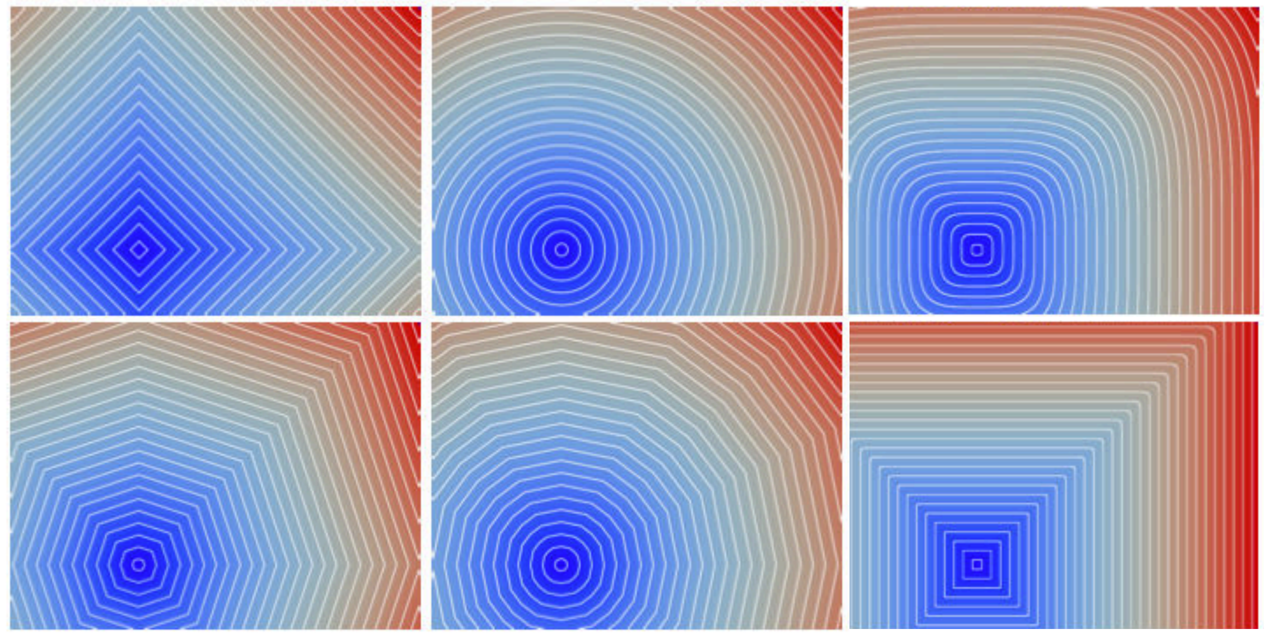
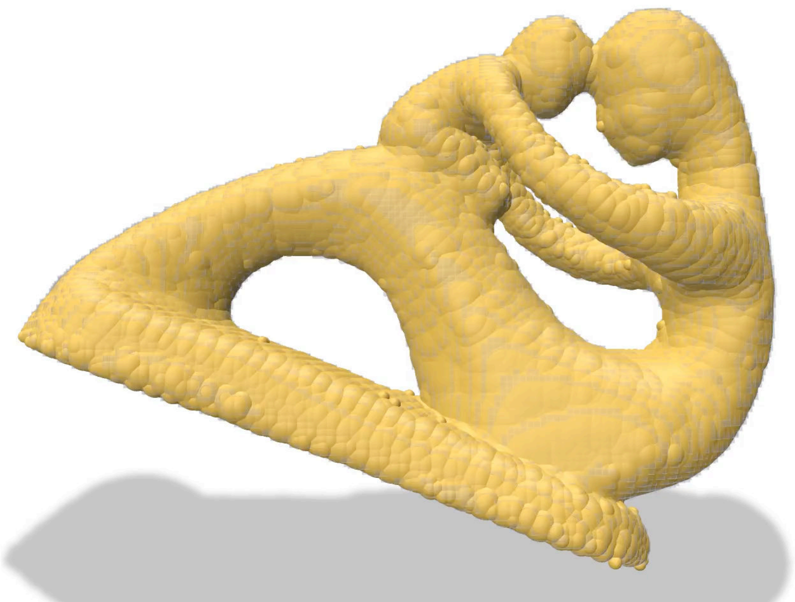
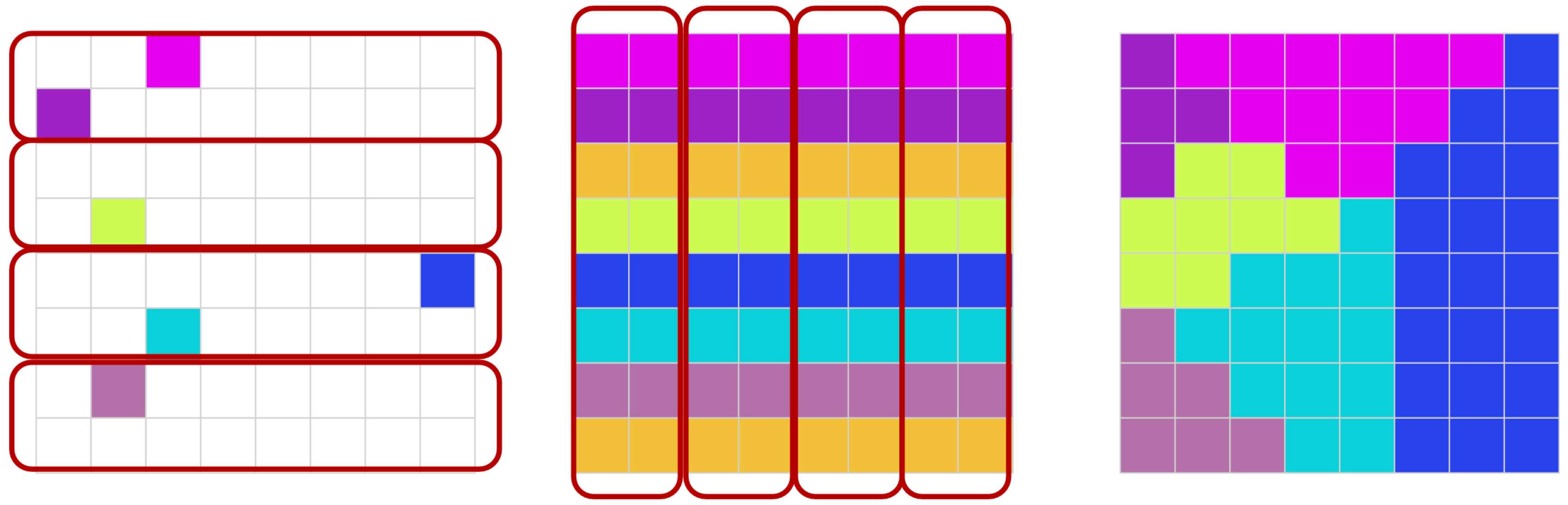
Exact and linear in time w.r.t. the number of grid points $O(d \cdot n^d)$ for l_2





Separable Volumetric approaches



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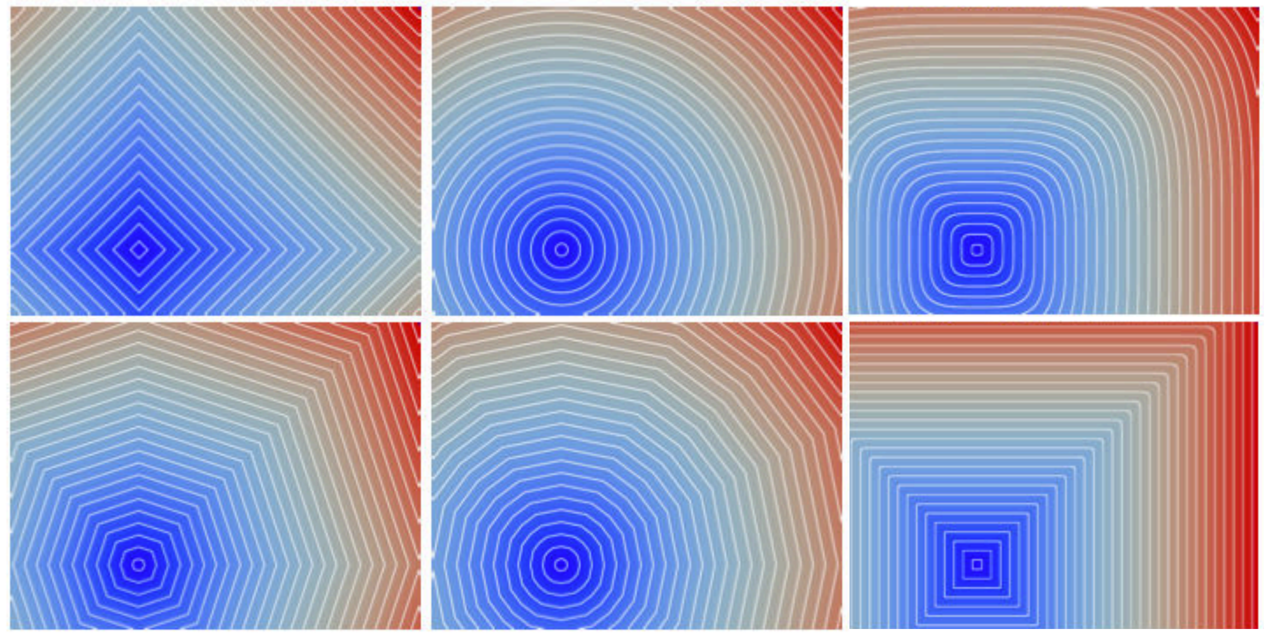

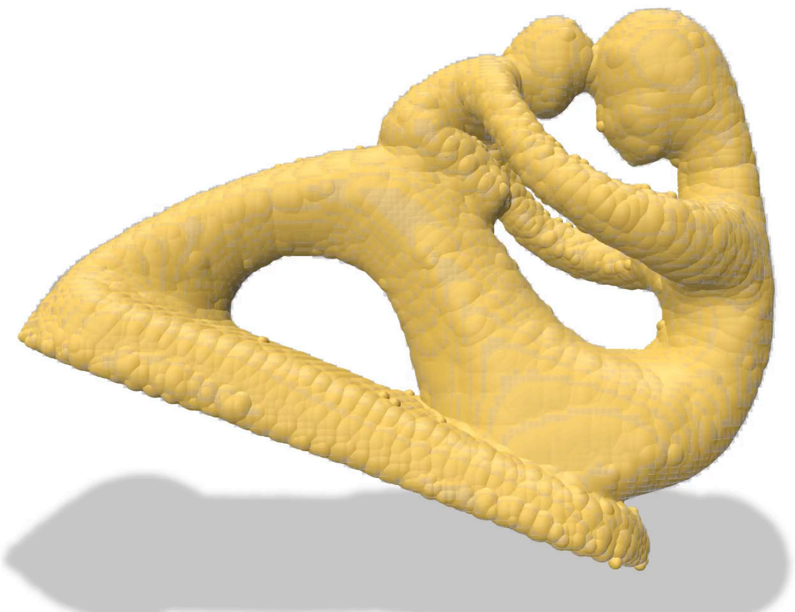

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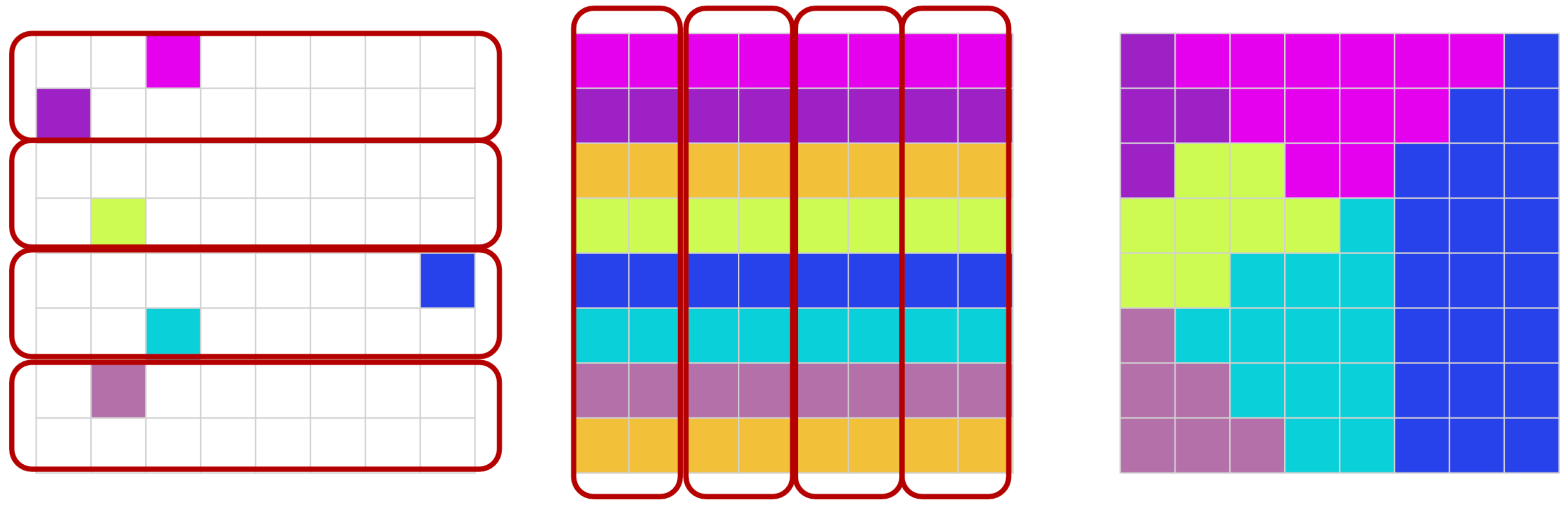

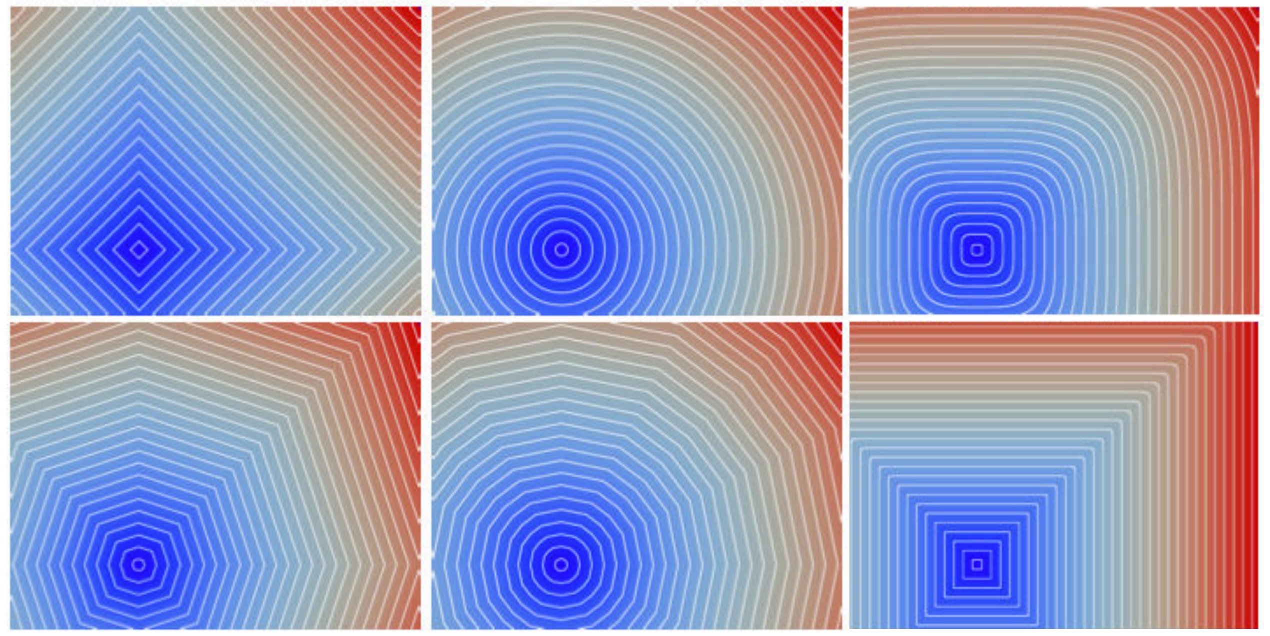

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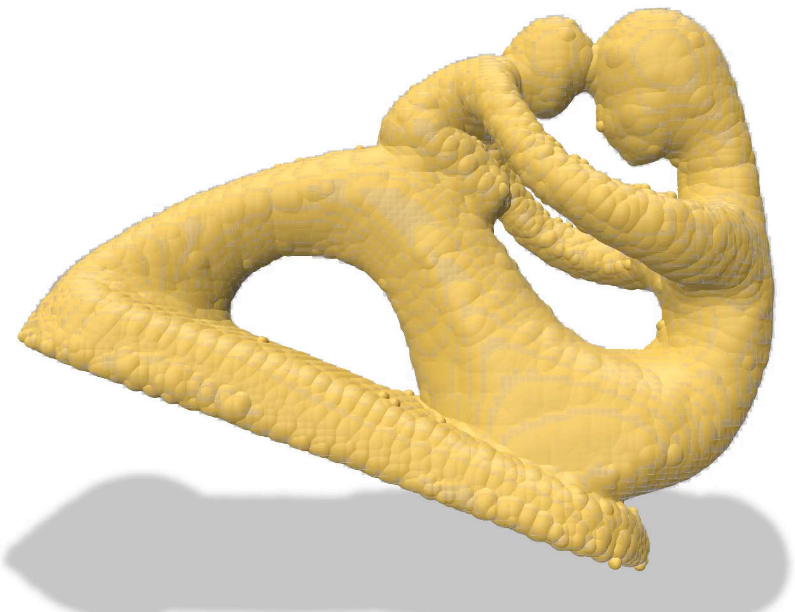

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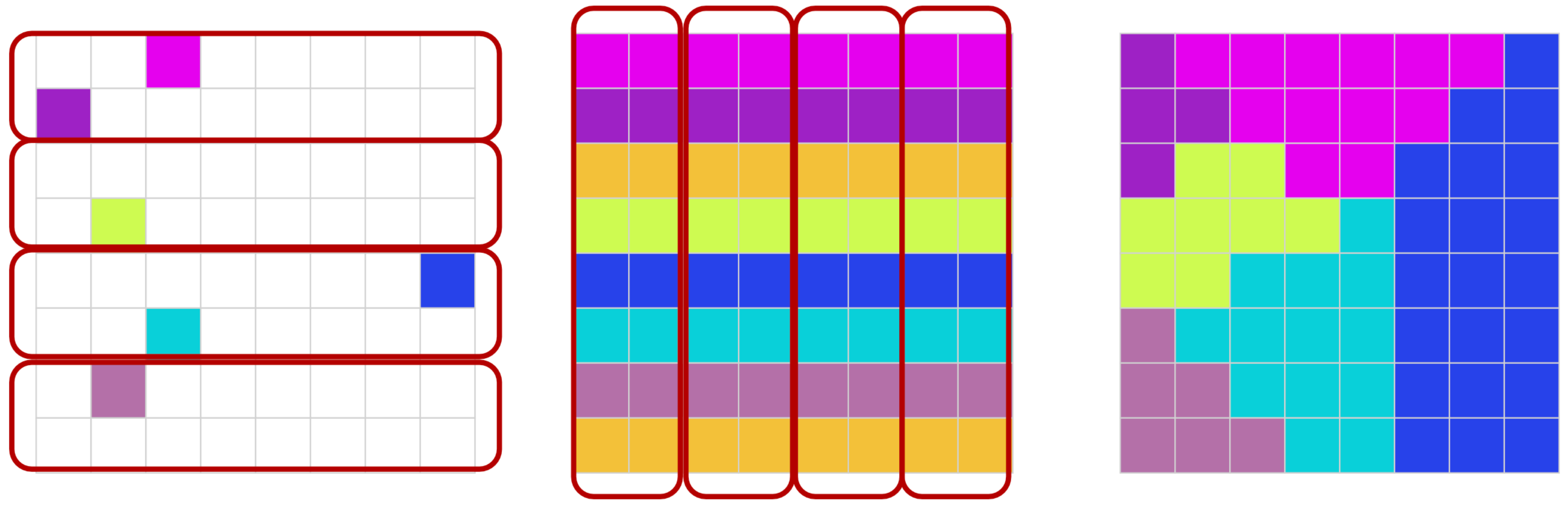

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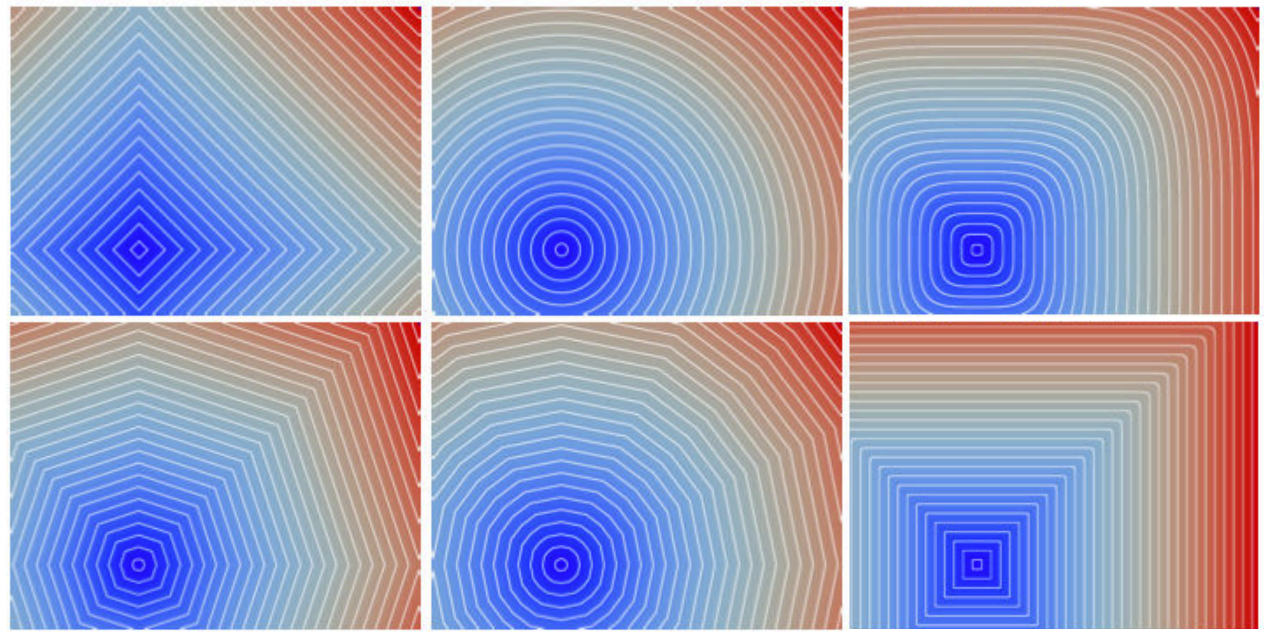

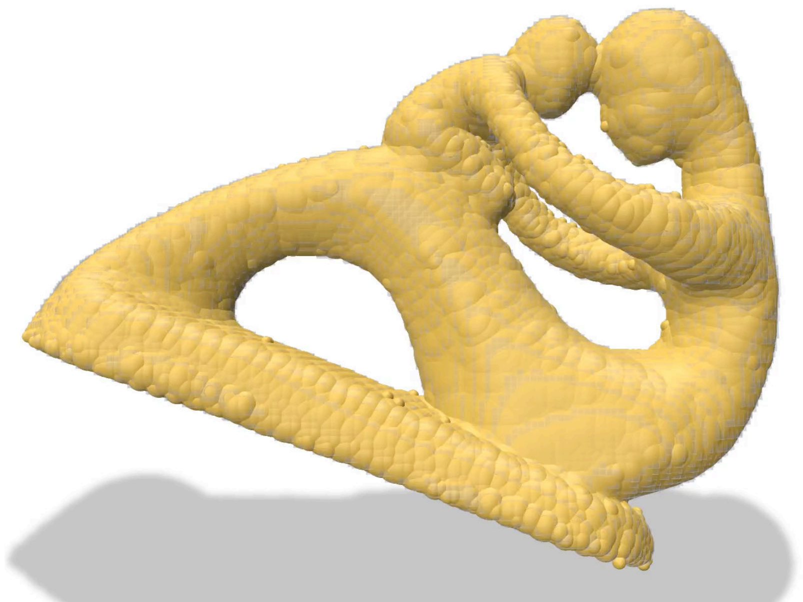

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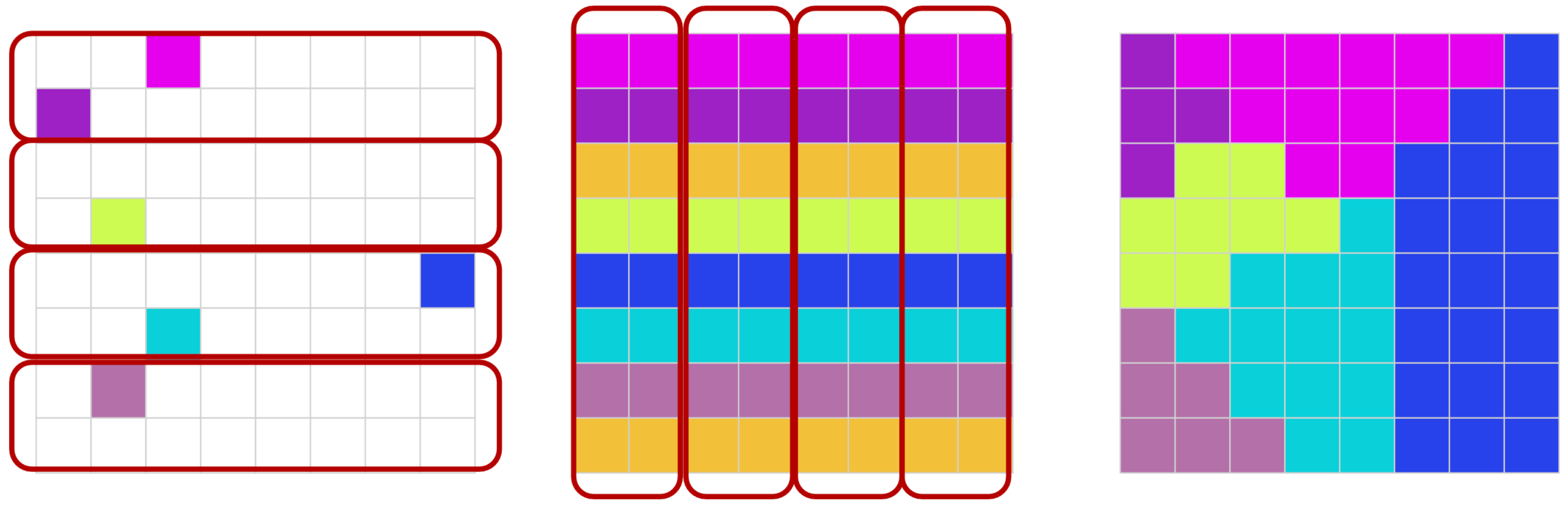

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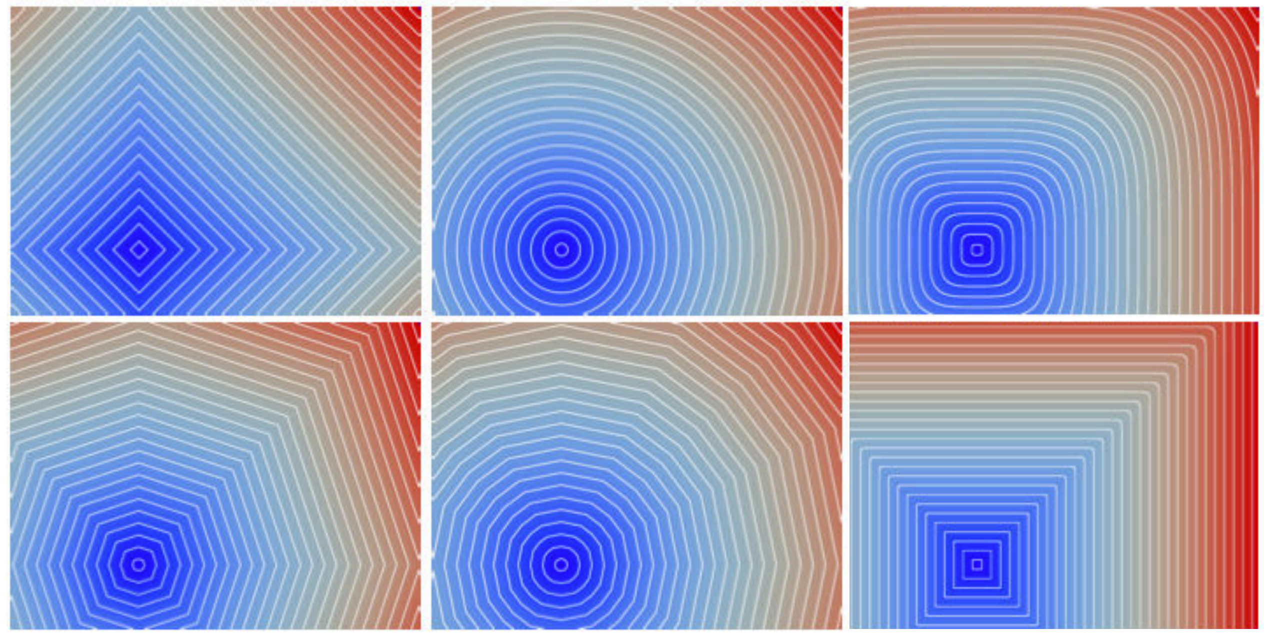

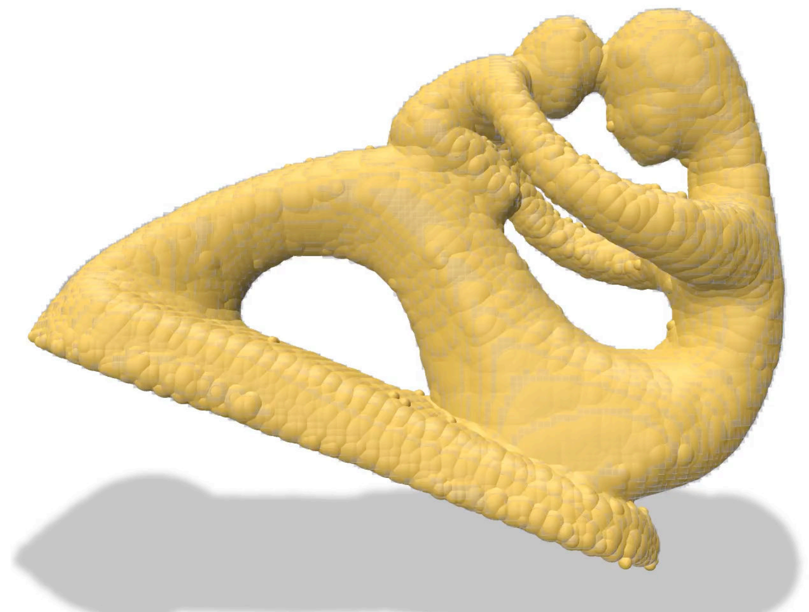

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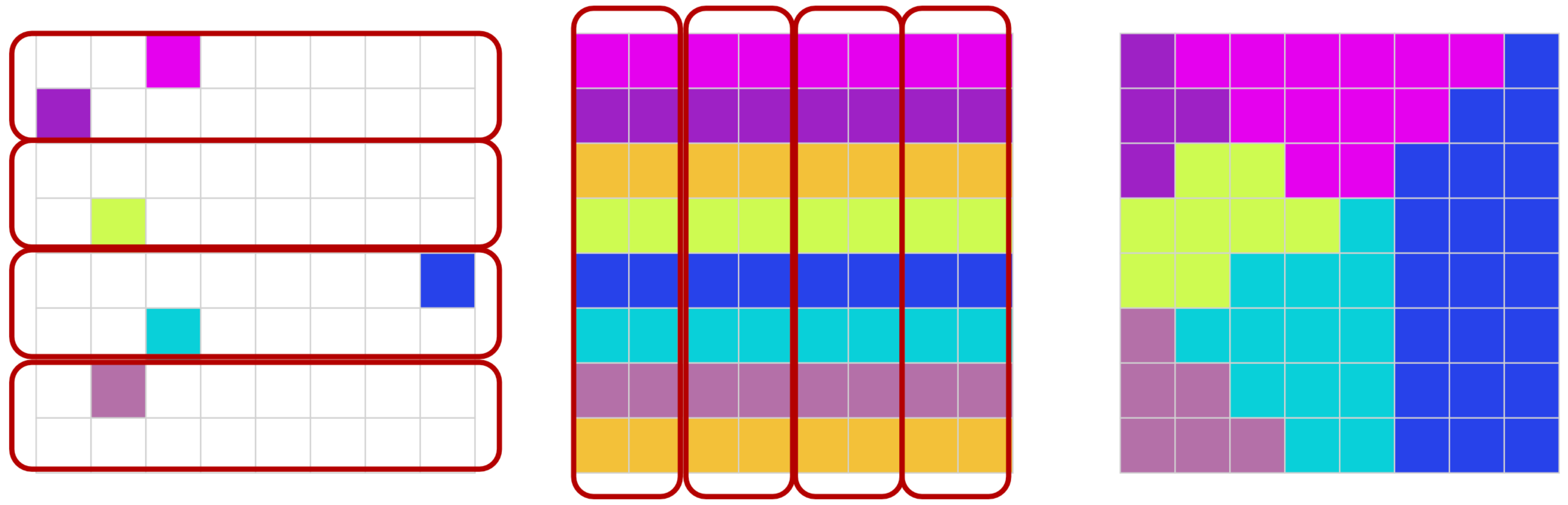

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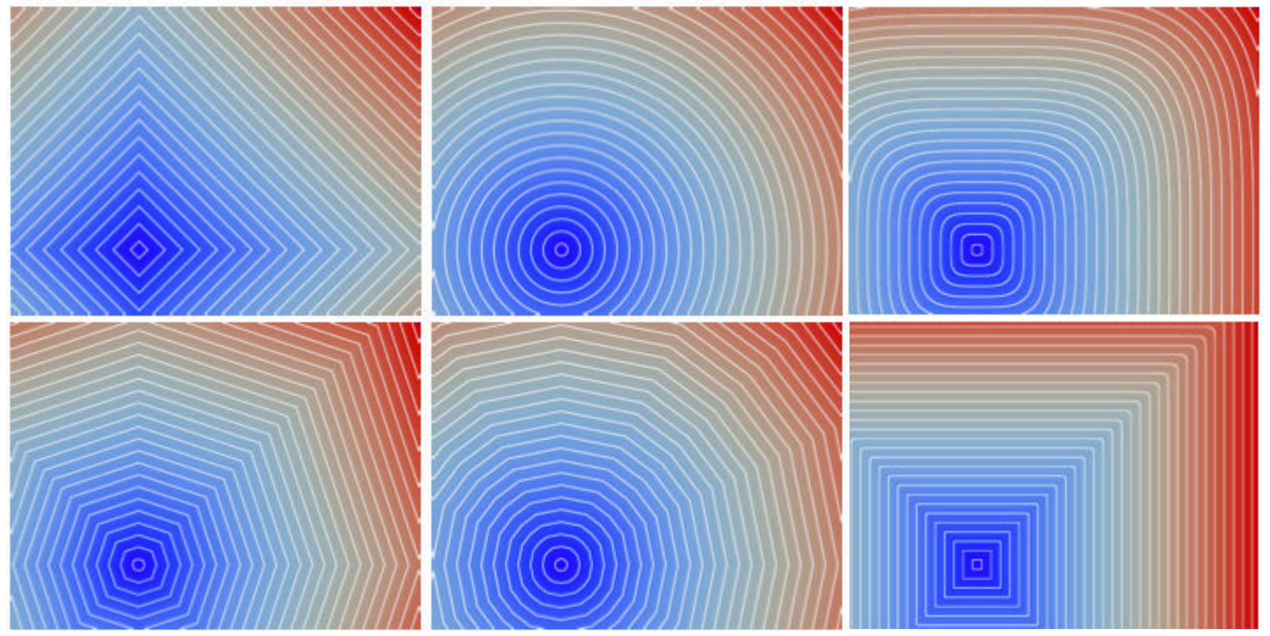

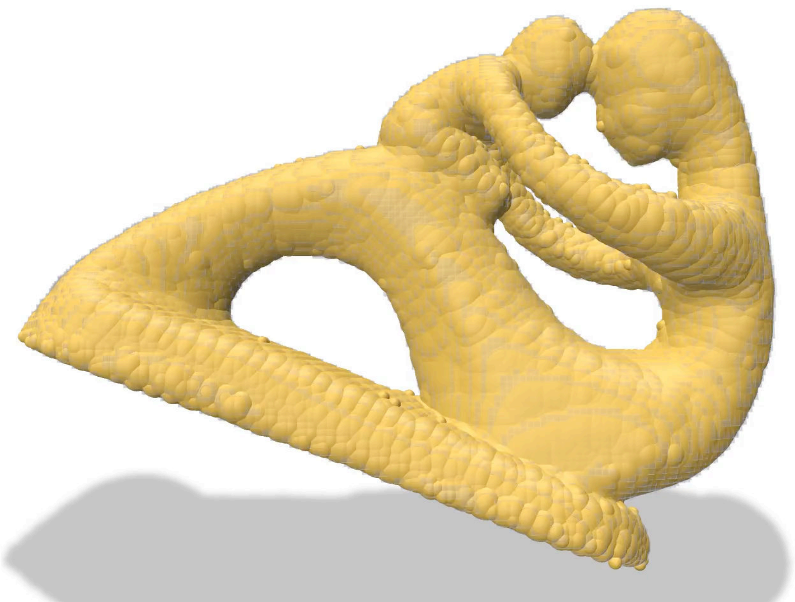

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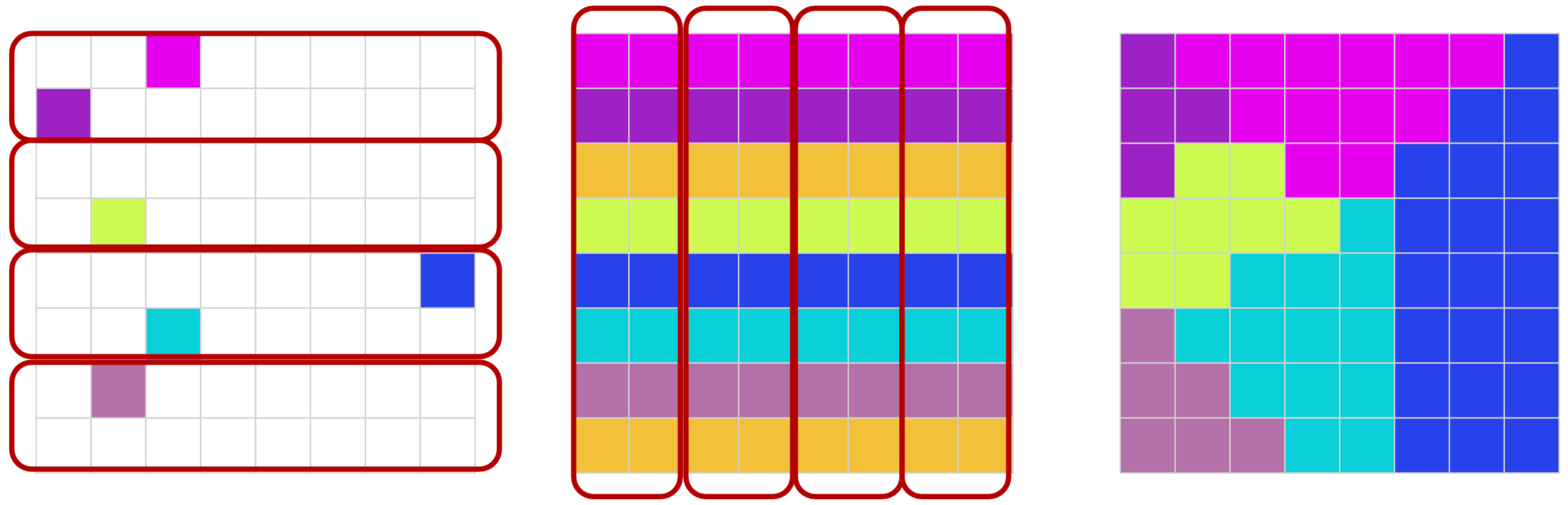

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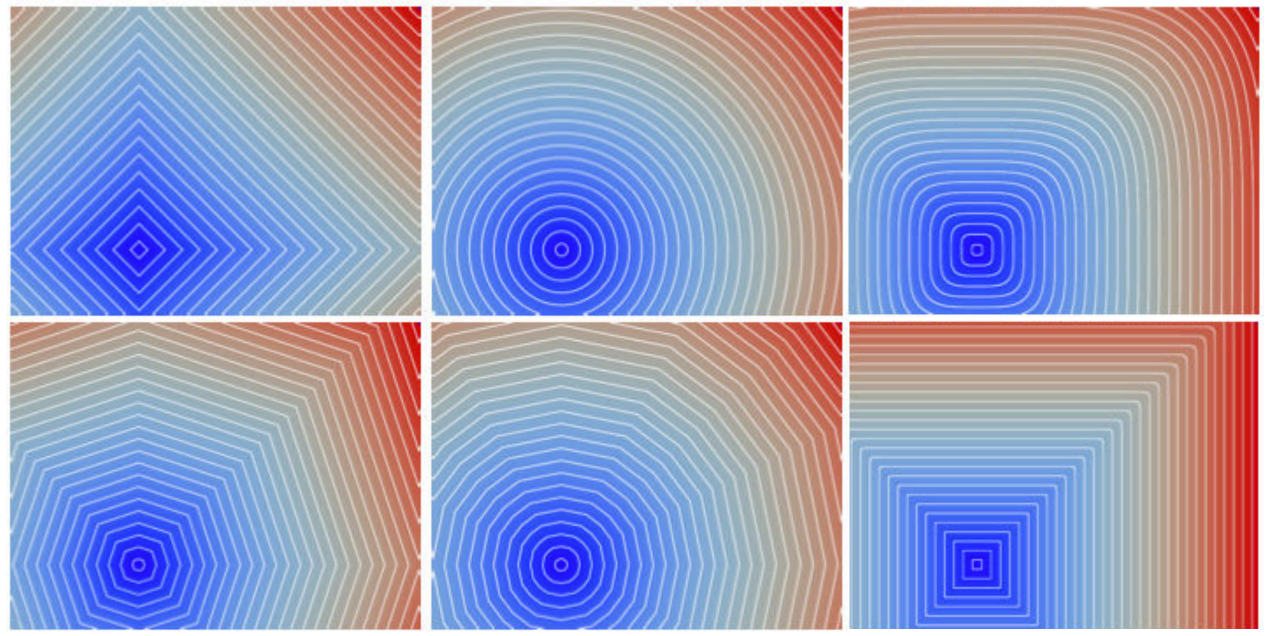

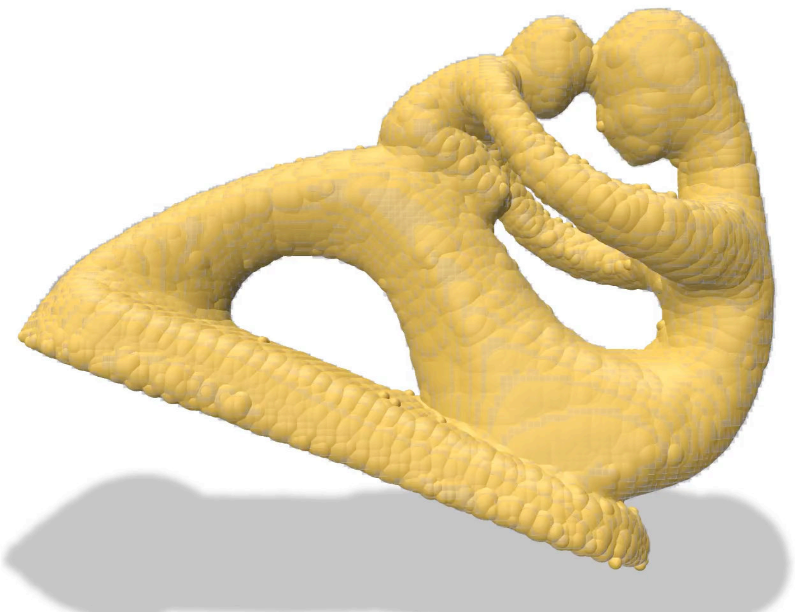

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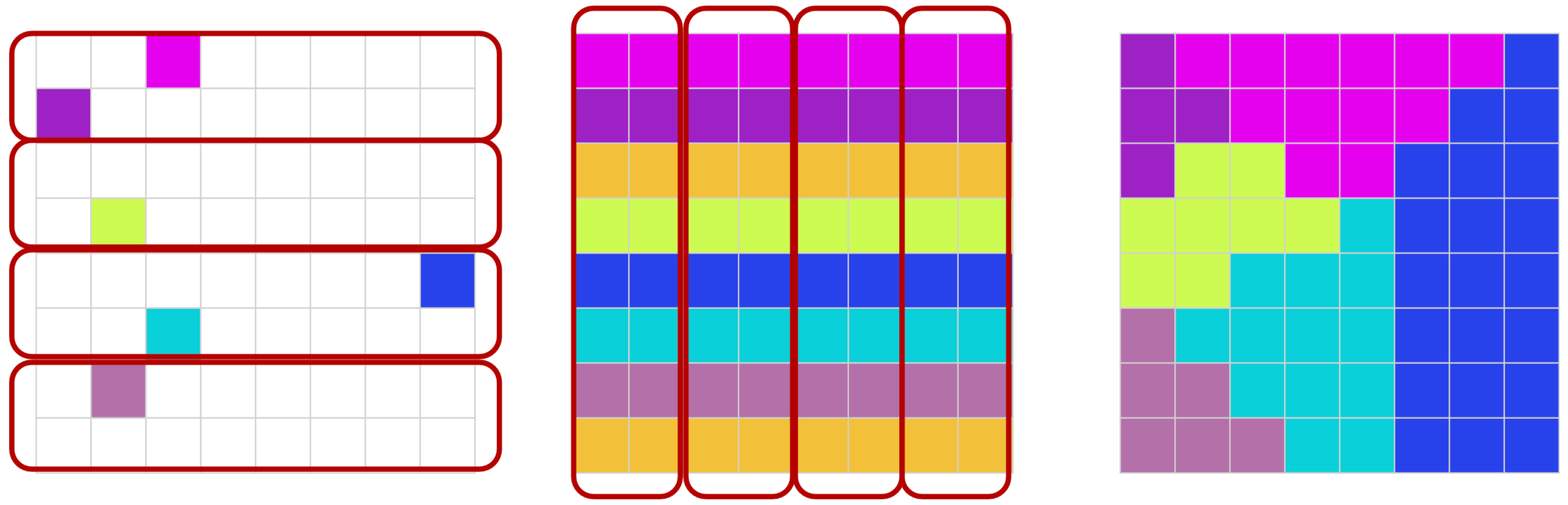

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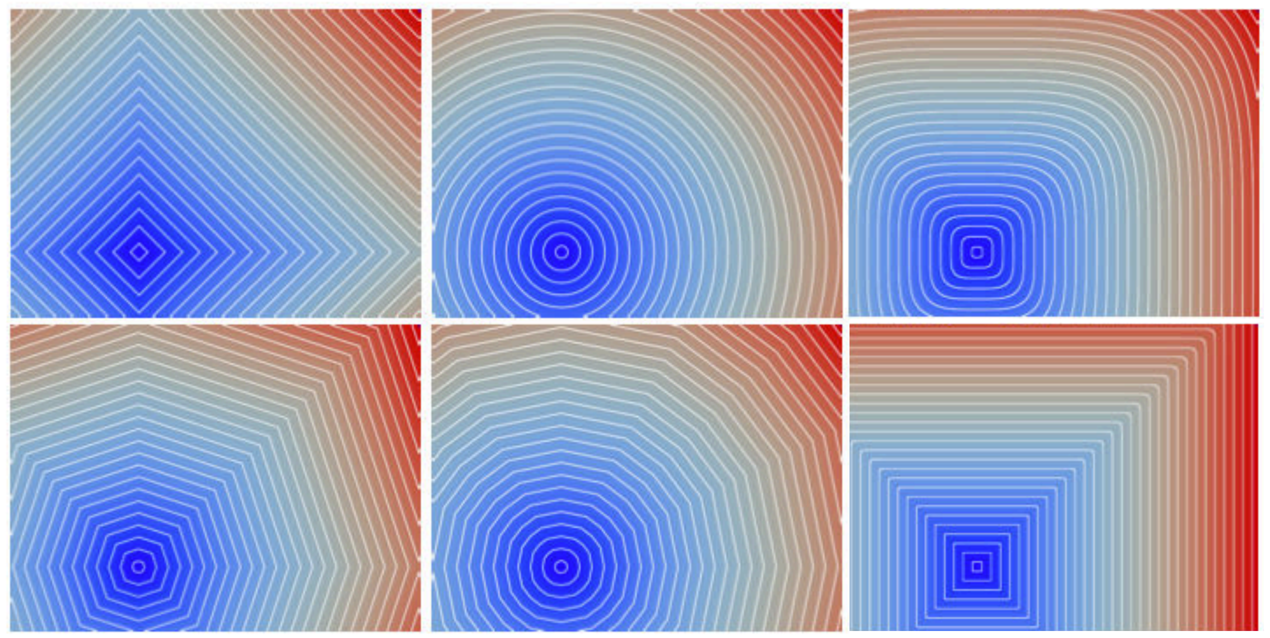

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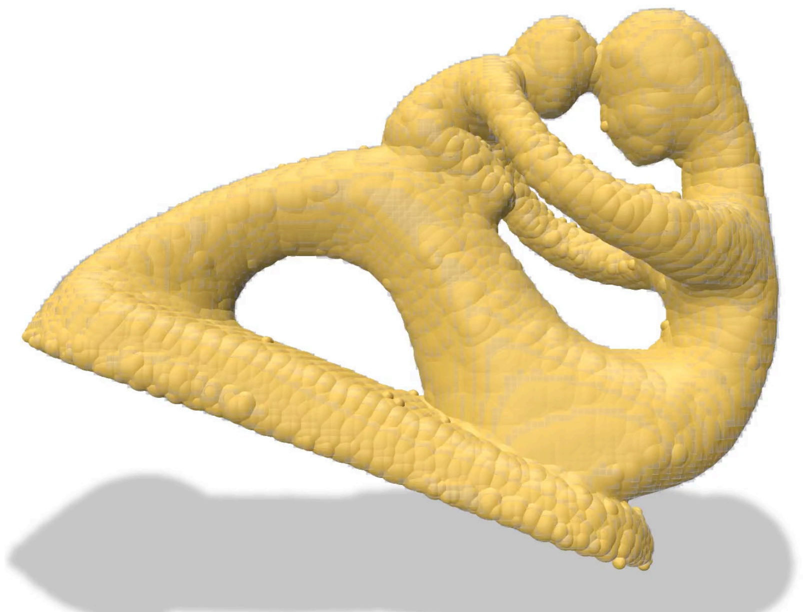

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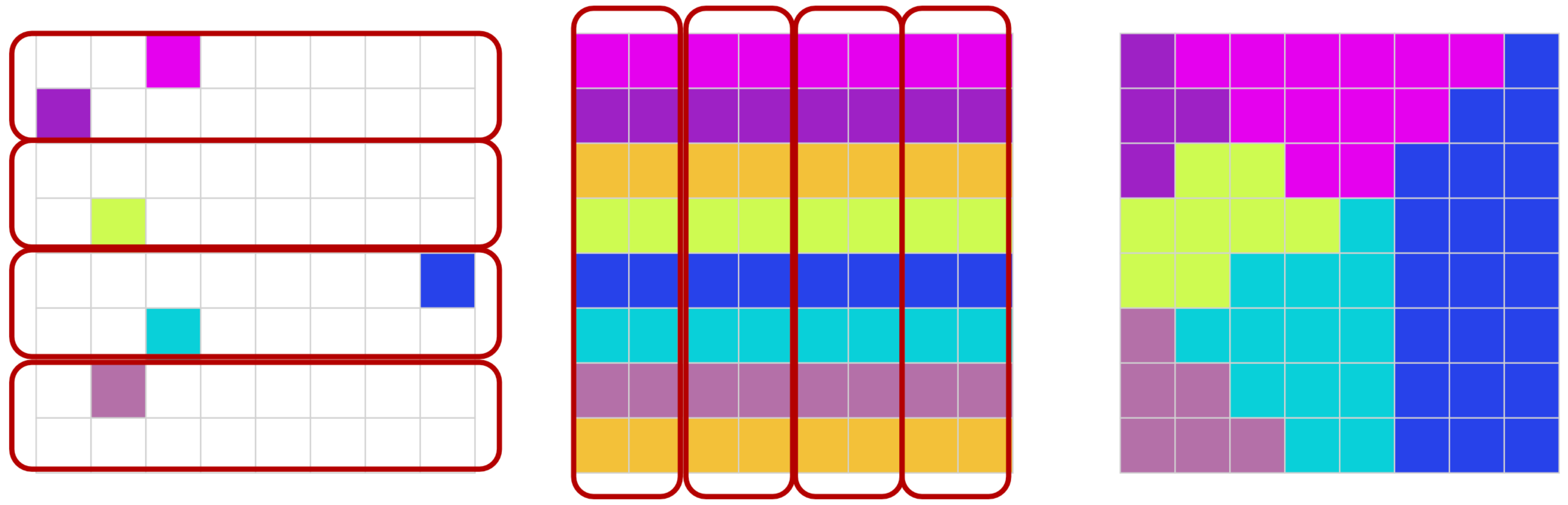

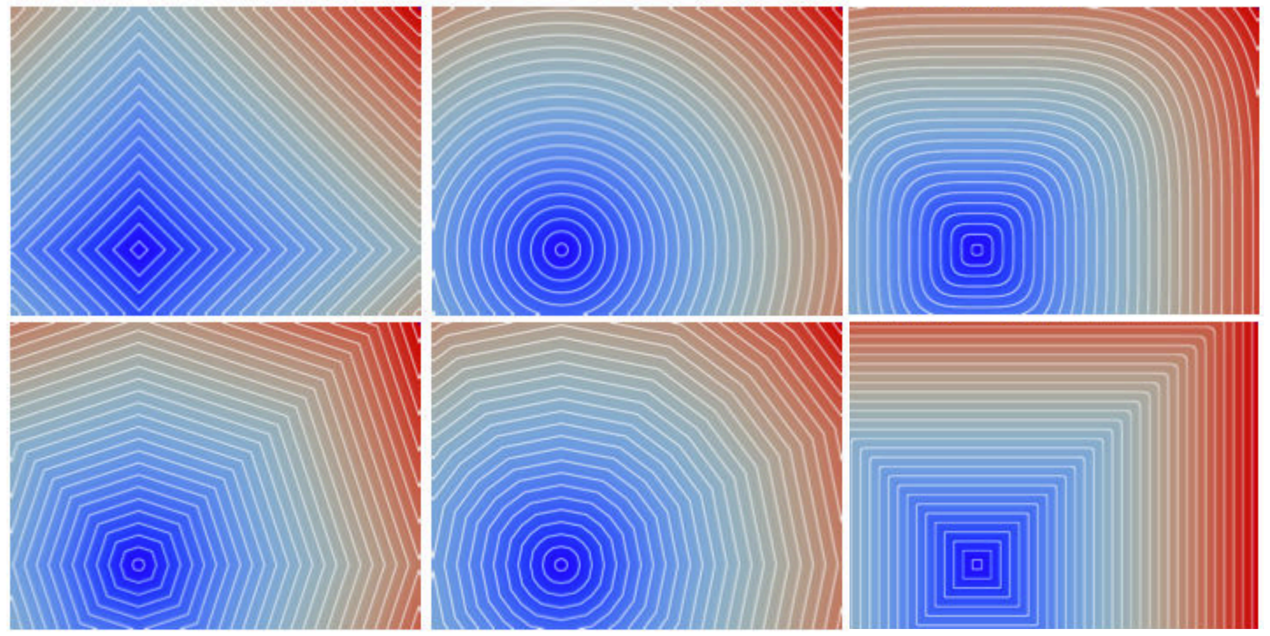

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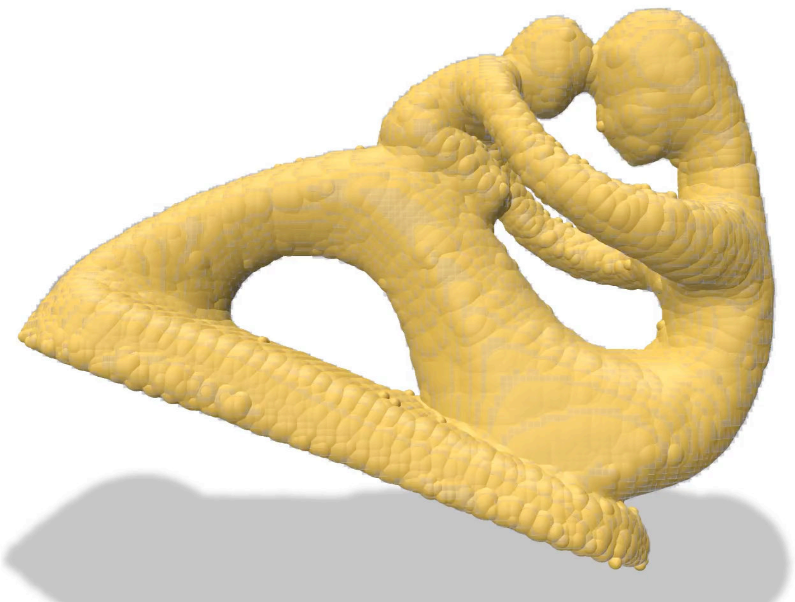

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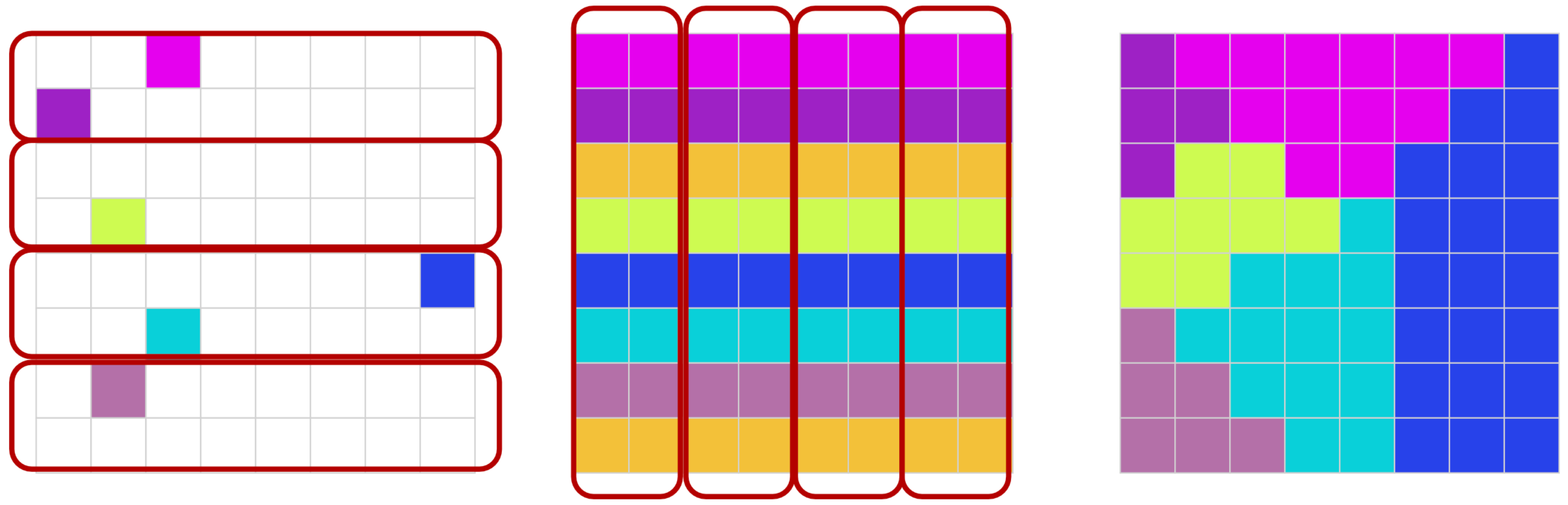

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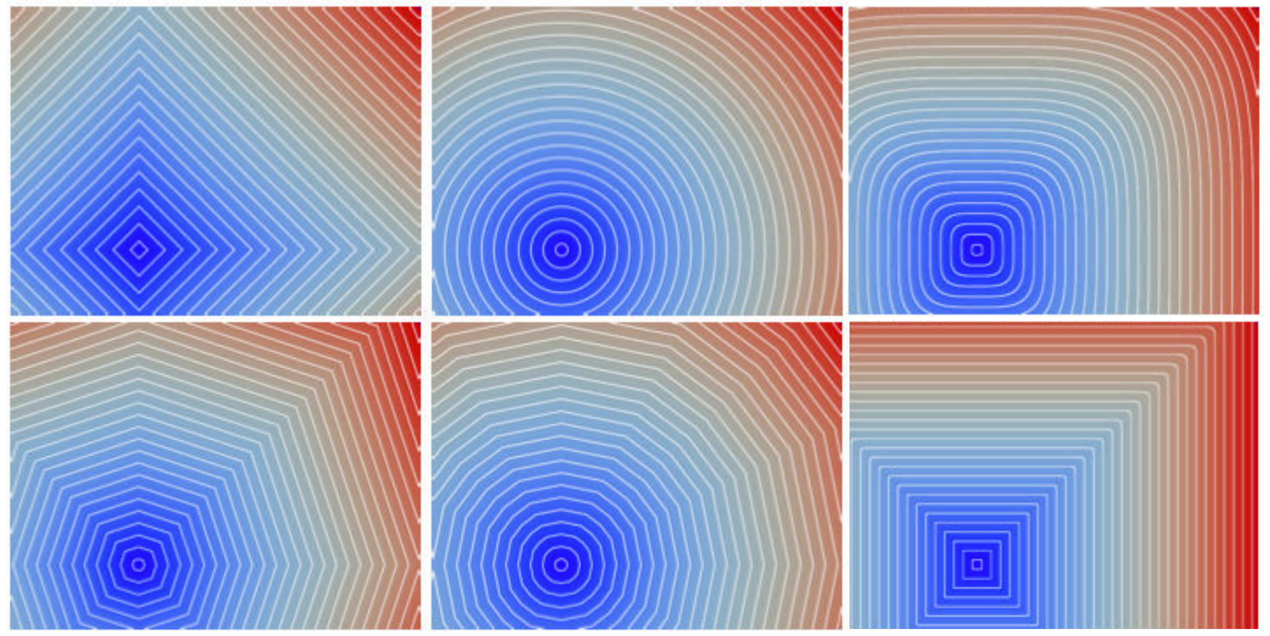

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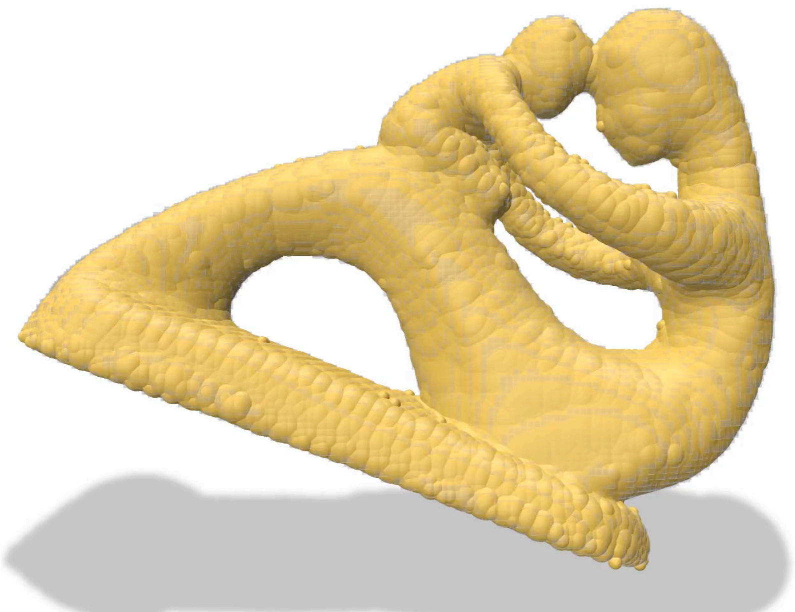

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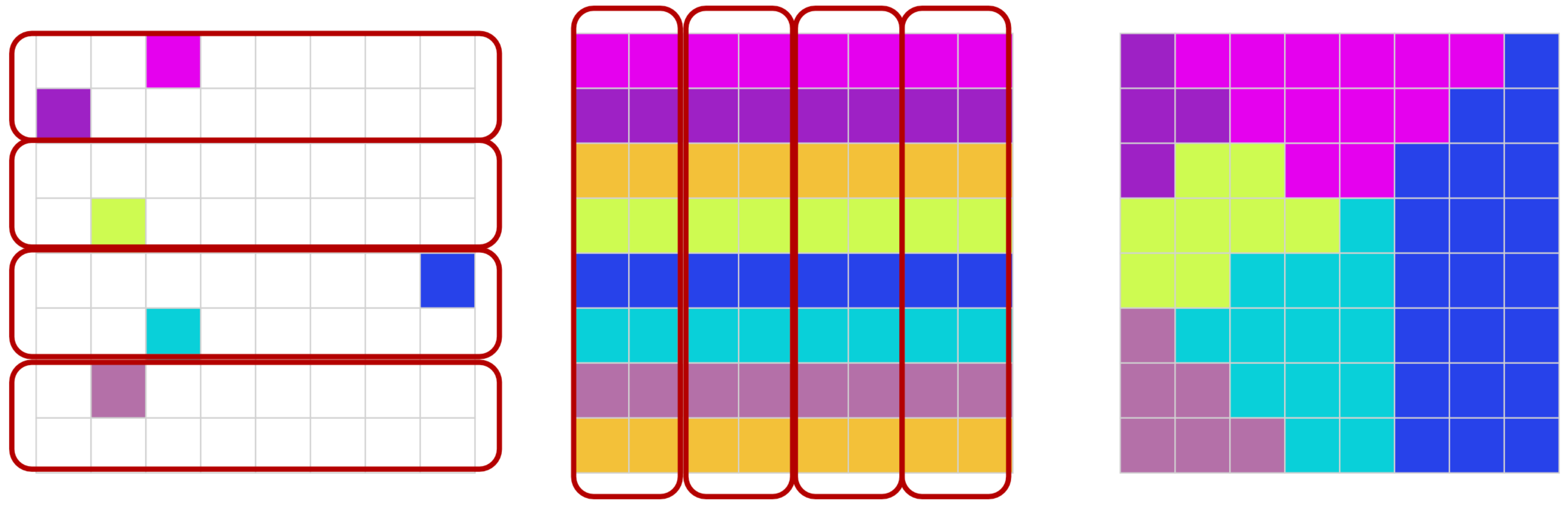

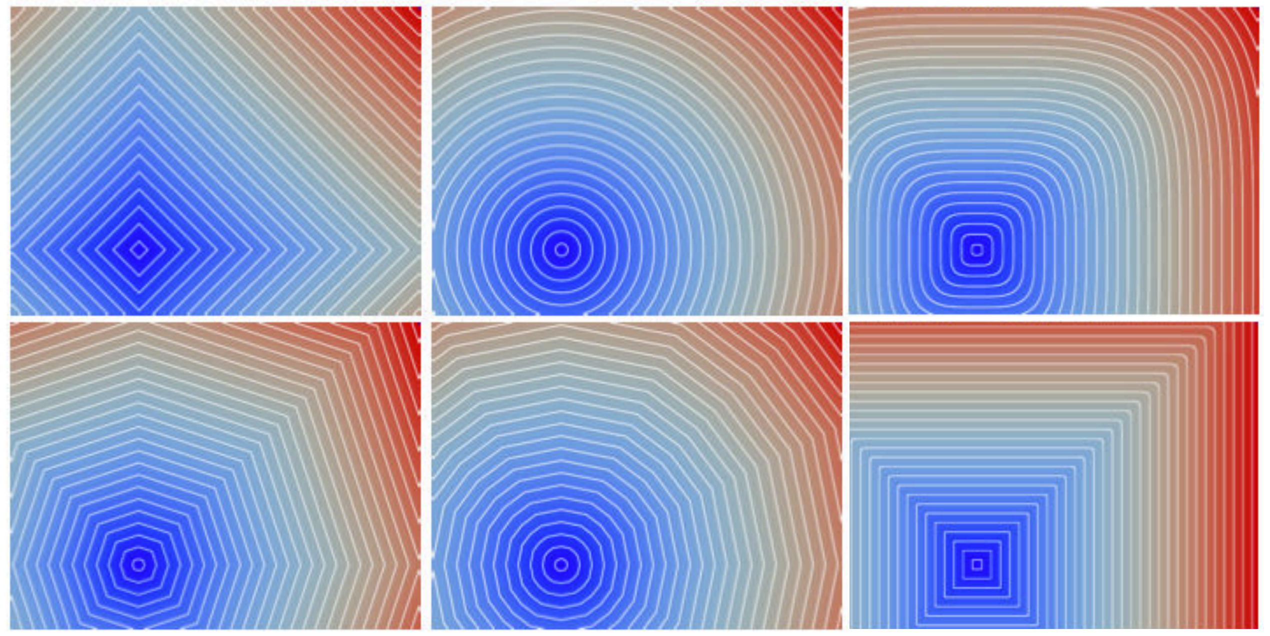

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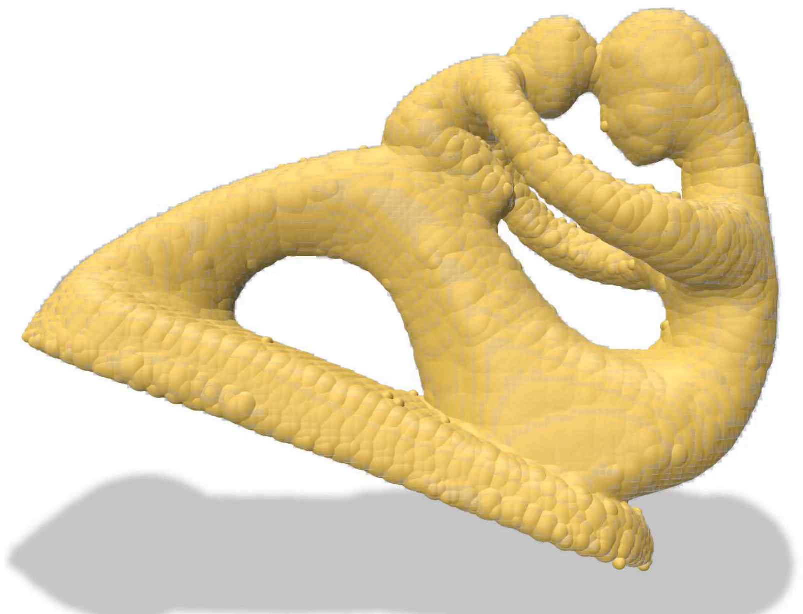

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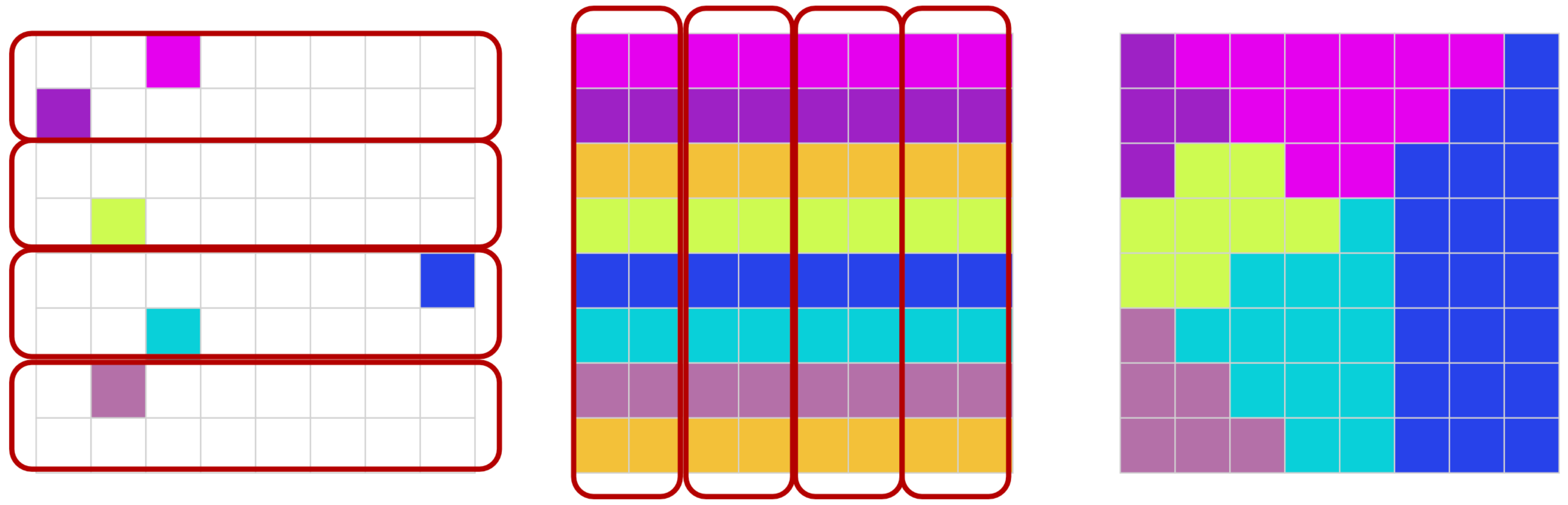

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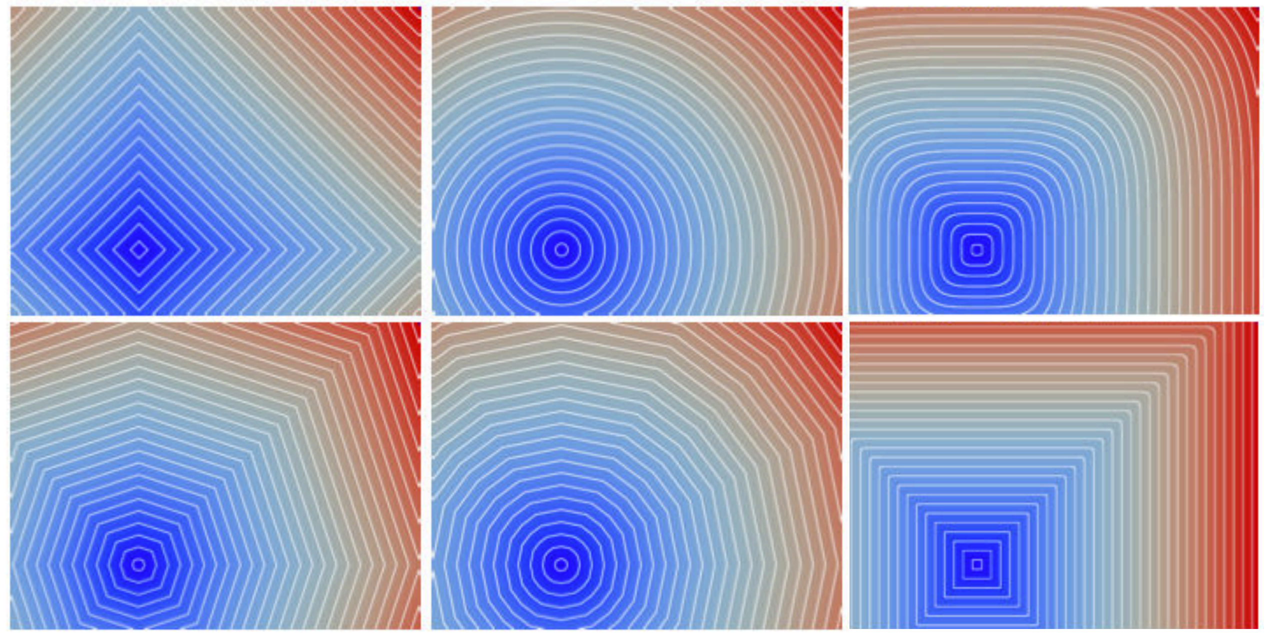

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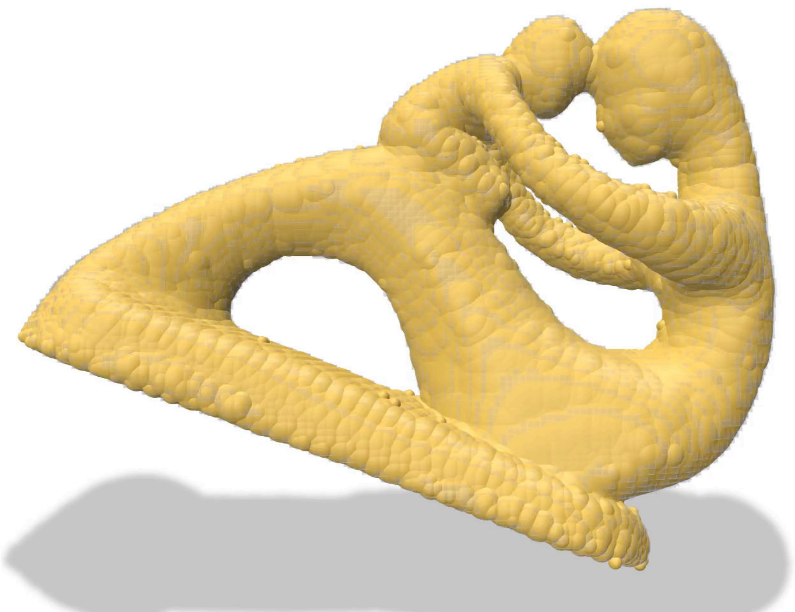

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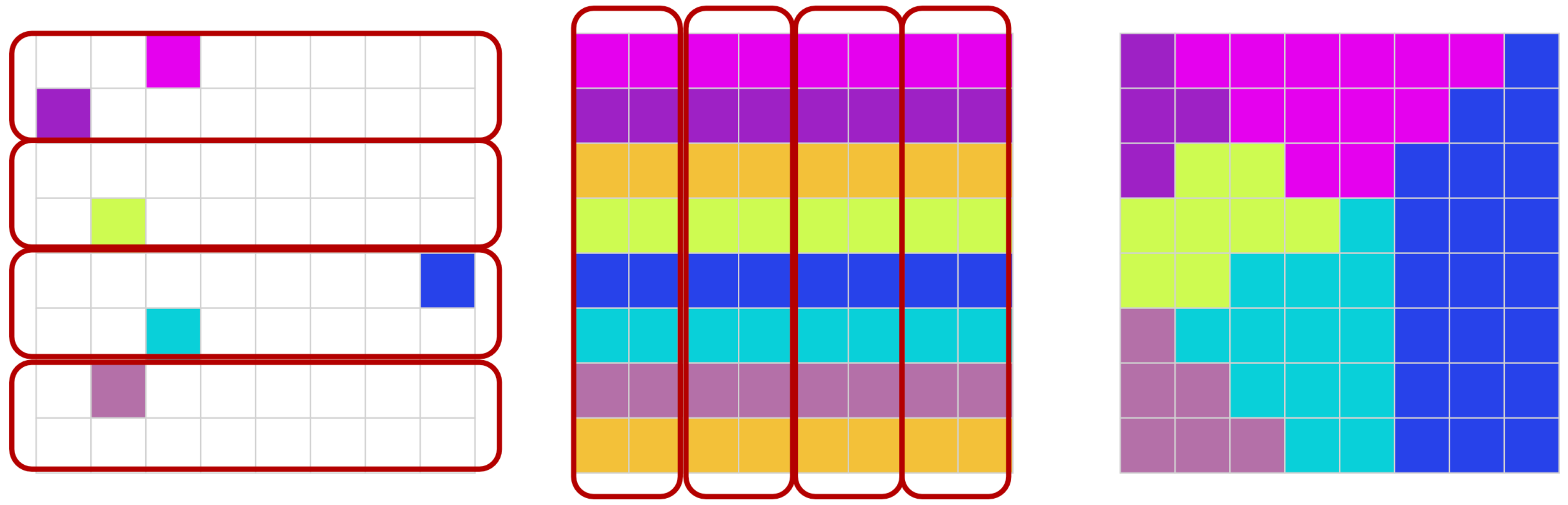

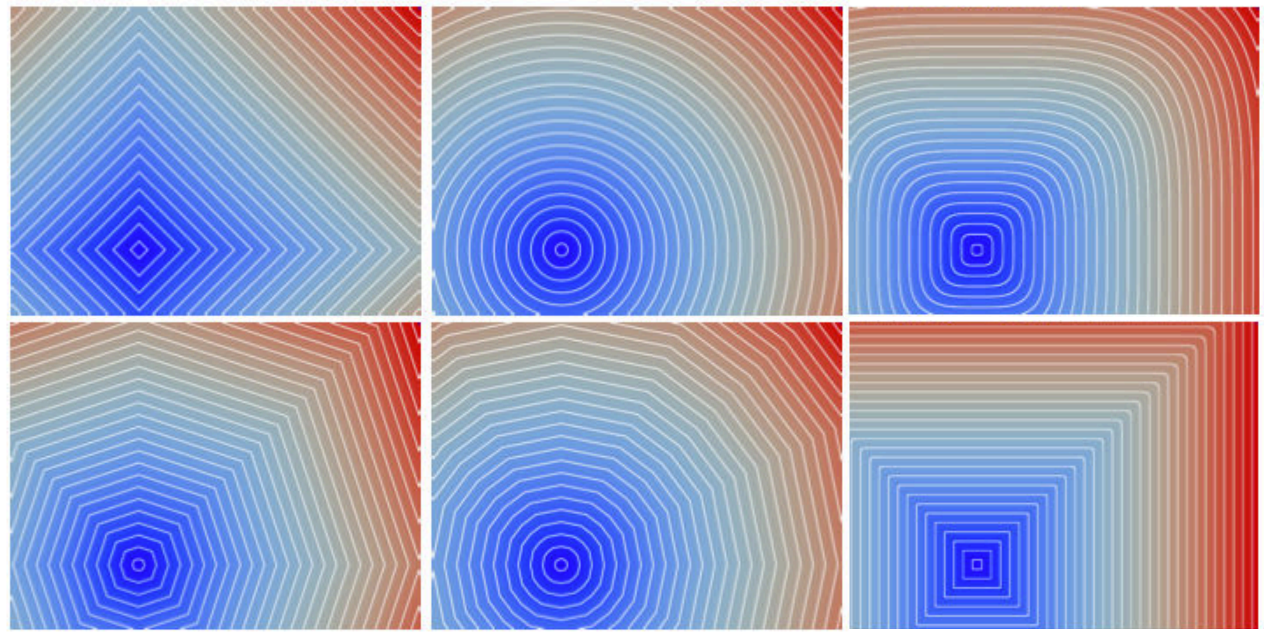

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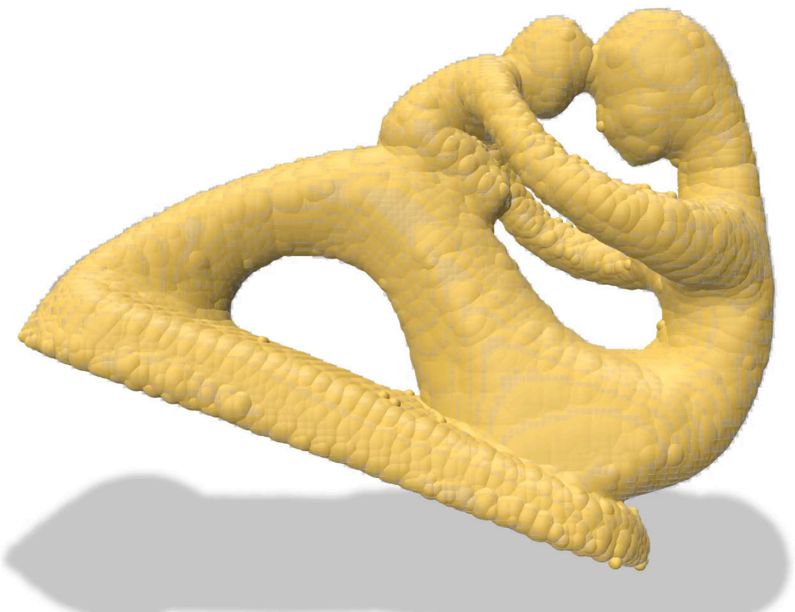
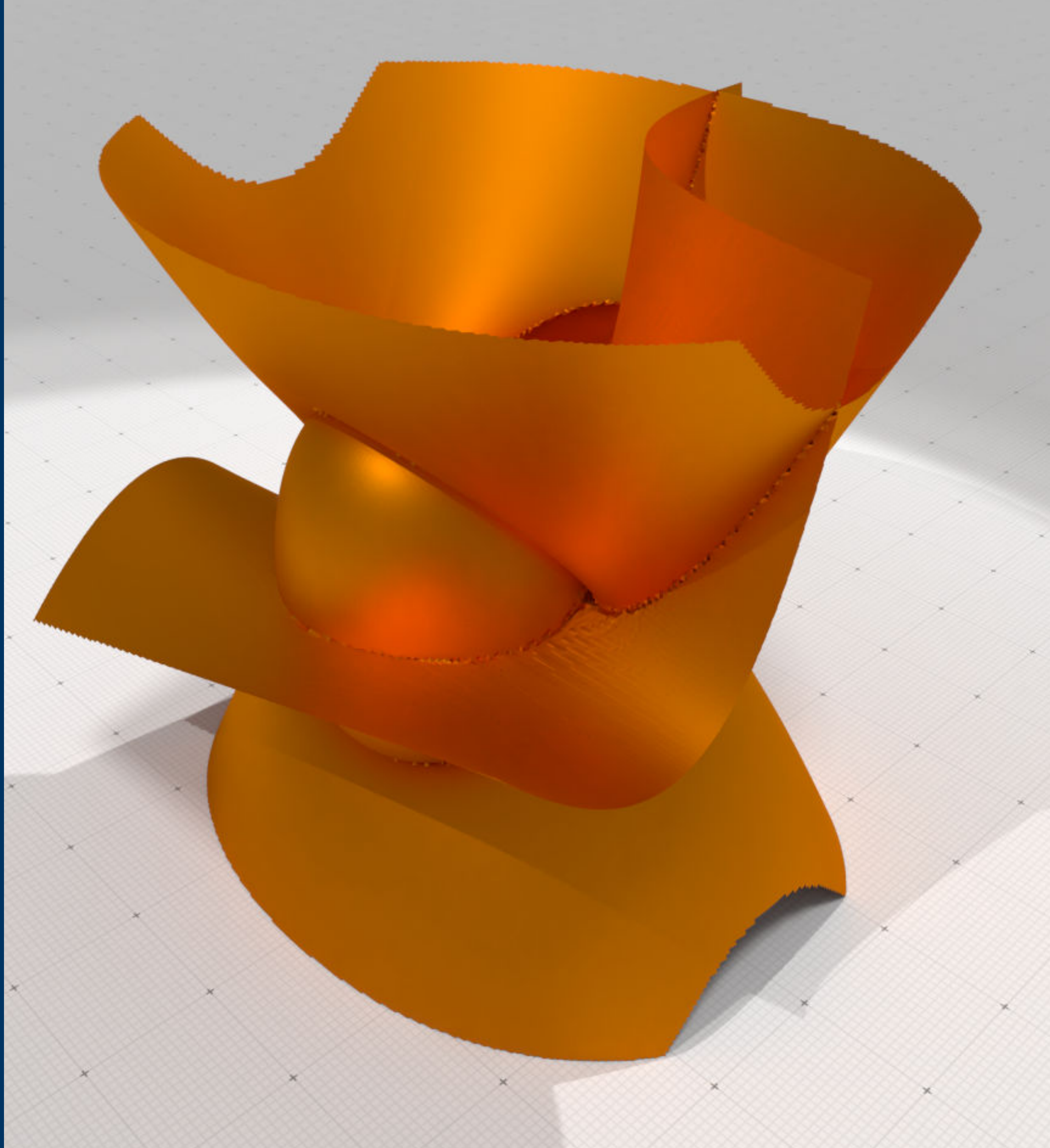
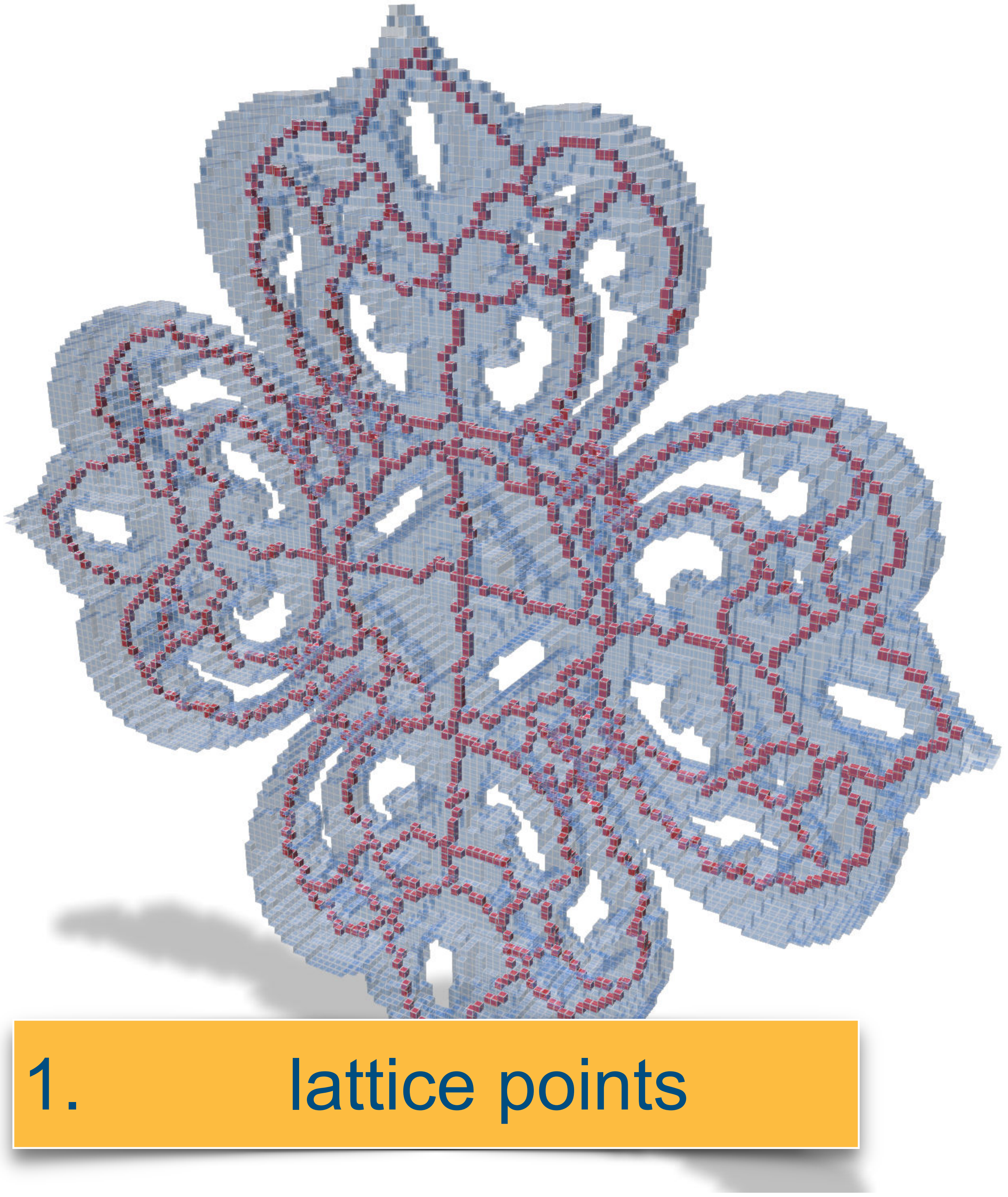


topology on $\mathbb{Z}^d$

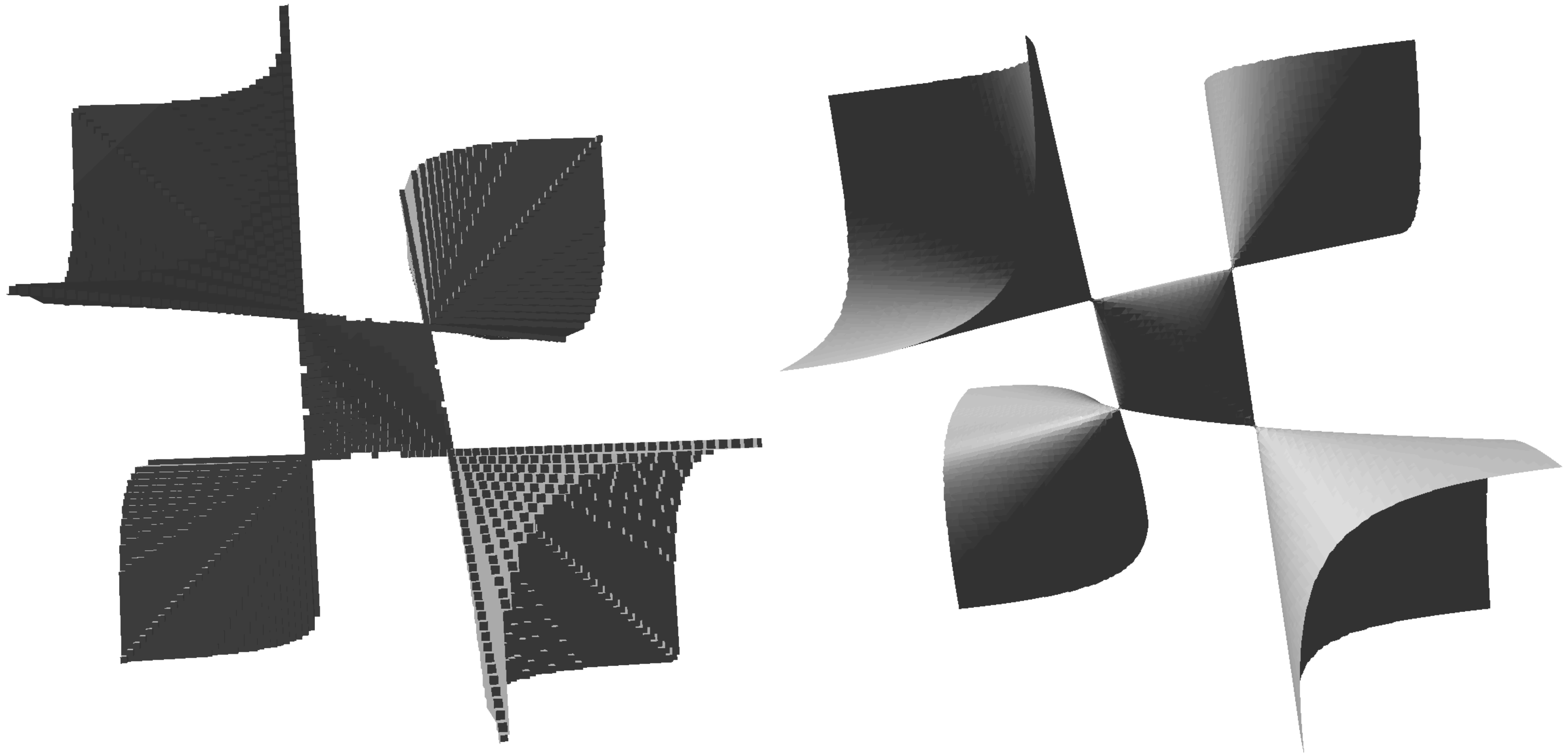


Before geometry : topological models for $\mathbb{Z}^d$

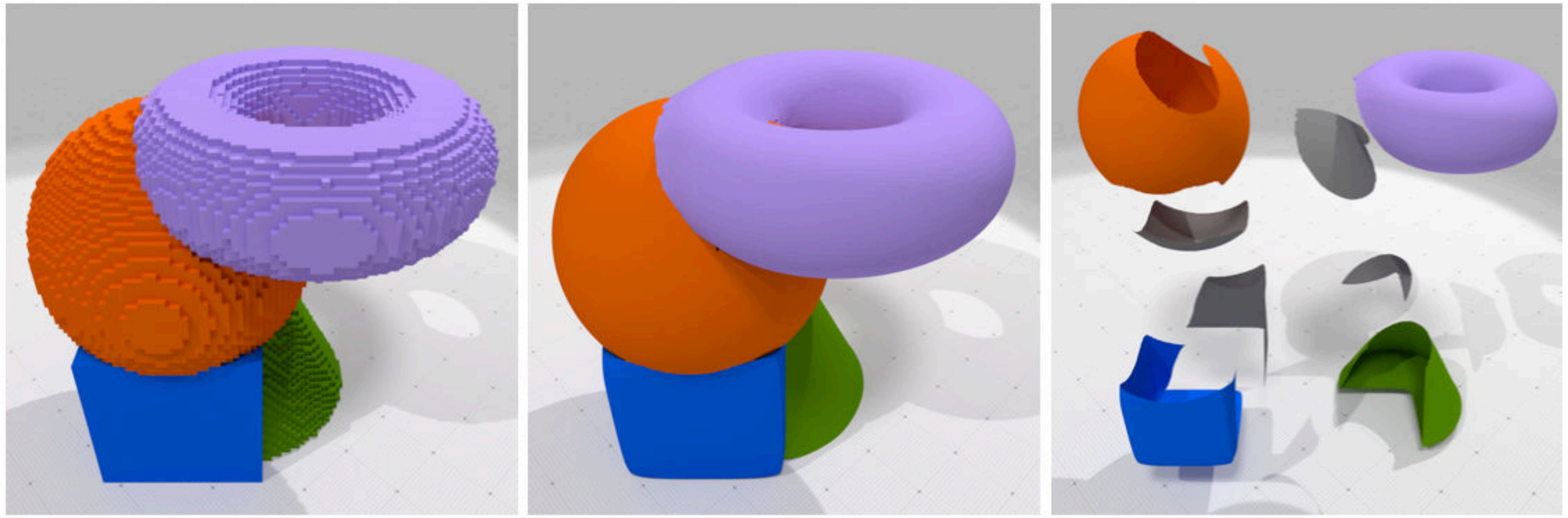
How to represent volumes, boundaries, curves, surfaces, partitions ?



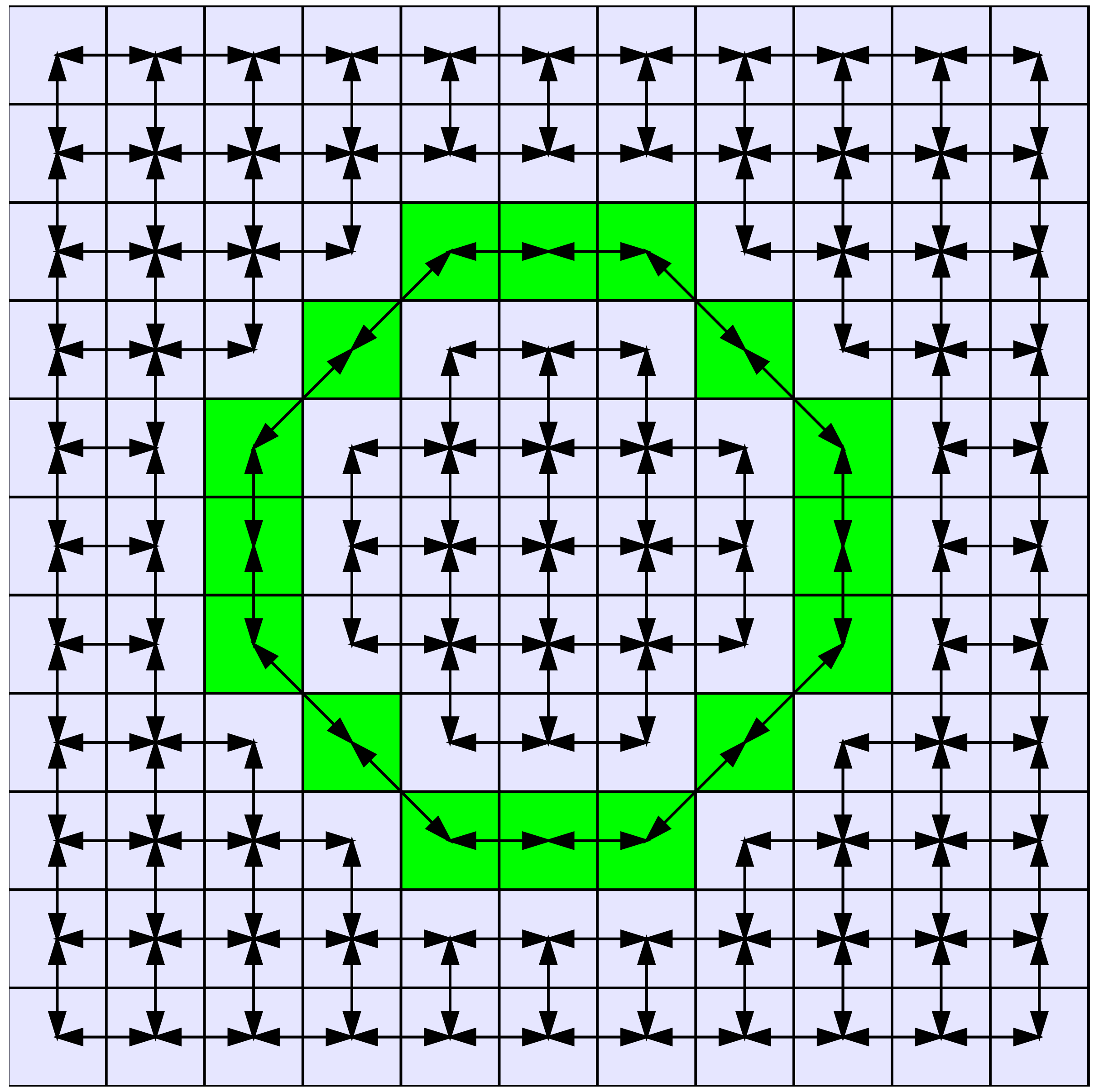
1. lattice points



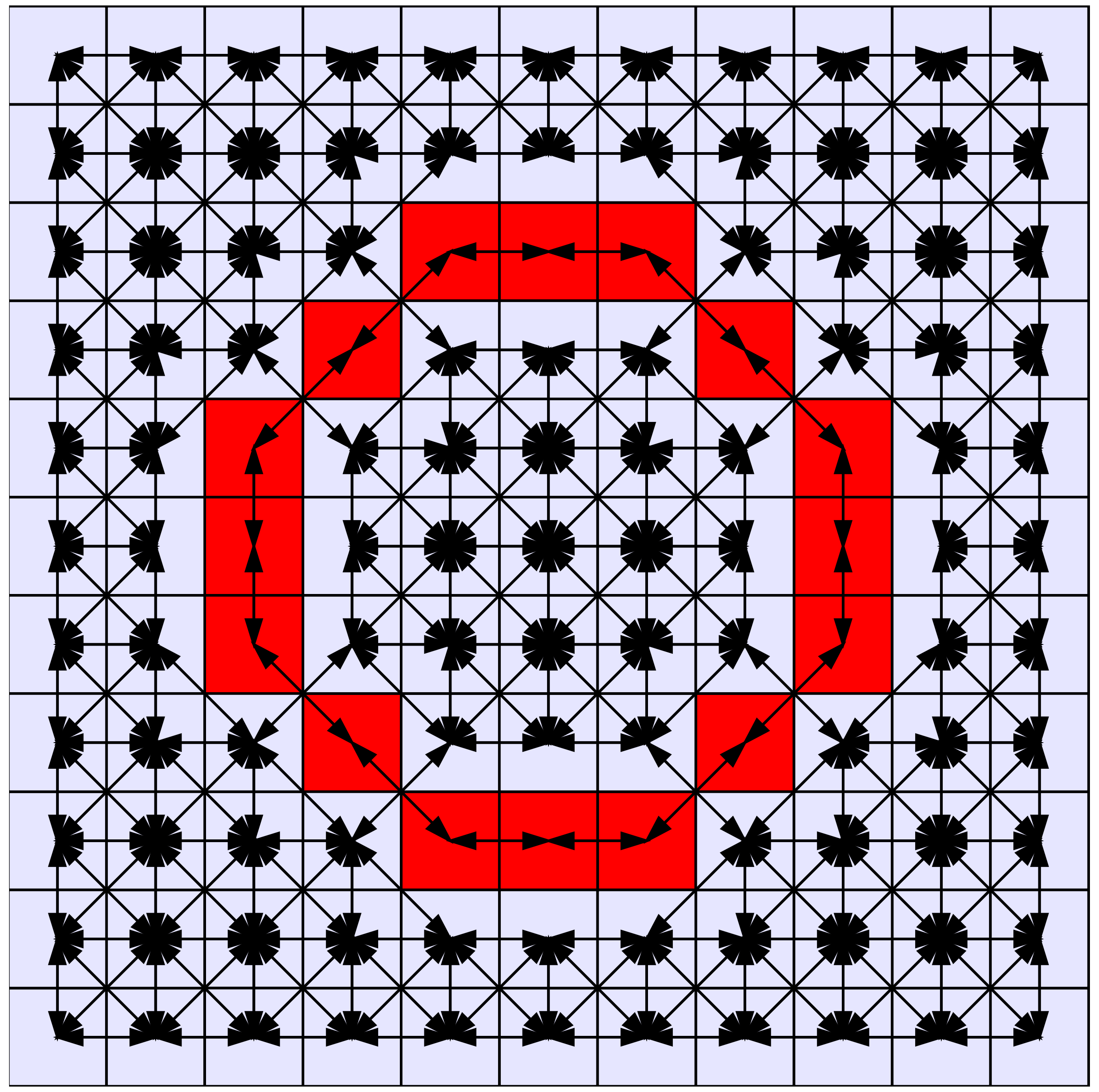
2. cubical complexes



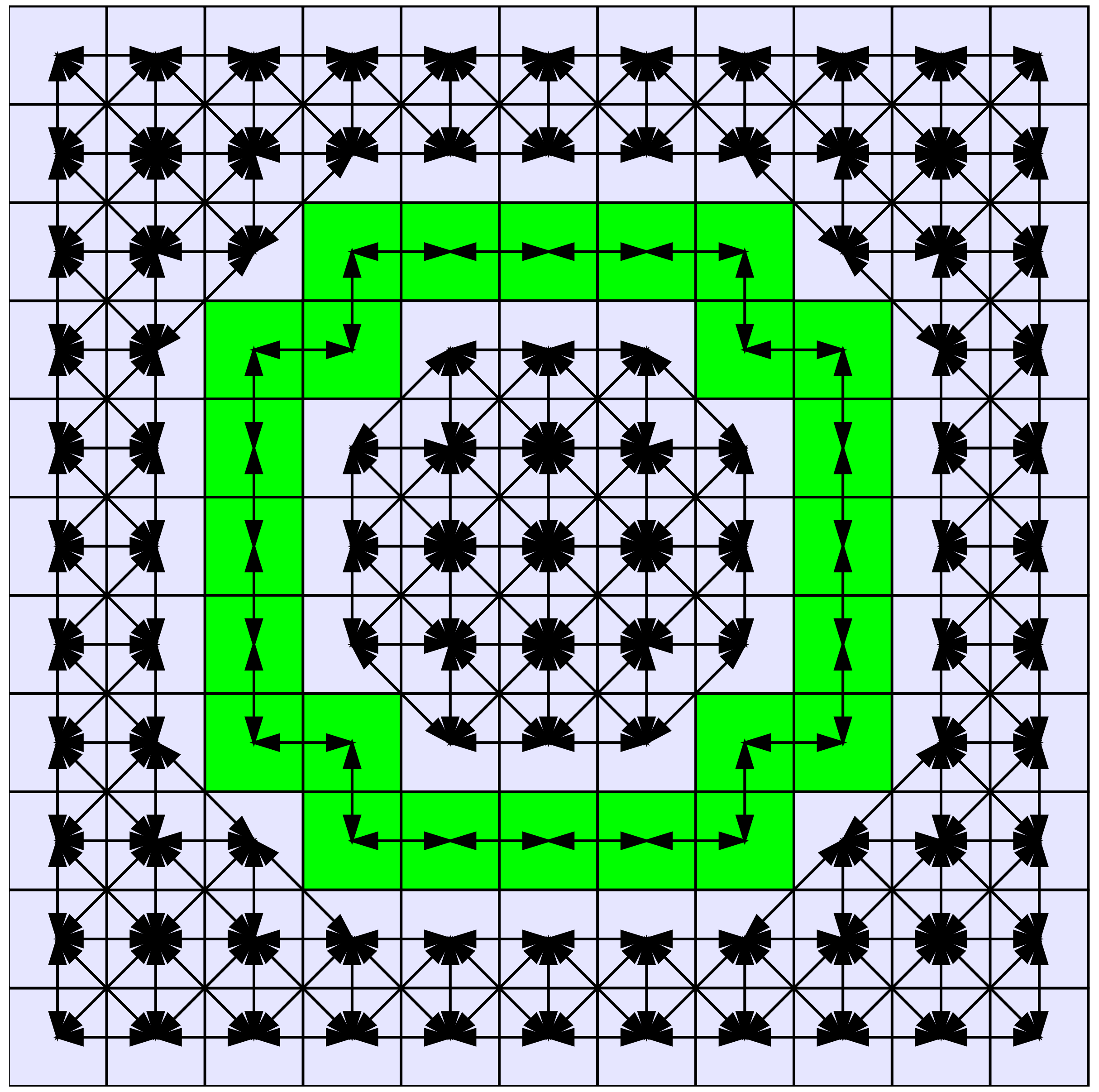
Digital topology



(8,4)-topology



(8,8)-topology

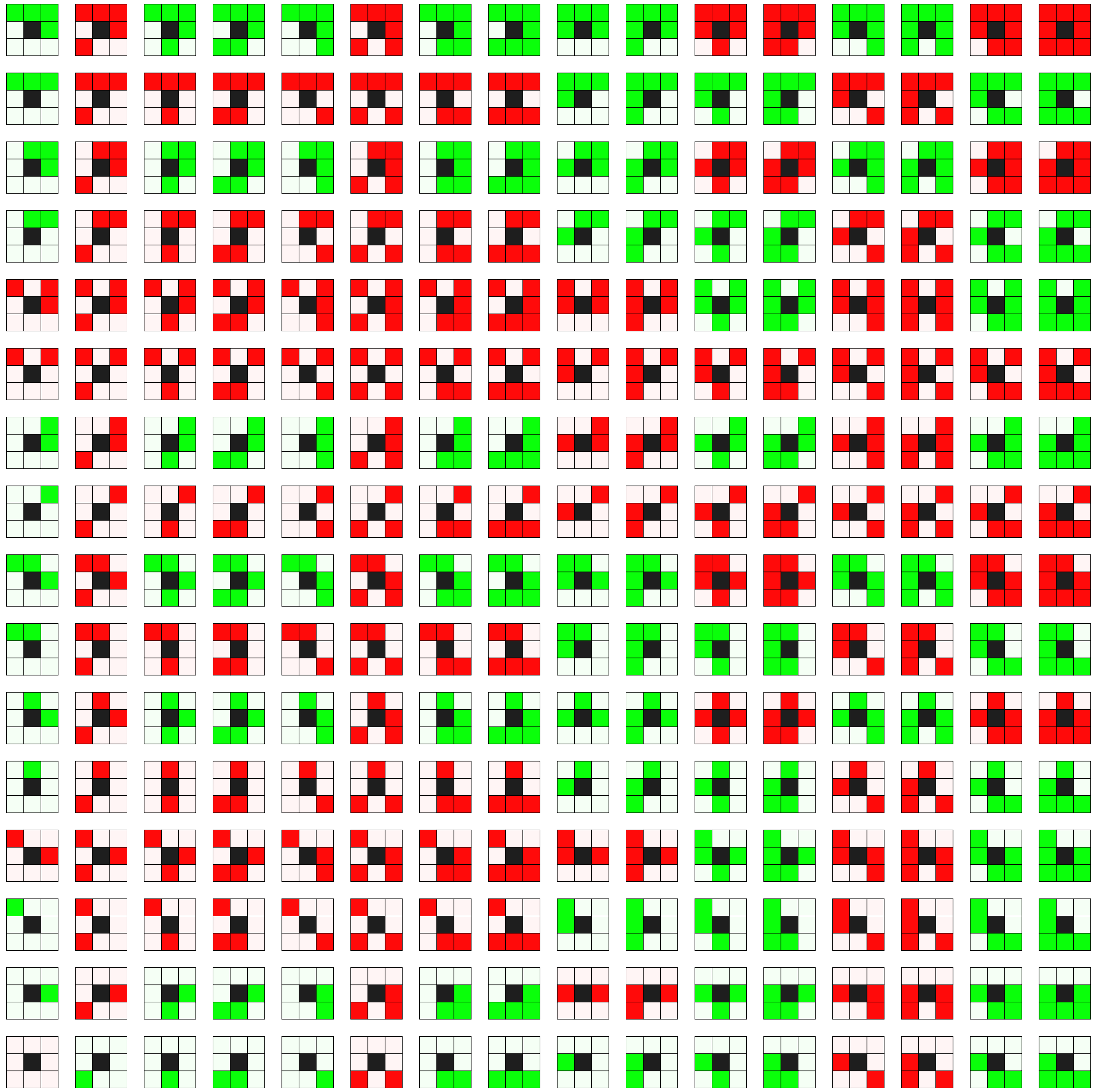


(4,8)-topology

Good adjacencies for object/background

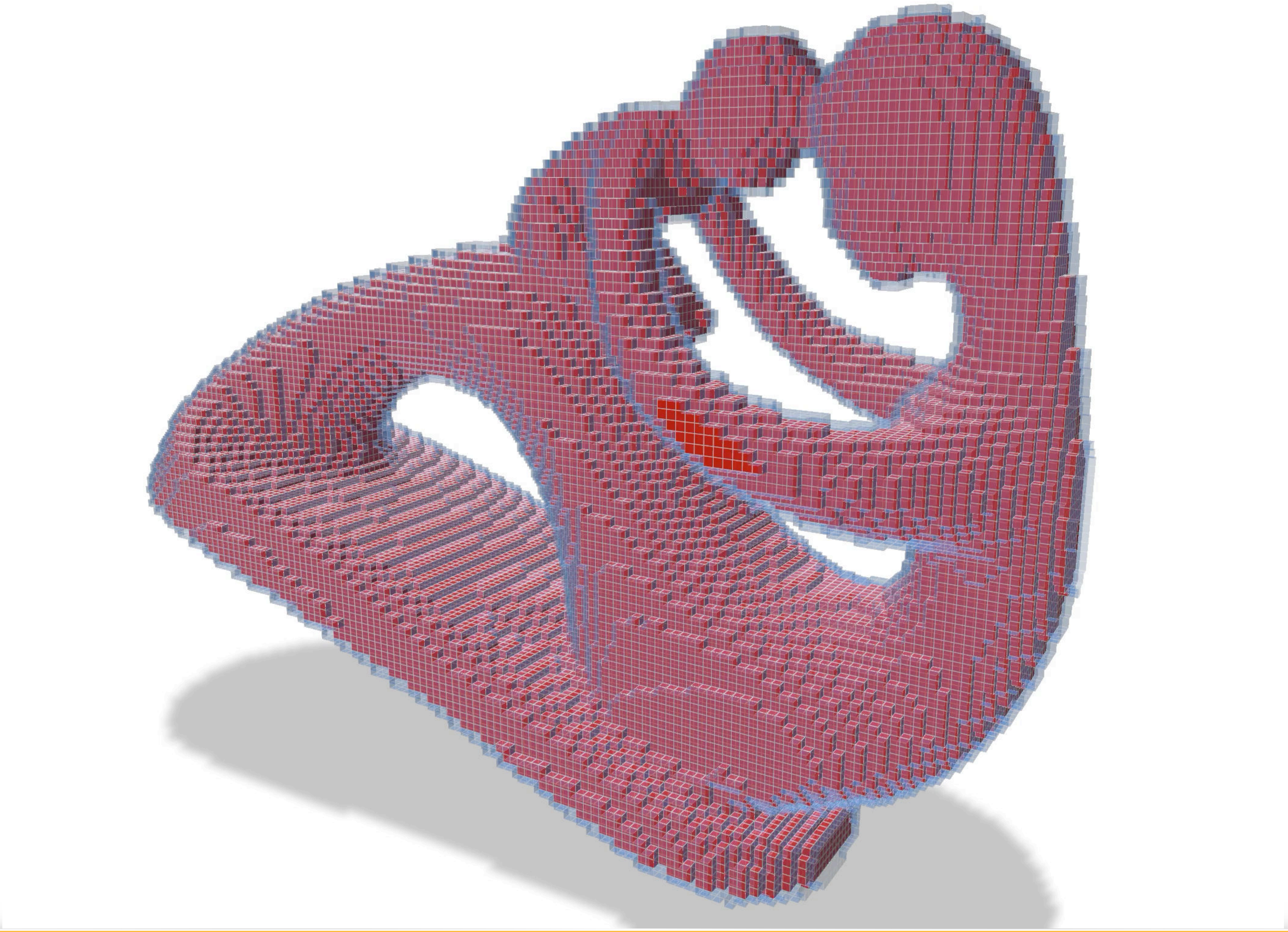
- Jordan separation theorem
- consistence borders and interior components
- definition of surfaces in $\mathbb{Z}^d$

Topology invariance: simple points



(8,4)-topology

locally keep connected components



Simple points: points whose removal preserves topology

- digital topology invariance of object and background
- very fast: look-up tables in 2D and 3D
- useful for skeleton extraction / coupled with medial axis

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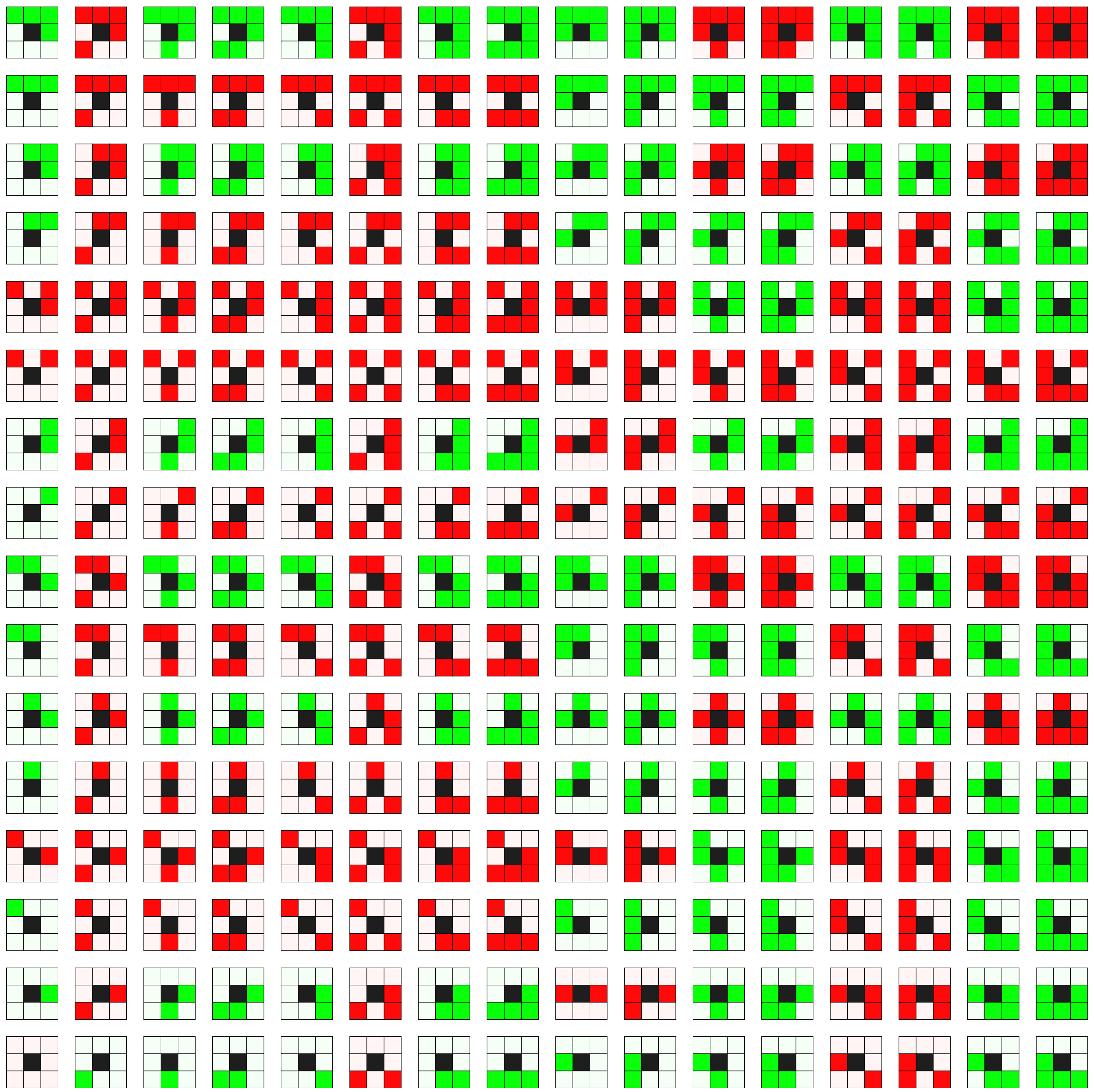

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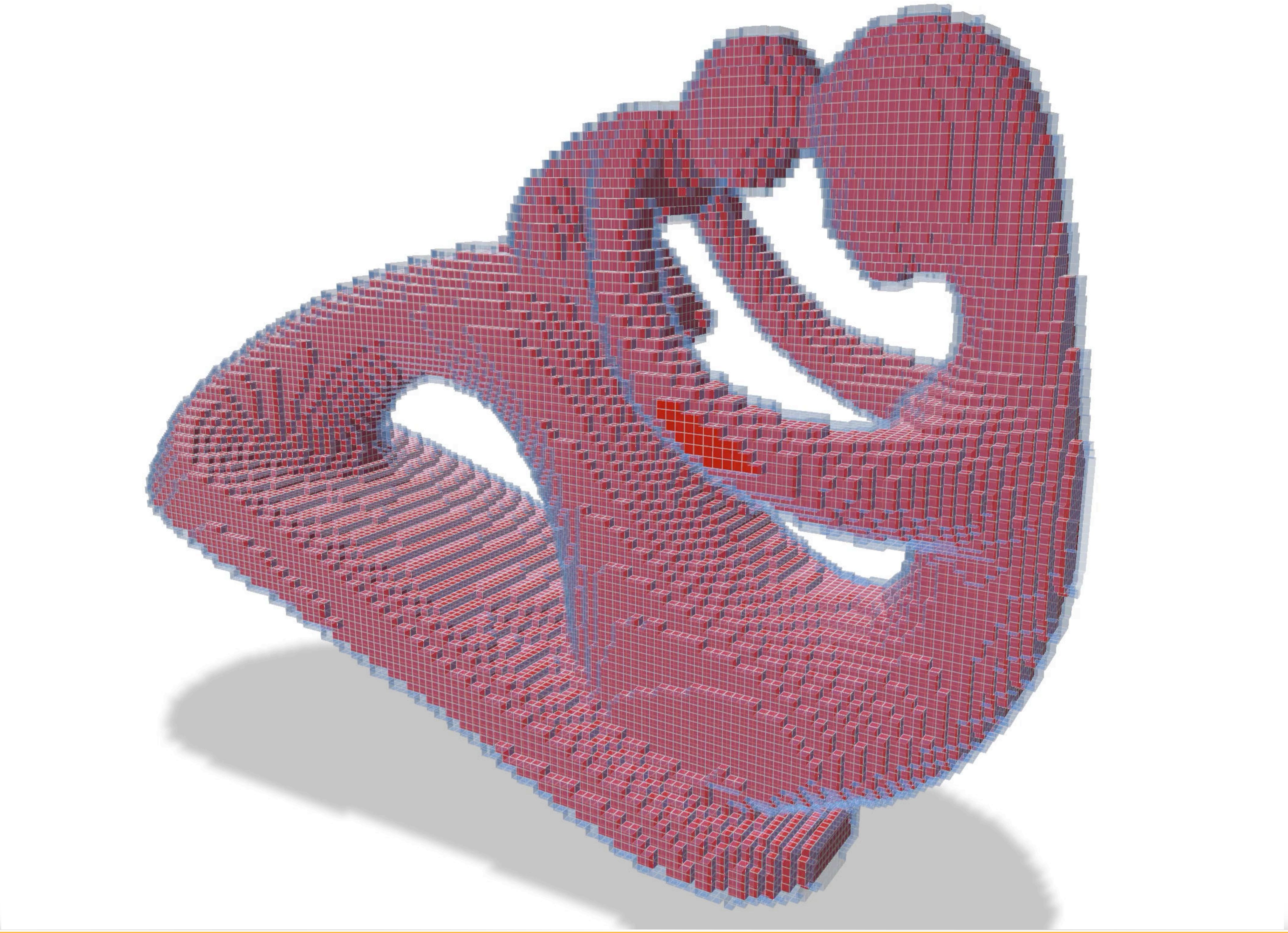

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hands on…

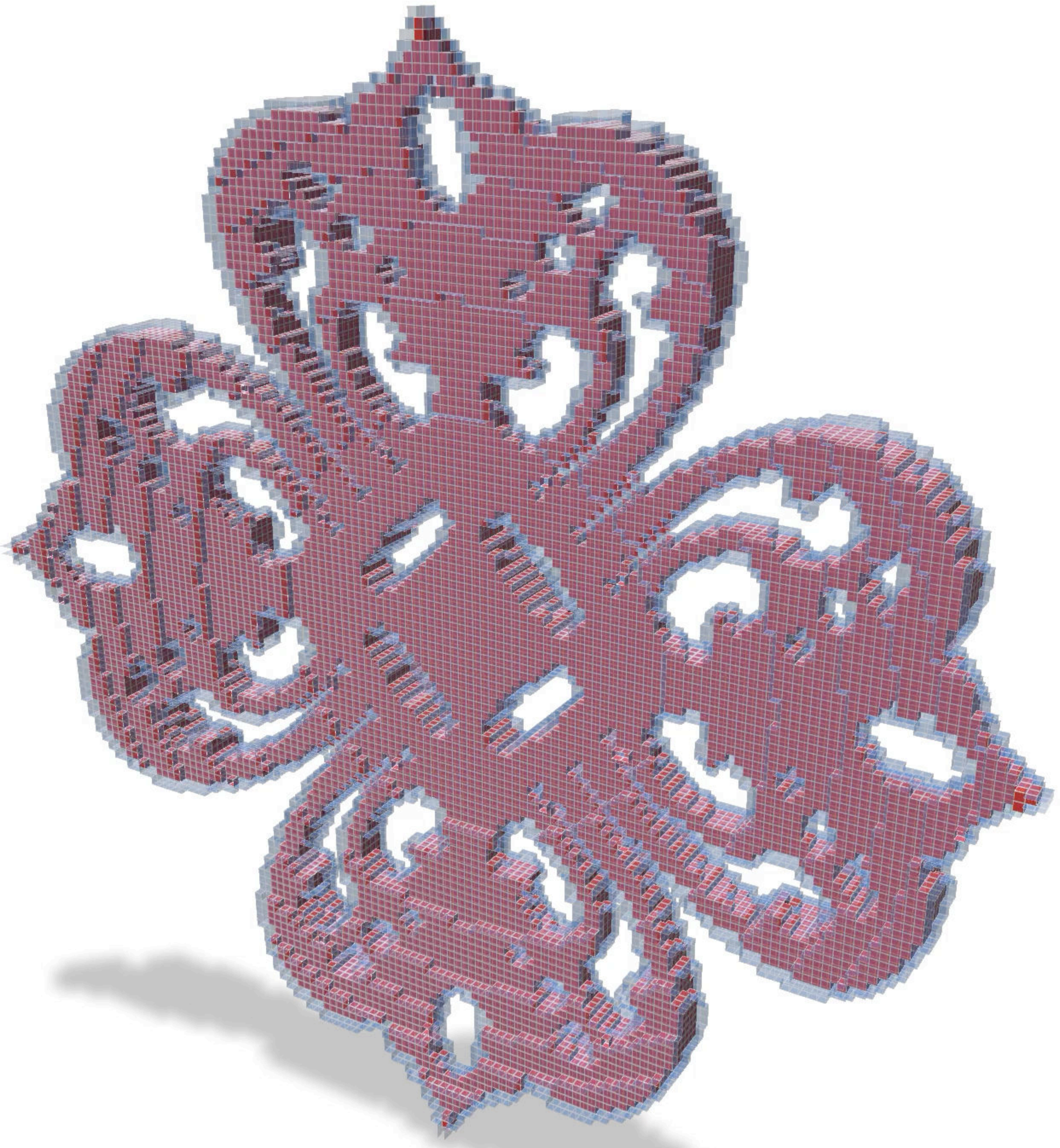
Create object with (26,6) topology from binary image

```
// Build object with digital topology
const auto K = SH3::getKSpace( binary_image );
Domain domain( K.lowerBound(), K.upperBound() );
Z3i::DigitalSet voxel_set( domain );
for ( auto p : domain )
  if ( (*binary_image)( p ) ) voxel_set.insertNew( p );
the_object = CountedPtr< Z3i::Object26_6 >( new Z3i::Object26_6( dt26_6, voxel_set ) );
the_object->setTable(functions::loadTable<3>(simplicity::tableSimple26_6));
```

Queue simple points

Remove simple points

```
// Removes a peel of simple points onto voxel object.
bool oneStep( CountedPtr< Z3i::Object26_6 > object )
{
  DigitalSet & S = object->pointSet();
  std::queue< Point > Q;
  for ( auto&& p : S )
    if ( object->isSimple( p ) )
      Q.push( p );
  int nb_simple = 0;
  while ( ! Q.empty() )
    {
      const auto p = Q.front();
      Q.pop();
      if ( object->isSimple( p ) )
        {
          S.erase( p );
          binary_image->setValue( p, false );
          ++nb_simple;
        }
    }
  trace.info() << "Removed " << nb_simple << " / " << S.size()
               << " points." << std::endl;
  registerDigitalSurface( binary_image, "Thinned object" );
  return nb_simple == 0;
}
```



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// Build object with digital topology
const auto K = SH3::getKSpace( binary_image );
Domain domain( K.lowerBound(), K.upperBound() );
Z3i::DigitalSet voxel_set( domain );
for ( auto p : domain )
  if ( (*binary_image)( p ) ) voxel_set.insertNew( p );
the_object = CountedPtr< Z3i::Object26_6 >( new Z3i::Object26_6( dt26_6, voxel_set ) );
the_object->setTable(functions::loadTable<3>(simplicity::tableSimple26_6));
```
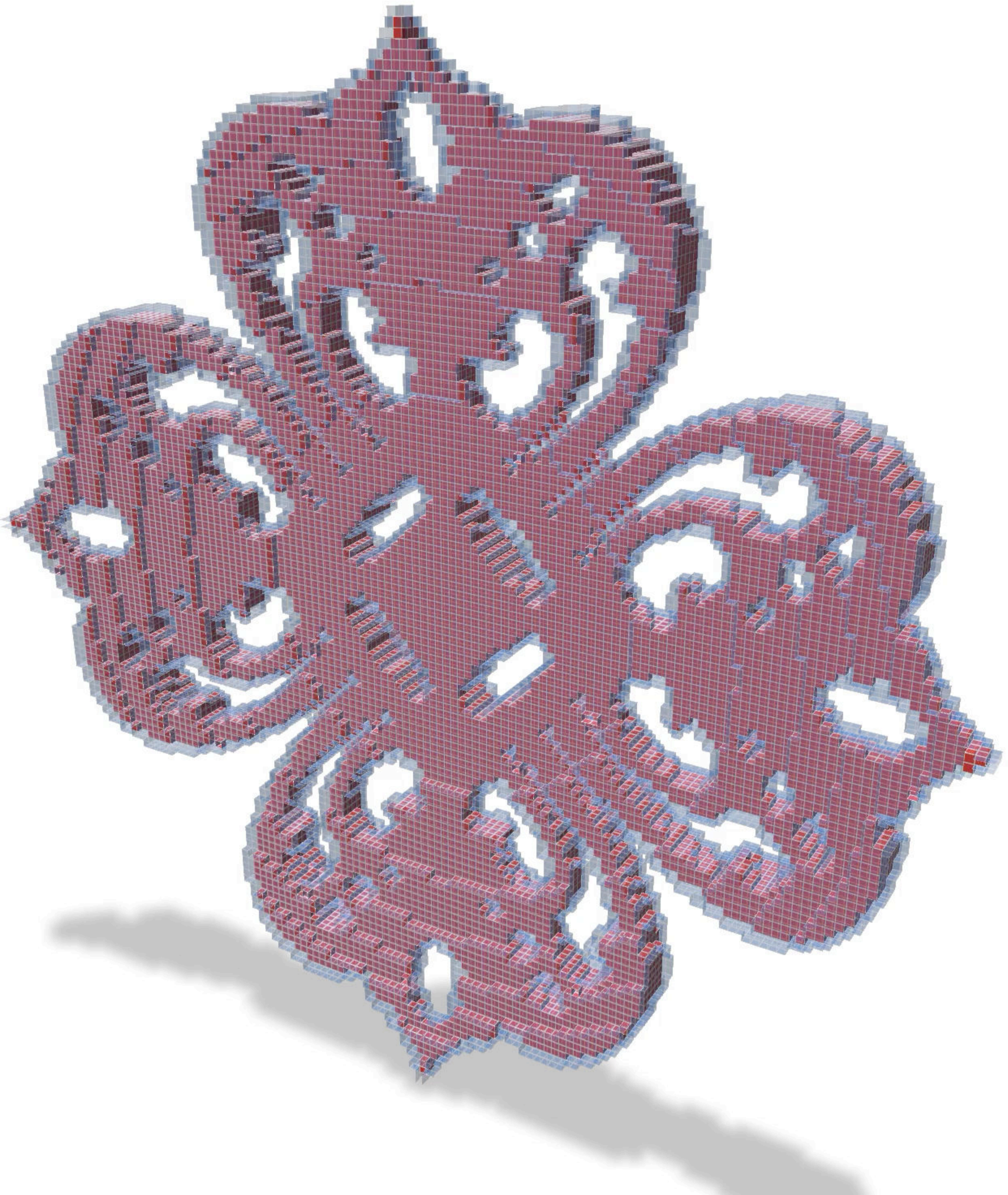

Create object with (26,6) topology from binary image

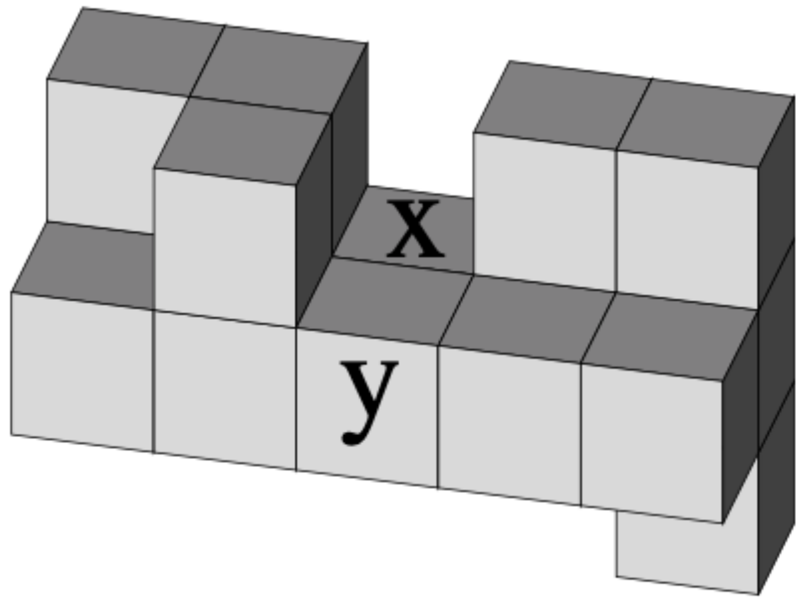
```cpp
// Removes a peel of simple points onto voxel object.
bool oneStep( CountedPtr< Z3i::Object26_6 > object )
{
  DigitalSet & S = object->pointSet();
  std::queue< Point > Q;
  for ( auto&& p : S )
    if ( object->isSimple( p ) )
      Q.push( p );
  int nb_simple = 0;
  while ( ! Q.empty() )
    {
      const auto p = Q.front();
      Q.pop();
      if ( object->isSimple( p ) )
        {
          S.erase( p );
          binary_image->setValue( p, false );
          ++nb_simple;
        }
    }
  trace.info() << "Removed " << nb_simple << " / " << S.size()
               << " points." << std::endl;
  registerDigitalSurface( binary_image, "Thinned object" );
  return nb_simple == 0;
}
```

Queue simple points

Remove simple points

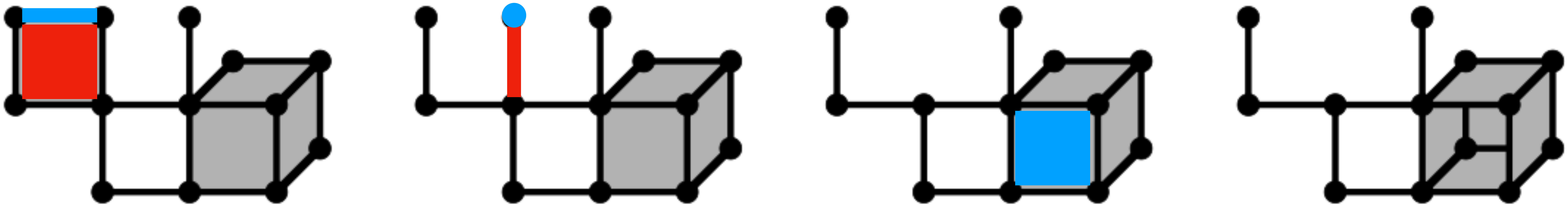


Homotopic collapses



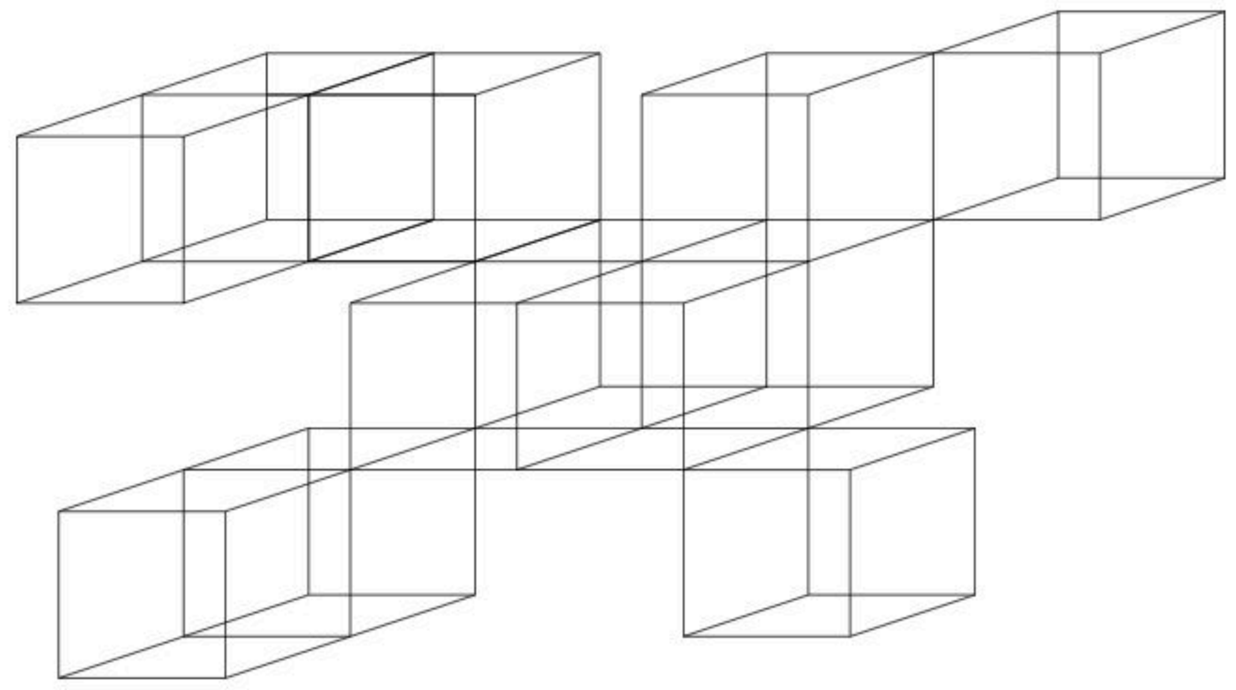
x and y are simple
but cannot be removed in parallel

Needs cubical complex representation

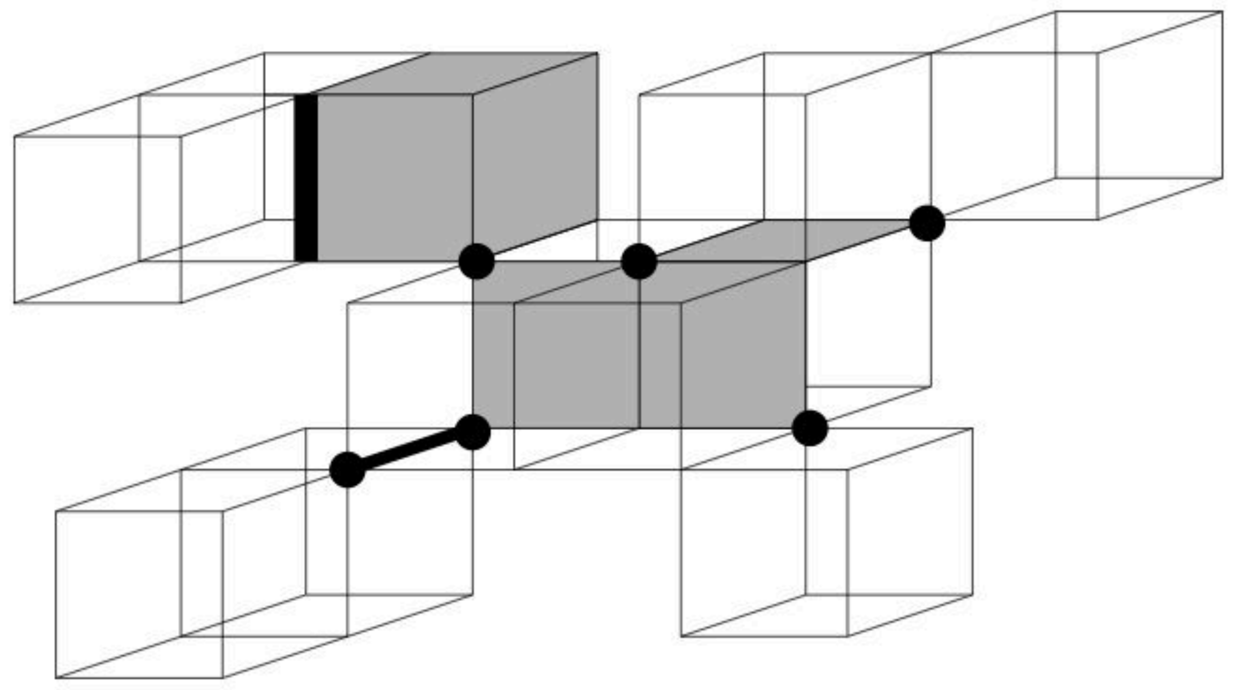


Elementary **collapse** : removing cell pairs (f,g) where g is free
preserves homotopy

Homotopic collapses and critical kernels

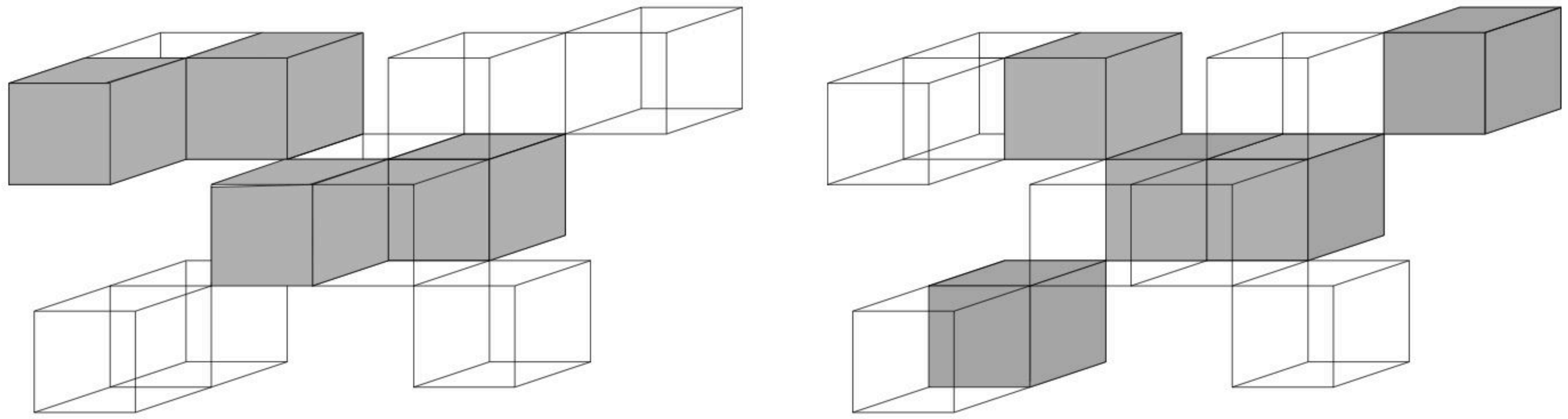


cubical complex X



Z := critical kernel of X

critical cells : cells that do not collapse onto their neighborhood

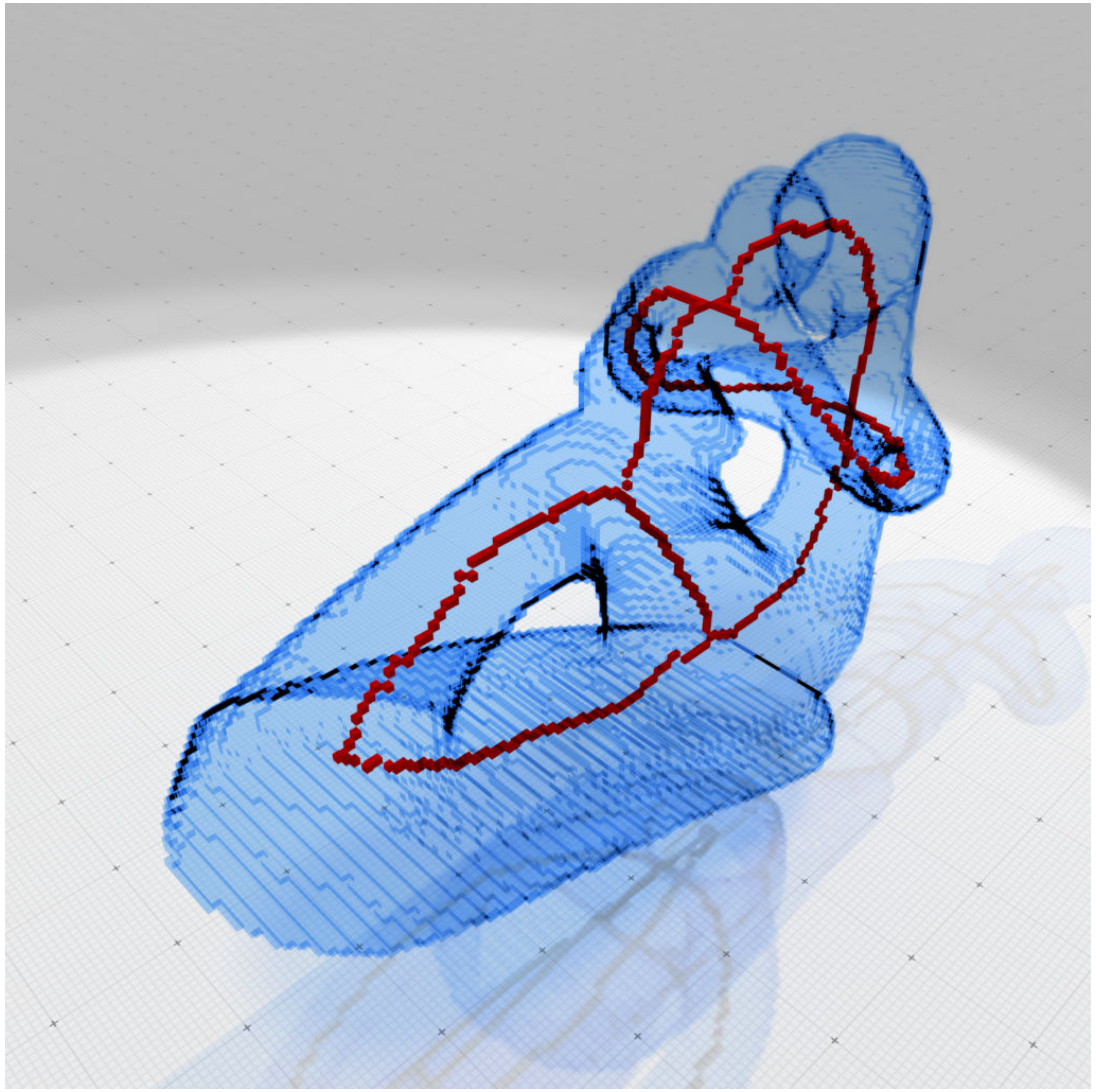


Both complexes Y_1, Y_2 are thinning, since $Z \subseteq Y_i \subseteq X$

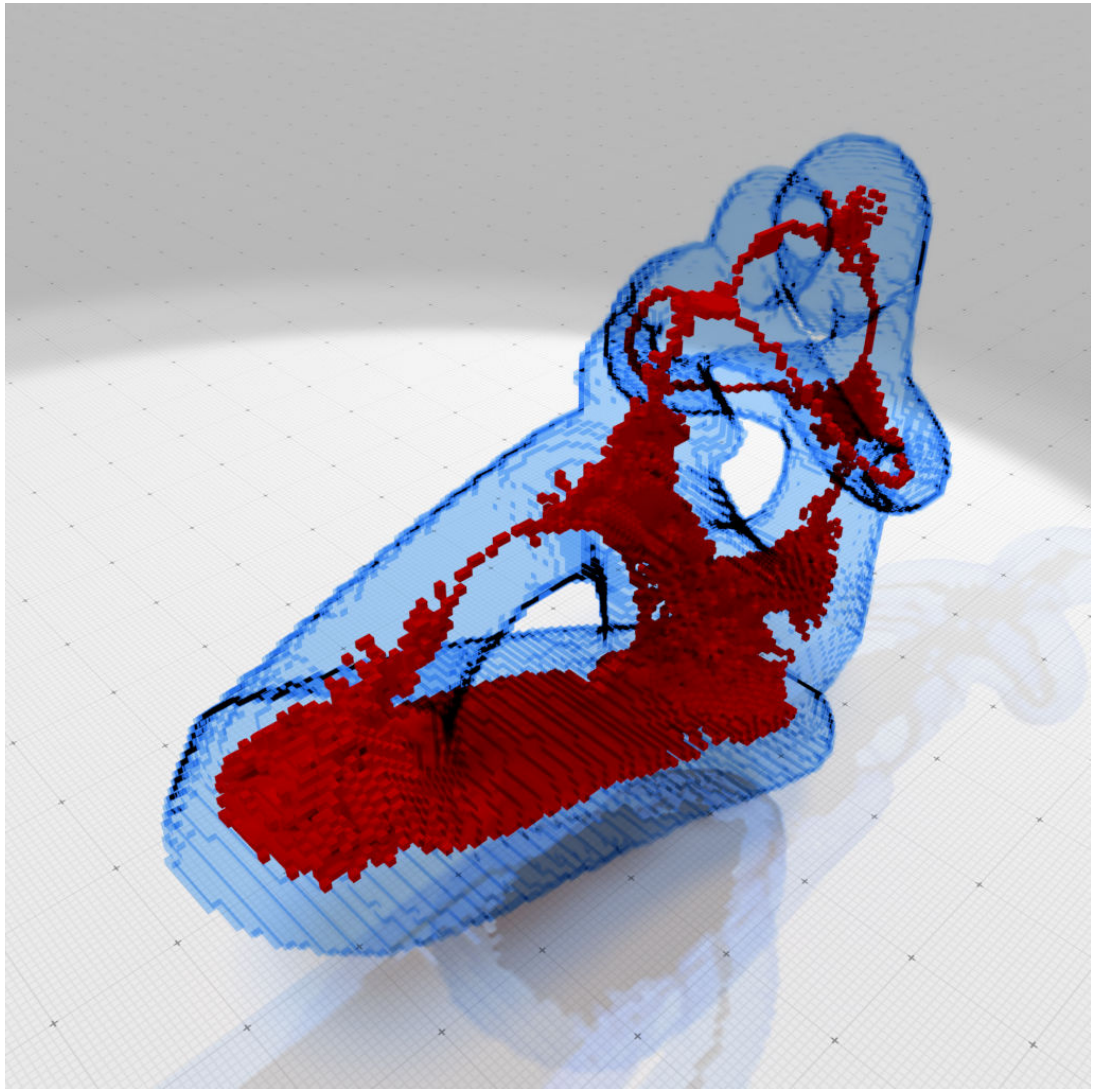
All complexes Y, such that $Z \subseteq Y \subseteq X$ are homotopic to X !

Allows parallel algorithms for extracting skeletons

Skeletons with critical kernels

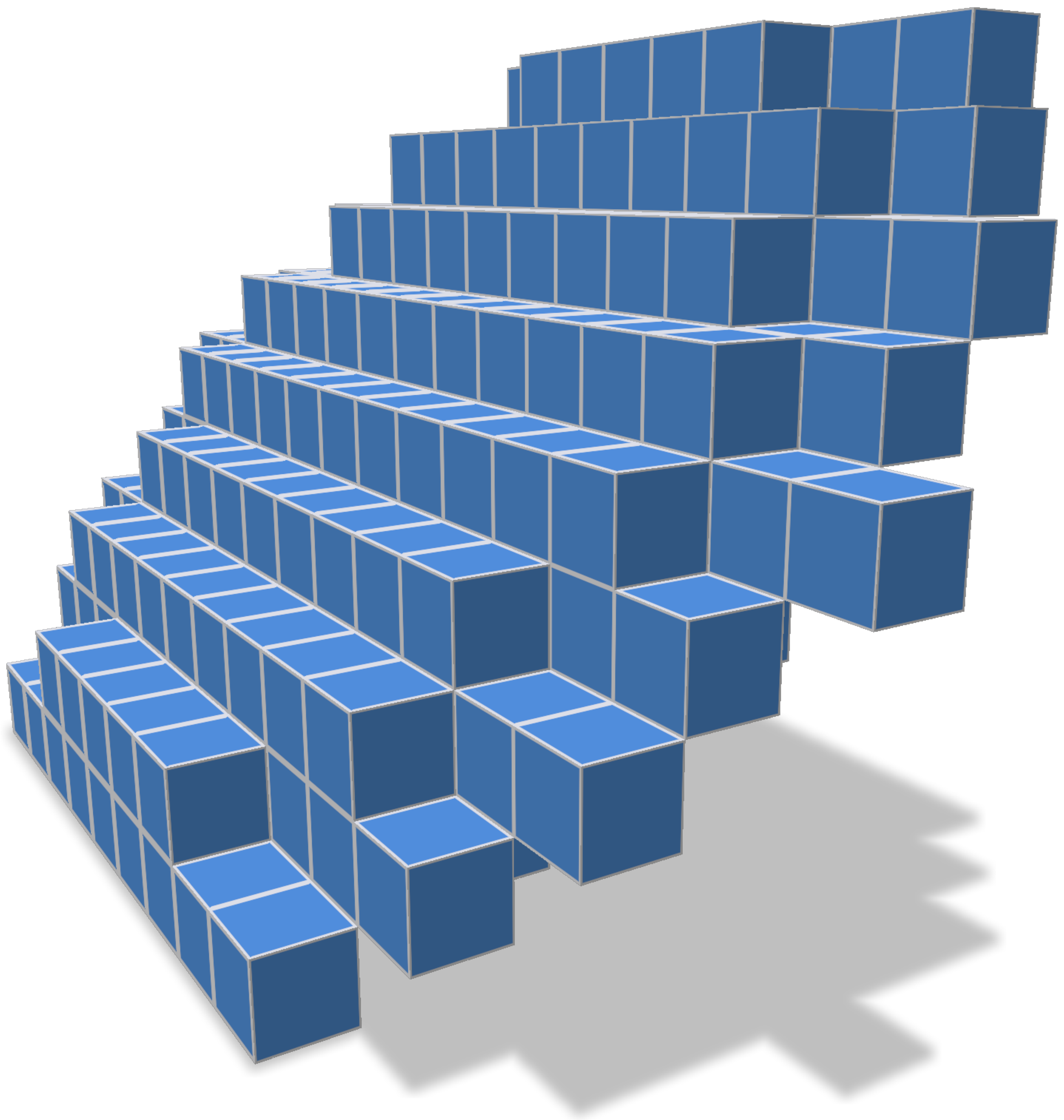


« curved » skeleton



« surface » skeleton

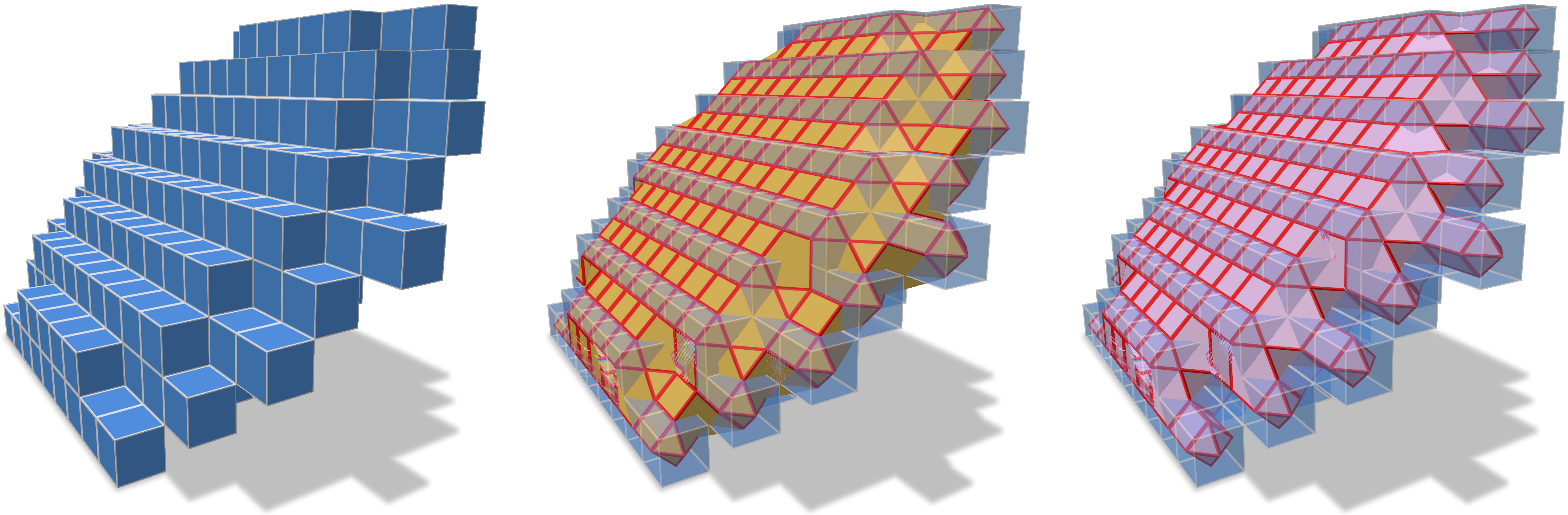
Digital surfaces



Primal surface
(here, digitization of some ellipsoid)

- digital surface ≈ set of faces of voxels
- in « ideal cases » 4-regular graph (3D)
  - vertices = surfels/faces
- generally not a manifold
  - pinched on edges and/or vertices
- not a sampling, only approximation
- only 6 different normals in 3D
  - even fine digital surface have poor normals

Digital surfaces + topology (primal ↔ dual)



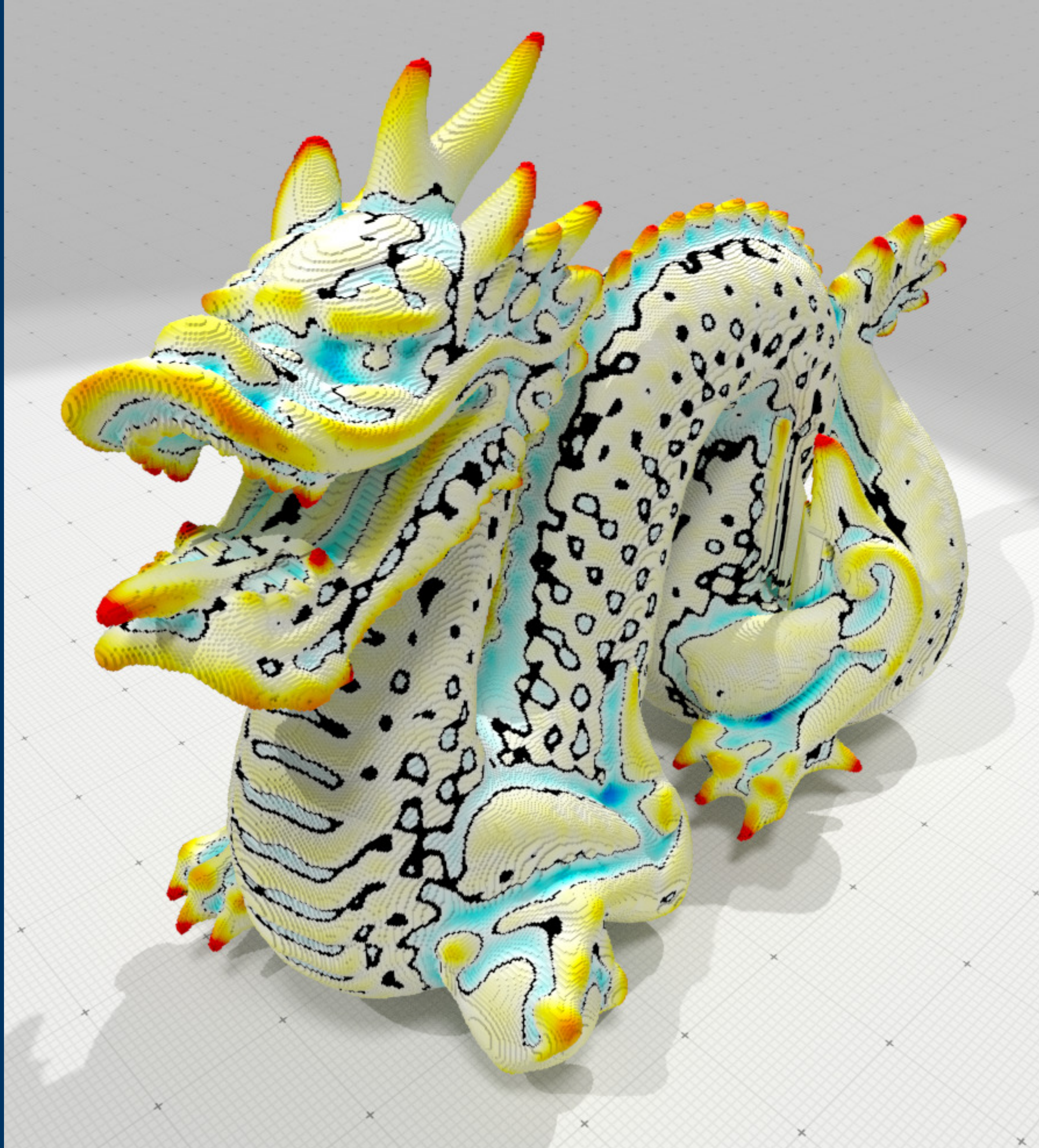
Primal surface

Dual surface
(26,6) topology

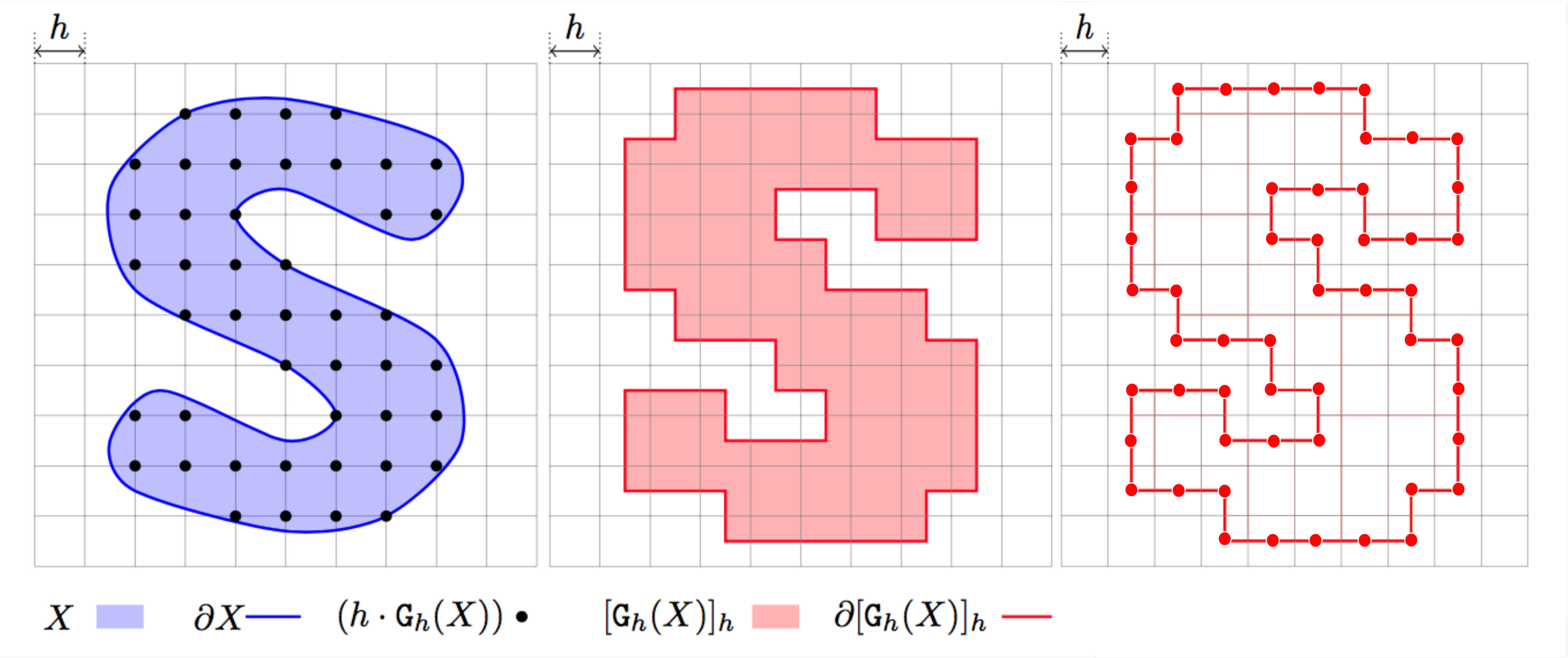
Dual surface
(6,26) topology

Adding object/background topology allows manifoldness in arbitrary dimensions
- exactly d-1 paths crossing at each point

digital surface geometry



Linking continuous and digital geometry : Gauss digitization with gridstep h

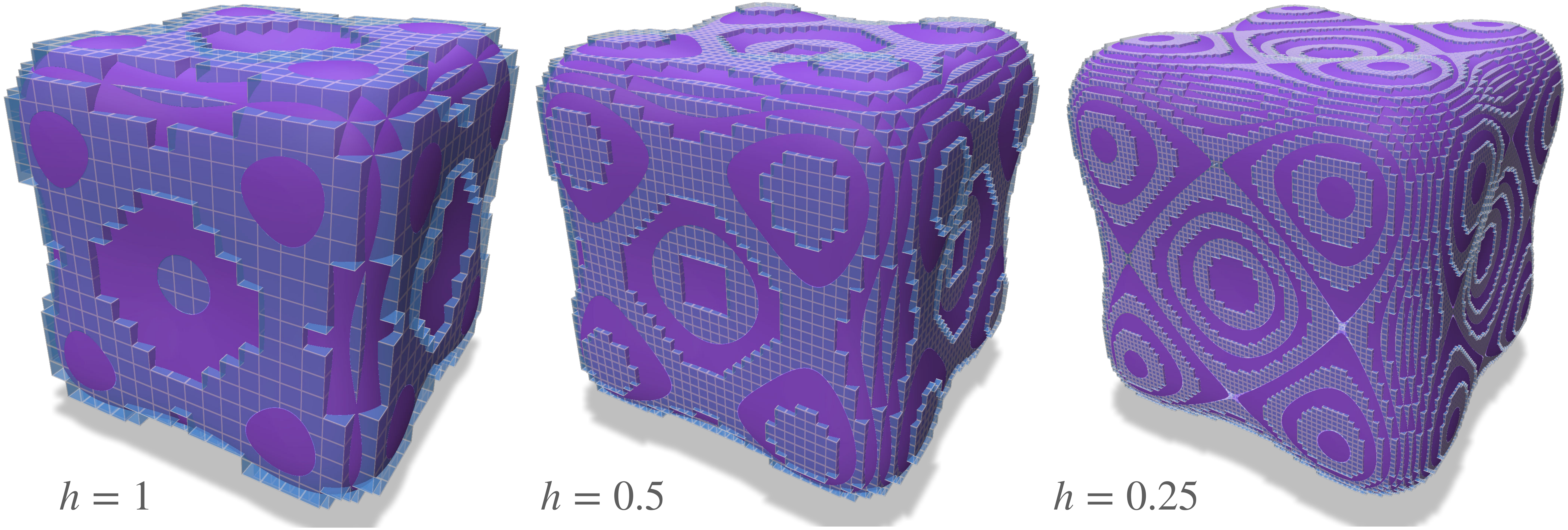


« digitization » « voxelization » « digitized surface »


What can we say for finer and finer digitization ? ($h \to 0$)

What can we say for finer and finer digitization ? ($h \to 0$)

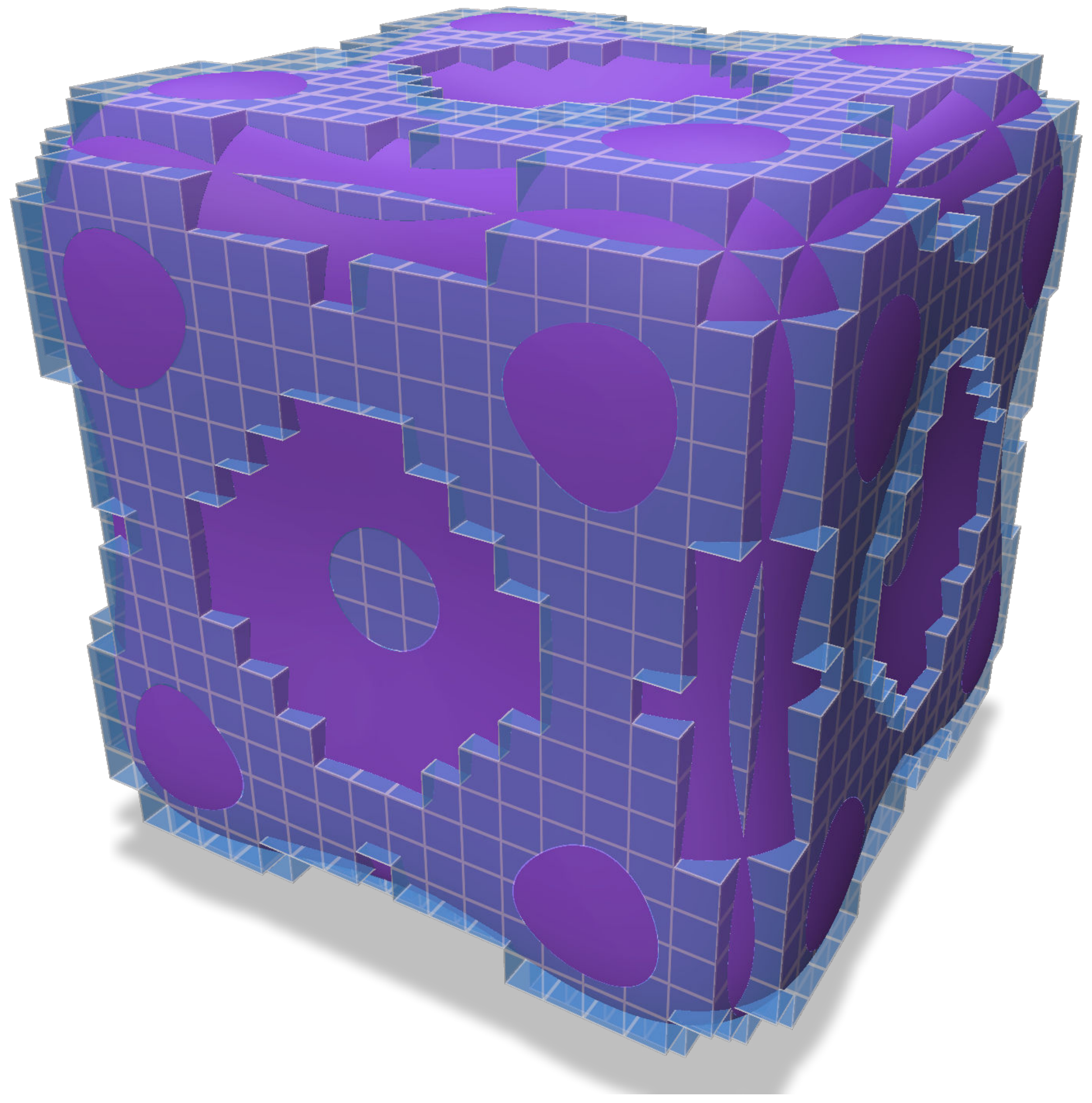
Hausdorff closeness of digitized shapes



For any compact domain $X \in \mathbb{R}^d$ such that ∂X has positive reach, and its digitization $X_h := [G_h(X)]_h$ on a grid with grid-step h, then $d_H(\partial X, \partial X_h) \leq h\sqrt{d}/2$ for small enough h

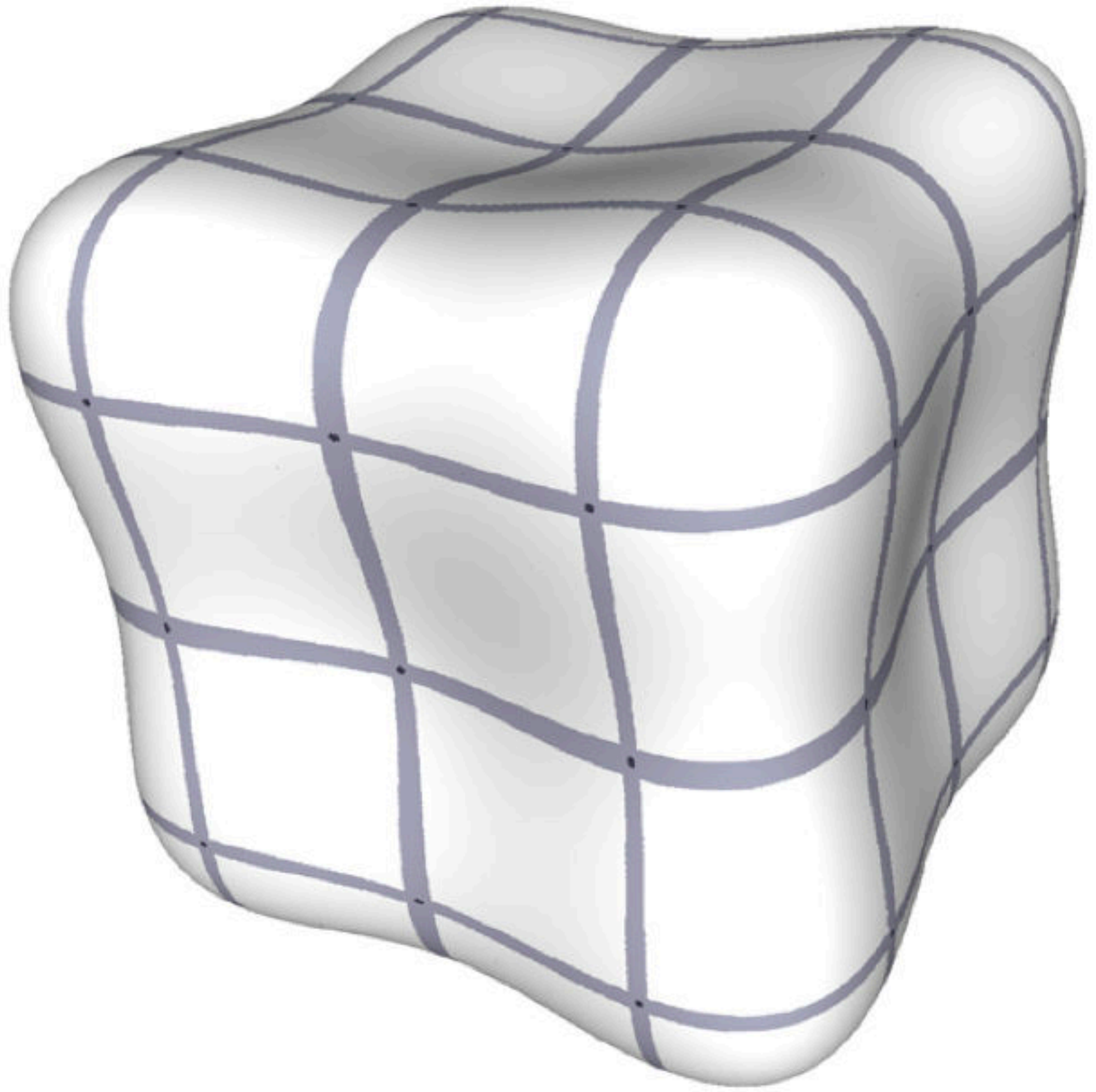
[LT16]

Bijectivity of projection and manifoldness




$h = 0.1$


$h = 0.05$


$h = 0.025$

[LT16]
If X has positive reach,
the size of the non-injective part of projection
$\pi_X : \partial X_h \to \partial X$ tends to zero as $h \to 0$.
(light gray + dark gray zones $\approx O(h)$)

[LT16]
If X has positive reach,
the size of the non-manifoldness part of ∂X_h
tends quickly to zero as $h \to 0$.
(dark gray zones $\approx O(h^2)$)

Multigrid convergence

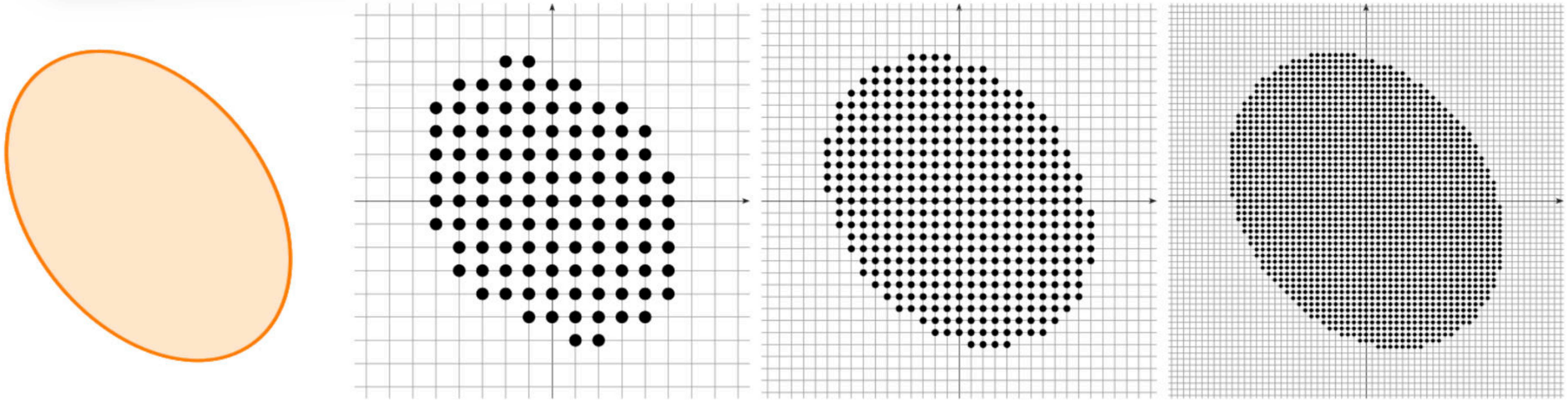
For digitization process G, the discrete geometric estimator $\hat{E}$ is multigrid convergent to the geometric quantity E for the family of shapes $\mathbb{X}$, iff, for any $X \in \mathbb{X}$, there exists a grid step $h_X > 0$, such that :

$\hat{E}(G_h(X), h)$ is defined for any $0 < h < h_X$,

$$|\hat{E}(G_h(X), h) - E(X)| < \tau_X(h)$$

where the speed of convergence $\tau_X(h)$ has null limit when $h \to 0$.

(Typically area, perimeter, integrals)



$M \in \mathbb{X}$ $\quad$ $\mathrm{G}_1(M)$ $\quad$ $\mathrm{G}_{0.5}(M)$ $\quad$ $\mathrm{G}_{0.25}(M)$

$\widehat{\mathrm{Area}}\,(G_h(X), h) := h^2 \#(G_h(X))$ tends toward $\mathrm{Area}(M)$ as $h \to 0$

Convergence speed is $O(h)$ and even $O(h^{\frac{22}{15}})$ for smooth enough M

Multigrid convergence (local version)

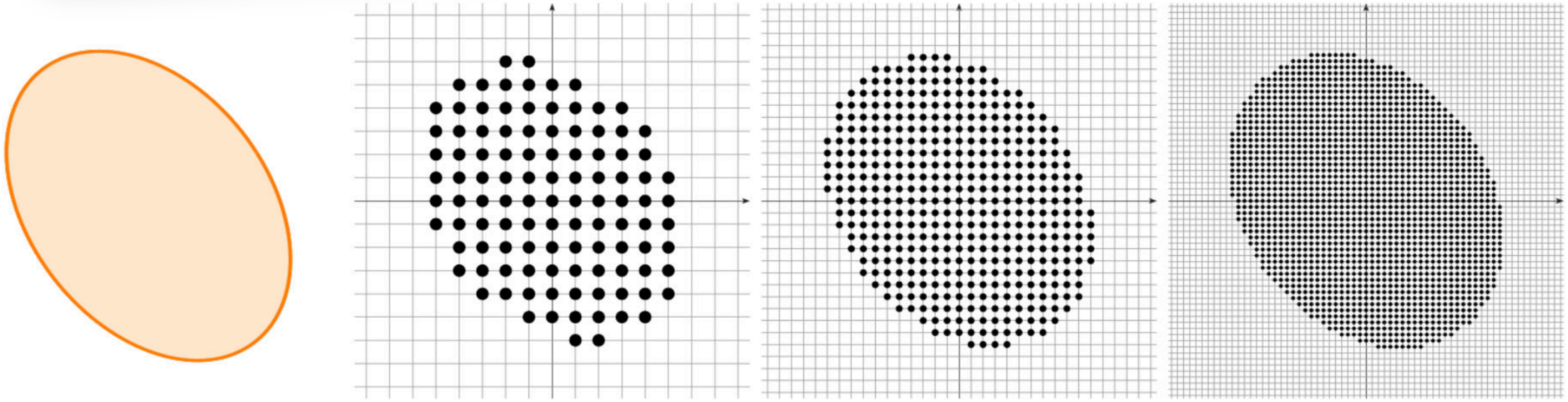
For digitization process G, the local discrete geometric estimator $\hat{E}$ is multigrid convergent to the geometric quantity E for the family of shapes $\mathbb{X}$, iff, for any $X \in \mathbb{X}$, there exists a grid step $h_X > 0$, such that :

$$\hat{E}(G_h(X), \hat{x}, h) \text{ is defined for any } \hat{x} \in \partial[G_h(X)]_h \text{ with } 0 < h < h_X,$$

$$\text{for any } x \in \partial X, \text{ for any } \hat{x} \in \partial[G_h(X)]_h \text{ with } \|x - \hat{x}\|_\infty \leq h, \quad |\hat{E}(G_h(X), \hat{x}, h) - E(X, x)| < \tau_X(h)$$

where the speed of convergence $\tau_X(h)$ has null limit when $h \to 0$.

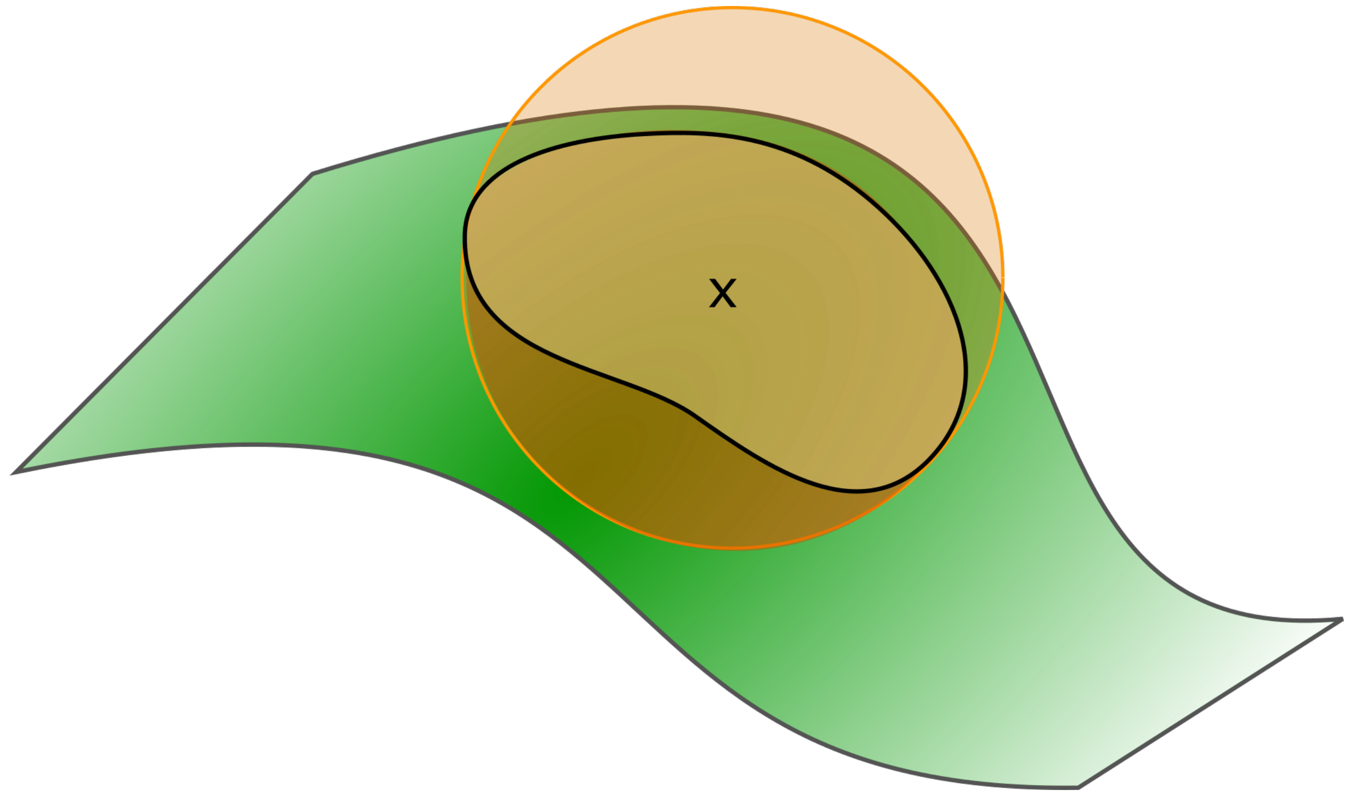
(Typically normal direction, curvatures, …)

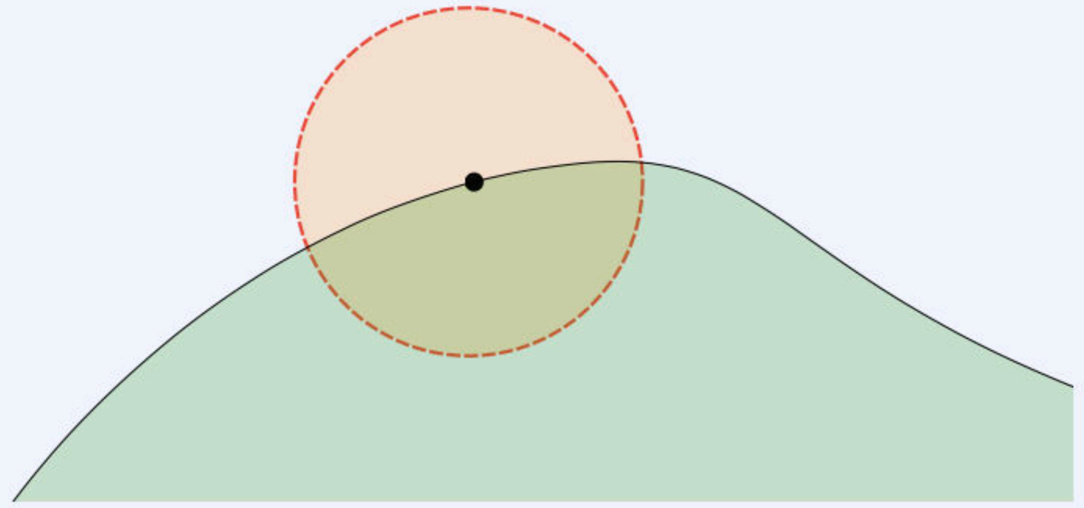


$M \in \mathbb{X}$ $\quad$ $G_1(M)$ $\quad$ $G_{0.5}(M)$ $\quad$ $G_{0.25}(M)$

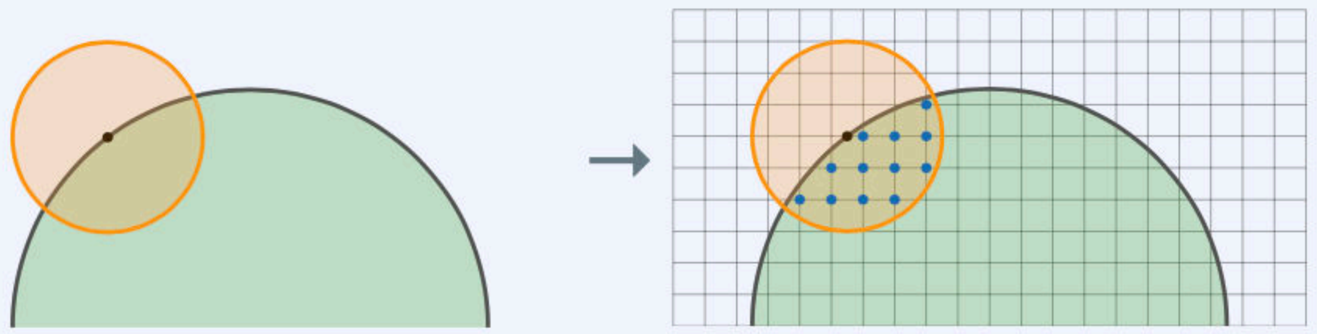
Normal vector and curvatures estimation

- Integral Invariants : analyzing set $B_R(x) \cap X$ gives normal vector, principal directions and curvatures [Pottmann et al. 2007]

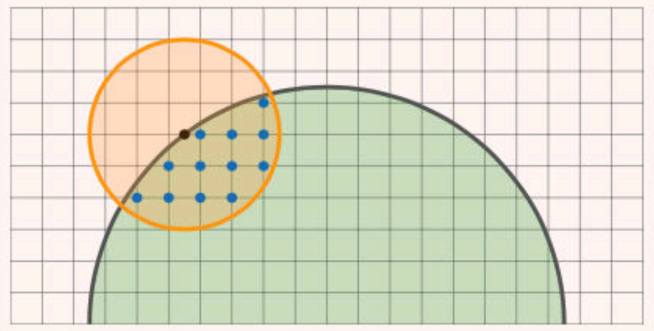




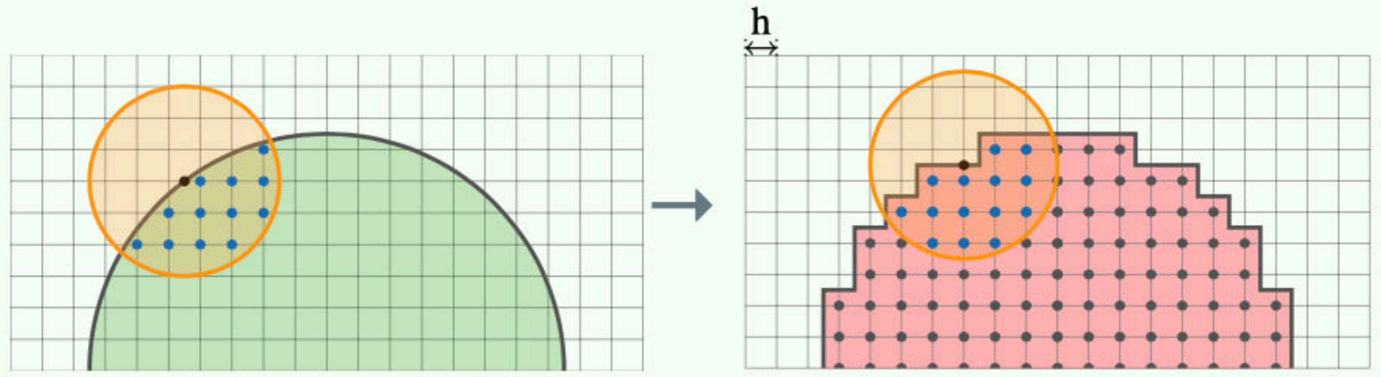
$$\kappa(M,\mathbf{x}) := \underbrace{\frac{3\pi}{2R} - \frac{3 \cdot A_R(M,\mathbf{x})}{R^3}}_{\kappa^R(M,\mathbf{x})} + O(R)$$ [Pottmann et al. 2007]



$$A_R(M,\mathbf{x}) \to \widehat{\mathrm{Area}}\left(B_{R/h}(\mathbf{x}/h) \cap \mathrm{G}_h(M)\right)$$



+ [Pottmann et al. 2007] $\hat{\kappa}^R(\mathrm{G}_h(M), \mathbf{x}, h)$



$$\hat{\kappa}^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) \to \kappa(M,\mathbf{x})$$

[C., Levallois, Lachaud]

Let M be a convex shape in $\mathbb{R}^2$ with a C^3 bounded positive curvature boundary.

$$\forall \mathbf{x} \in \partial M, \forall \hat{\mathbf{x}} \in \partial[\mathrm{G}_h(M)]_h, \|\hat{x} - x\|_\infty \le h \Rightarrow$$

$$|\hat{\kappa}^R(\mathrm{G}_h(M), \hat{\mathbf{x}}, h) - \kappa(M,\mathbf{x})| = O(R) + O\left(\frac{h^\beta}{R^{1+\beta}}\right) + O\left(\frac{h^{\alpha'}}{R^2}\right) + O\left(h^{\alpha'}\right) + O\left(\frac{h^{2\alpha'}}{R^2}\right)$$

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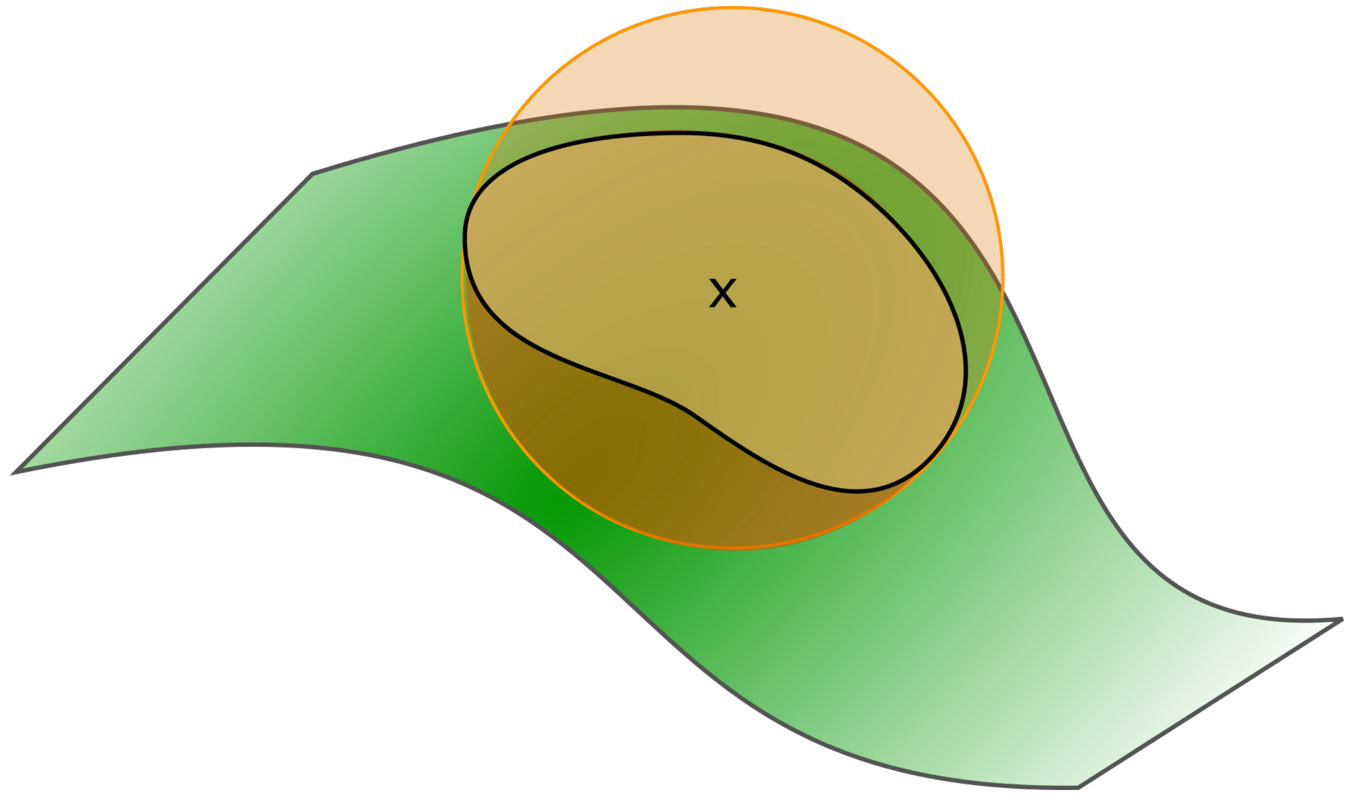

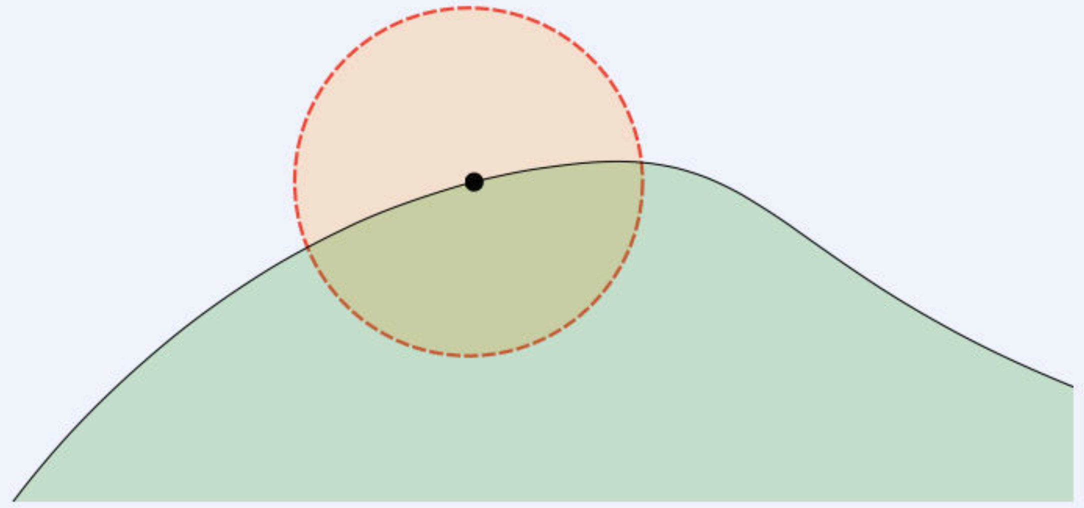

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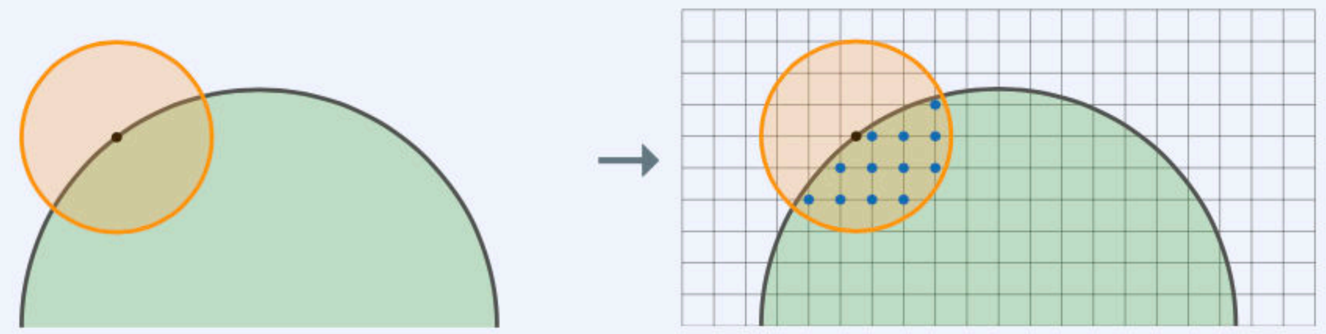
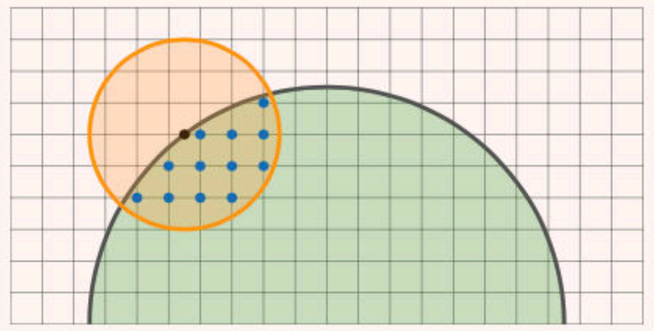
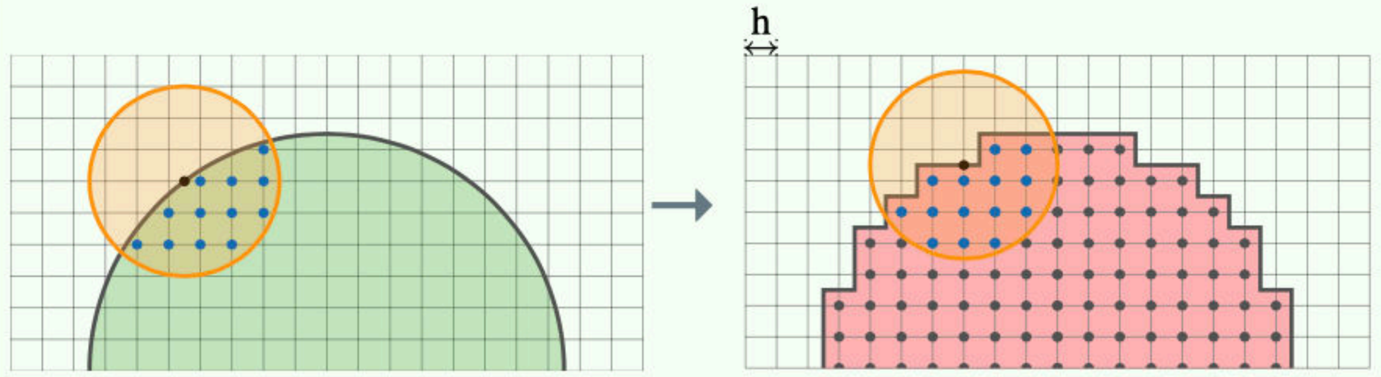


$$A_R(M,\mathbf{x}) \to \widehat{\text{Area}}\,(B_{R/h}(\mathbf{x}/h) \cap \mathtt{G}_h(M))$$



+ [Pottmann et al. 2007]

$$\kappa^R(\mathtt{G}_h(M), \mathbf{x}, h)$$



$$\kappa^R(\mathtt{G}_h(M), \hat{\mathbf{x}}, h) \to \kappa(M,\mathbf{x})$$

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Let M be a convex shape in $\mathbb{R}^2$ with a C^3 bounded positive curvature boundary.

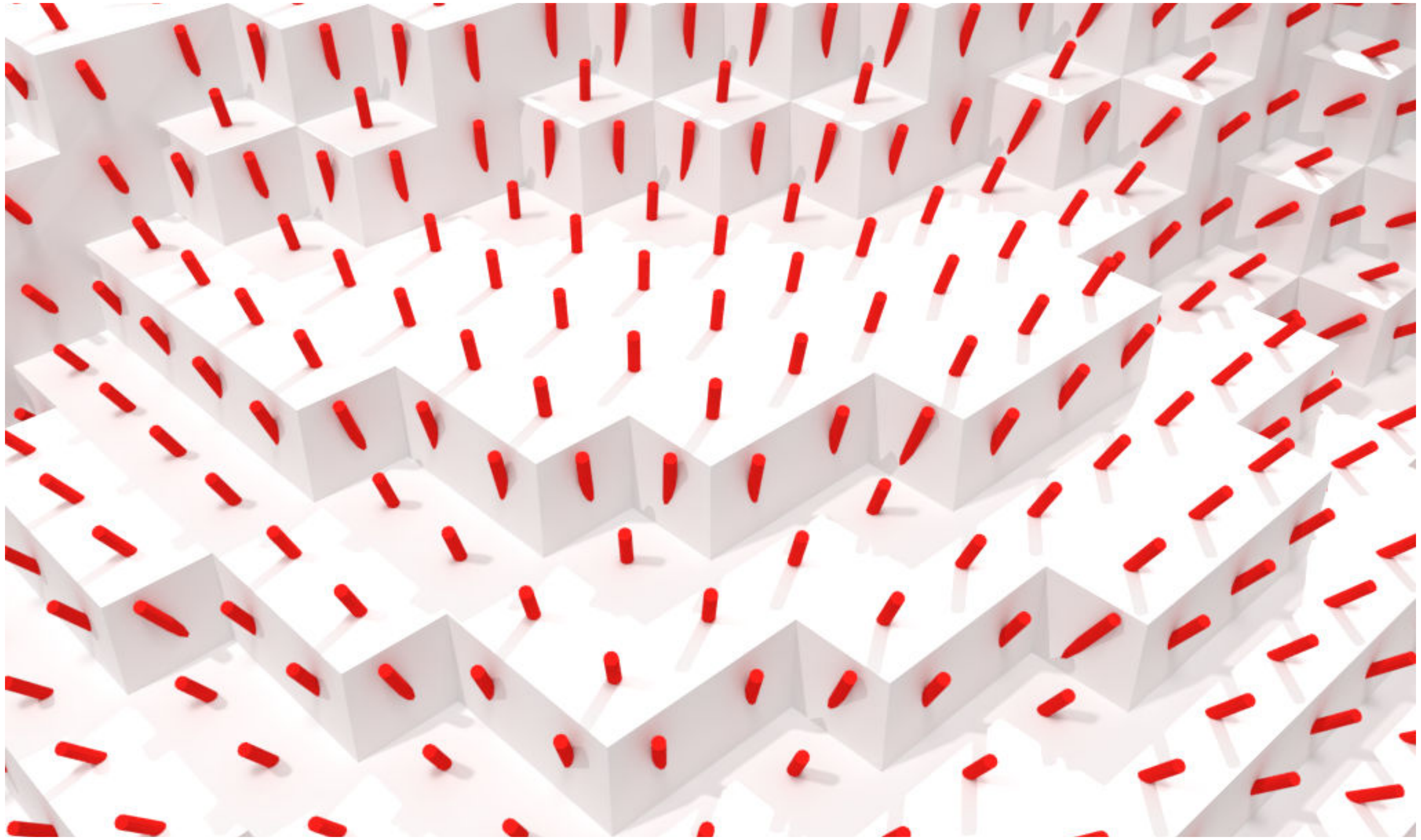
$$\forall \mathbf{x} \in \partial M, \forall \hat{\mathbf{x}} \in \partial[\mathtt{G}_h(M)]_h, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$

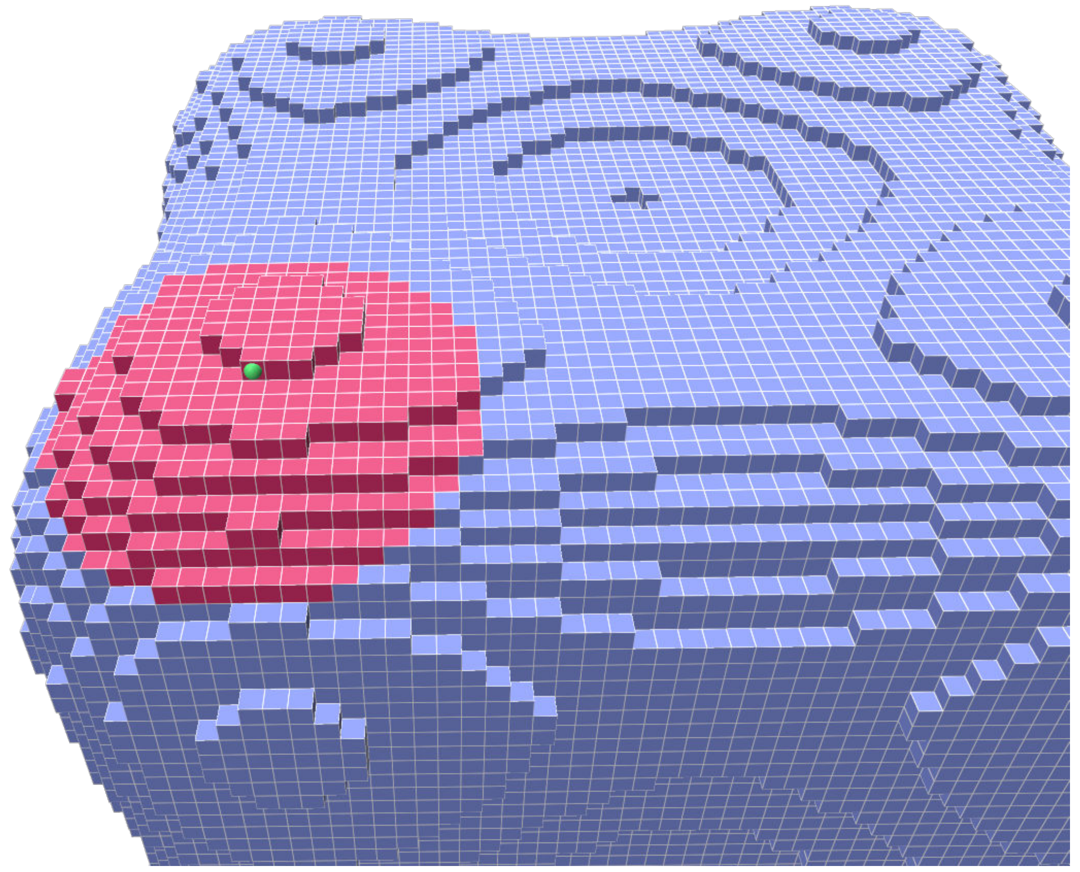
$$|\kappa^R(\mathtt{G}_h(M), \hat{\mathbf{x}}, h) - \kappa(M,\mathbf{x})| = O(R) + O\left(\frac{h^\beta}{R^{1+\beta}}\right) + O\left(\frac{h^{\alpha'}}{R^2}\right) + O\left(h^{\alpha'}\right) + O\left(\frac{h^{2\alpha'}}{R^2}\right)$$

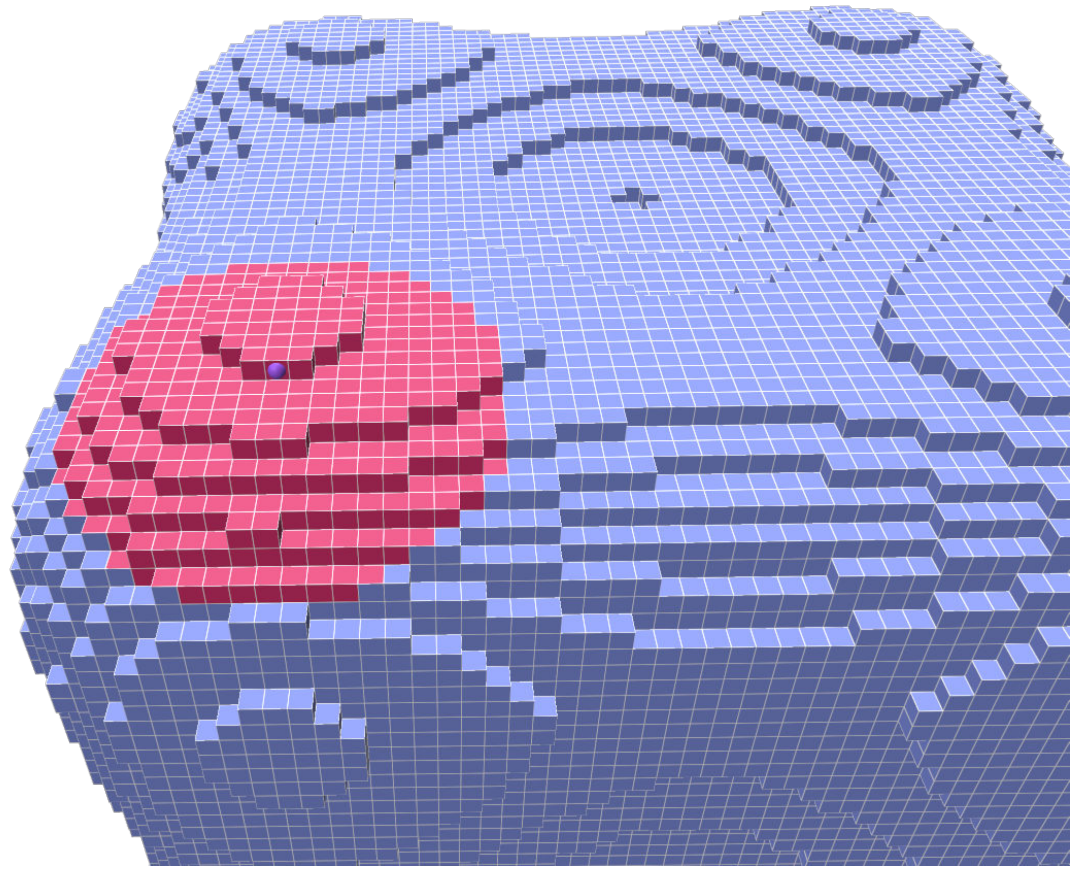
With optimal radius $R = O(h^{\frac{1}{3}})$, then :

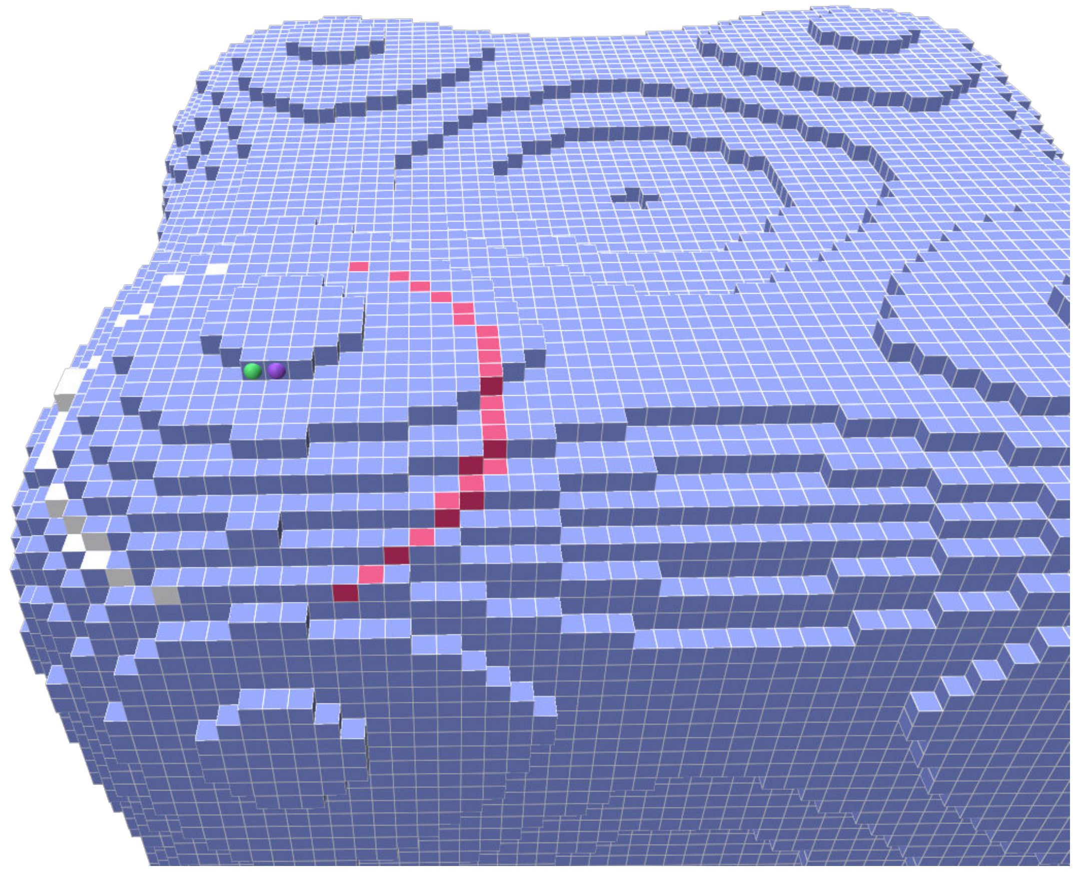
- normals $\left\| \hat{\mathbf{n}}(G_h(M), \xi(x), h)) - \mathbf{n}(M,x) \right\| \leq C \cdot h^{\frac{2}{3}}$
- mean curvature $\left\| \hat{\kappa}(M_h, \xi(x))) - \kappa(M,x) \right\|_2 \leq C \cdot h^{\frac{1}{3}}$
- … [CLL2014], [LCL2017]

Normal vector field estimation









Incremental computation : estimate at y nearby x only requires preceding result + looking at points within $B_R(y) \ominus B_R(x)$

hands on…

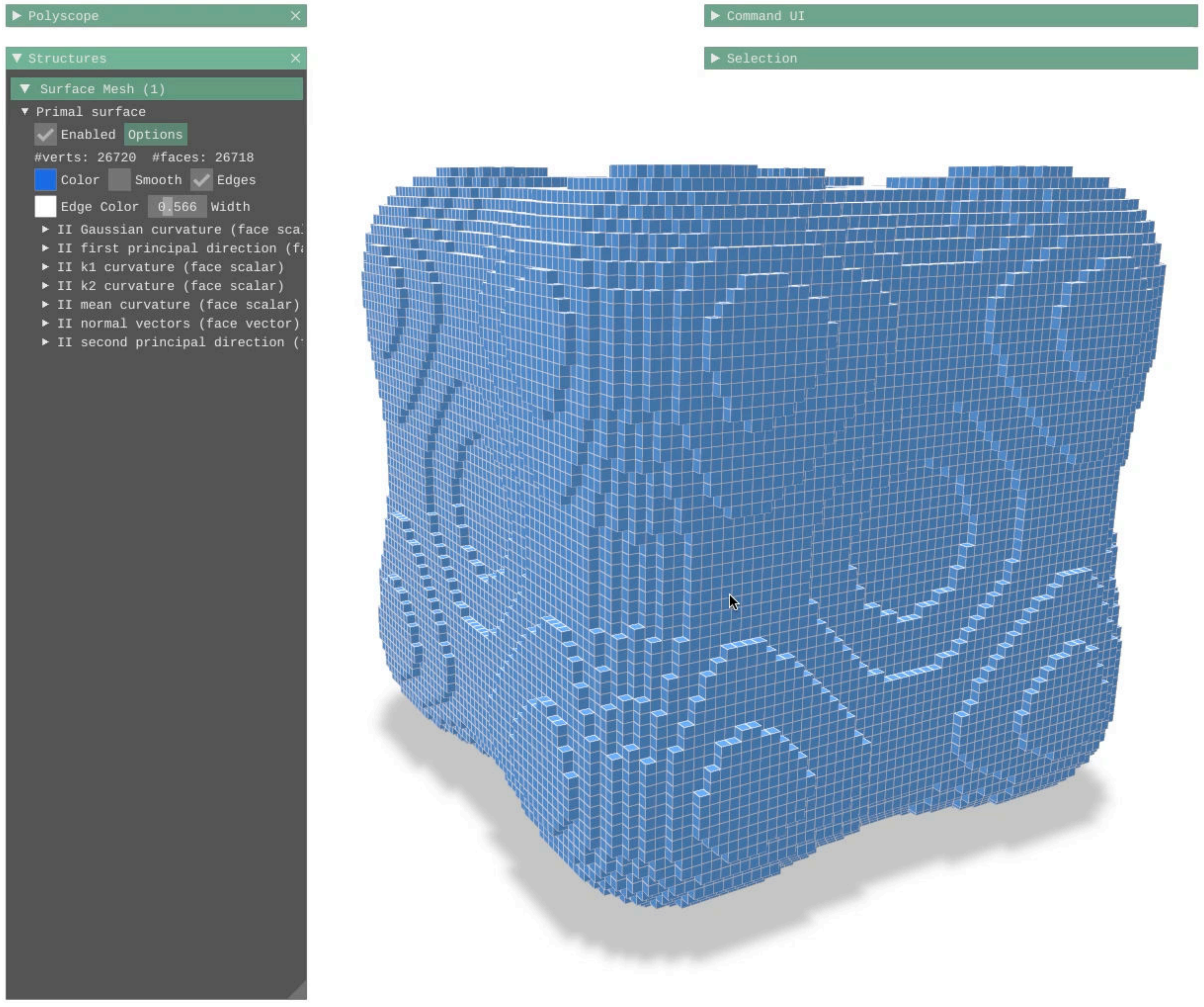
```cpp
void oneStepAll(double h)
{
  auto params = SH3::defaultParameters() | SHG3::defaultParameters() |  SHG3::parametersGeometryEstimation();
  params( "polynomial", "goursat" )( "gridstep", h );
  auto implicit_shape  = SH3::makeImplicitShape3D  ( params );
  auto digitized_shape = SH3::makeDigitizedImplicitShape3D( implicit_shape, params );
  auto K               = SH3::getKSpace( params );
  auto binary_image    = SH3::makeBinaryImage( digitized_shape, params );
  auto surface         = SH3::makeDigitalSurface( binary_image, K, params );
  auto embedder        = SH3::getCellEmbedder( K );
  SH3::Cell2Index c2i;
  auto surfels         = SH3::getSurfelRange( surface, params );
  auto primalSurface   = SH3::makePrimalPolygonalSurface(c2i, surface);

  //Need to convert the faces
  std::vector<std::vector<std::size_t>> faces;
  for(auto &face: primalSurface->allFaces())
    faces.push_back(primalSurface->verticesAroundFace( face ));
  auto digsurf = polyscope::registerSurfaceMesh("Primal surface", primalSurface->positions(), faces);
  digsurf->rescaleToUnit(); digsurf->setEdgeWidth(h*h);  digsurf->setEdgeColor({1.,1.,1.});

  //Computing some differential quantities
  params("r-radius", 5*std::pow(h,-2.0/3.0));
  auto Mcurv      = SHG3::getIIMeanCurvatures(binary_image, surfels, params);
  auto normalsII  = SHG3::getIINormalVectors(binary_image, surfels, params);
  auto KTensor    = SHG3::getIIPrincipalCurvaturesAndDirections(binary_image, surfels, params);  //Recomputing...

  std::vector<double> Gcurv(surfels.size()),k1(surfels.size()),k2(surfels.size());
  std::vector<RealVector> d1(surfels.size()),d2(surfels.size());
  auto i=0;
  for(auto &t: KTensor) //AOS->SOA
  {
    k1[i]    = std::get<0>(t);
    k2[i]    = std::get<1>(t);
    d1[i]    = std::get<2>(t);
    d2[i]    = std::get<3>(t);
    Gcurv[i] = k1[i]*k2[i];
    ++i;
  }

  //Attaching quantities
  digsurf->addFaceVectorQuantity("II normal vectors", normalsII, polyscope::VectorType::AMBIENT);
  digsurf->addFaceScalarQuantity("II mean curvature", Mcurv);
  digsurf->addFaceScalarQuantity("II Gaussian curvature", Gcurv);
  digsurf->addFaceScalarQuantity("II k1 curvature", k1);
  digsurf->addFaceScalarQuantity("II k2 curvature", k2);
  digsurf->addFaceVectorQuantity("II first principal direction", d1, polyscope::VectorType::AMBIENT);
  digsurf->addFaceVectorQuantity("II second principal direction", d2, polyscope::VectorType::AMBIENT);
}
```



```cpp
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{
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  auto digitized_shape = SH3::makeDigitizedImplicitShape3D( implicit_shape, params );
  auto K               = SH3::getKSpace( params );
  auto binary_image    = SH3::makeBinaryImage( digitized_shape, params );
  auto surface         = SH3::makeDigitalSurface( binary_image, K, params );
  auto embedder        = SH3::getCellEmbedder( K );
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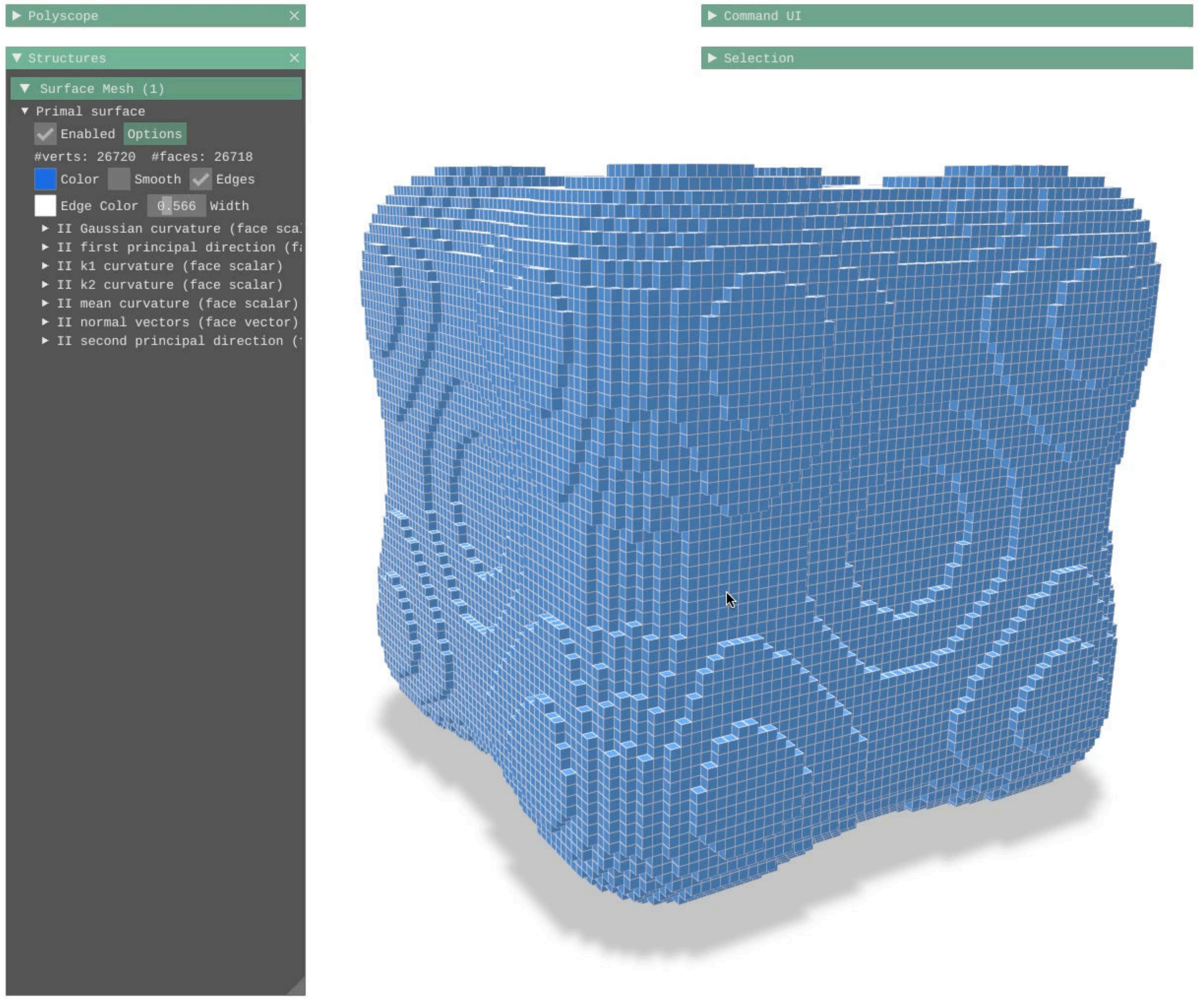
  auto digsurf = polyscope::registerSurfaceMesh("Primal surface", primalSurface->positions(), faces);
  digsurf->rescaleToUnit(); digsurf->setEdgeWidth(h*h);  digsurf->setEdgeColor({1.,1.,1.});

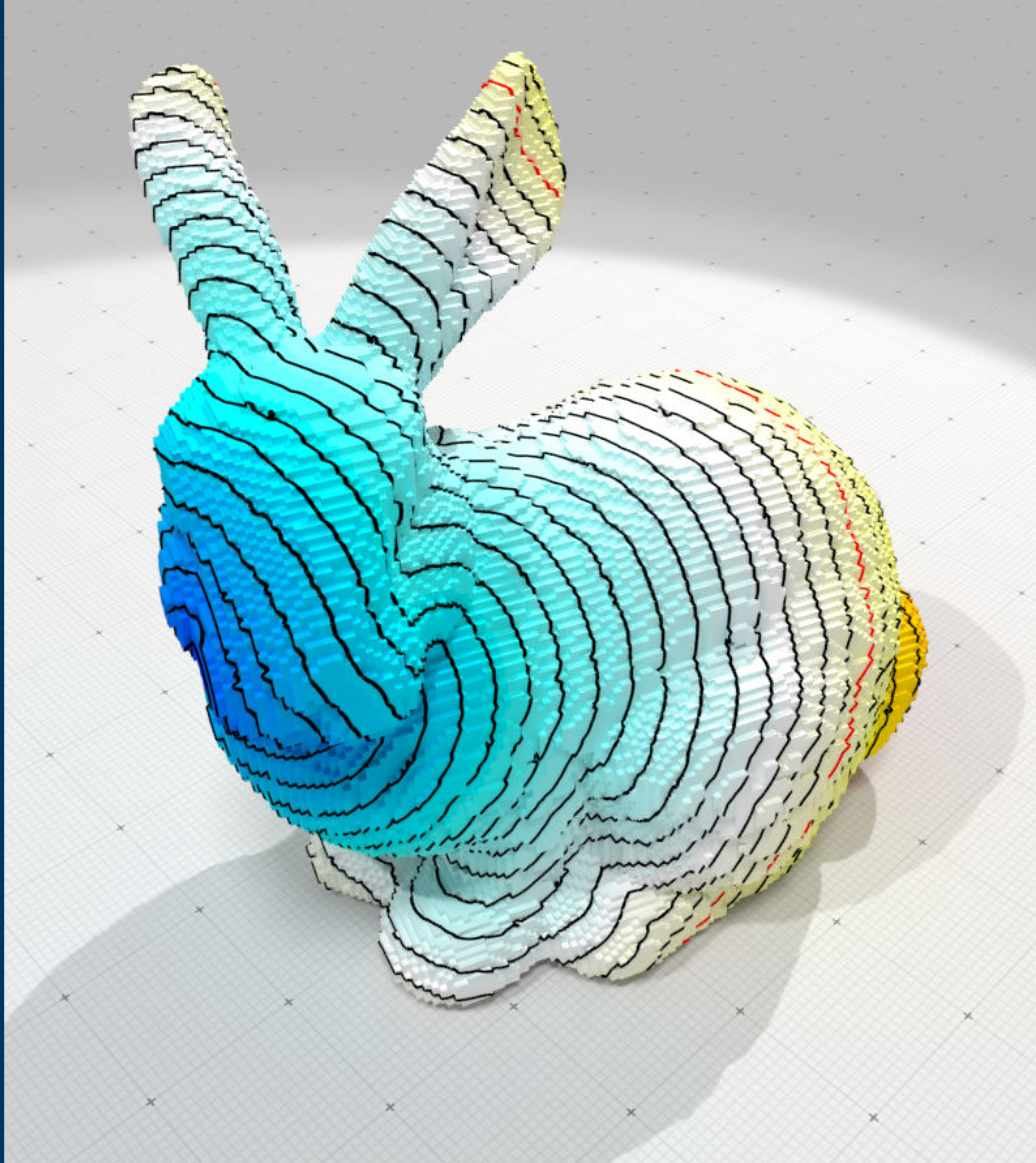
  //Computing some differential quantities
  params("r-radius", 5*std::pow(h,-2.0/3.0));
  auto Mcurv     = SHG3::getIIMeanCurvatures(binary_image, surfels, params);
  auto normalsII = SHG3::getIINormalVectors(binary_image, surfels, params);
  auto KTensor   = SHG3::getIIPrincipalCurvaturesAndDirections(binary_image, surfels, params);  //Recomputing...

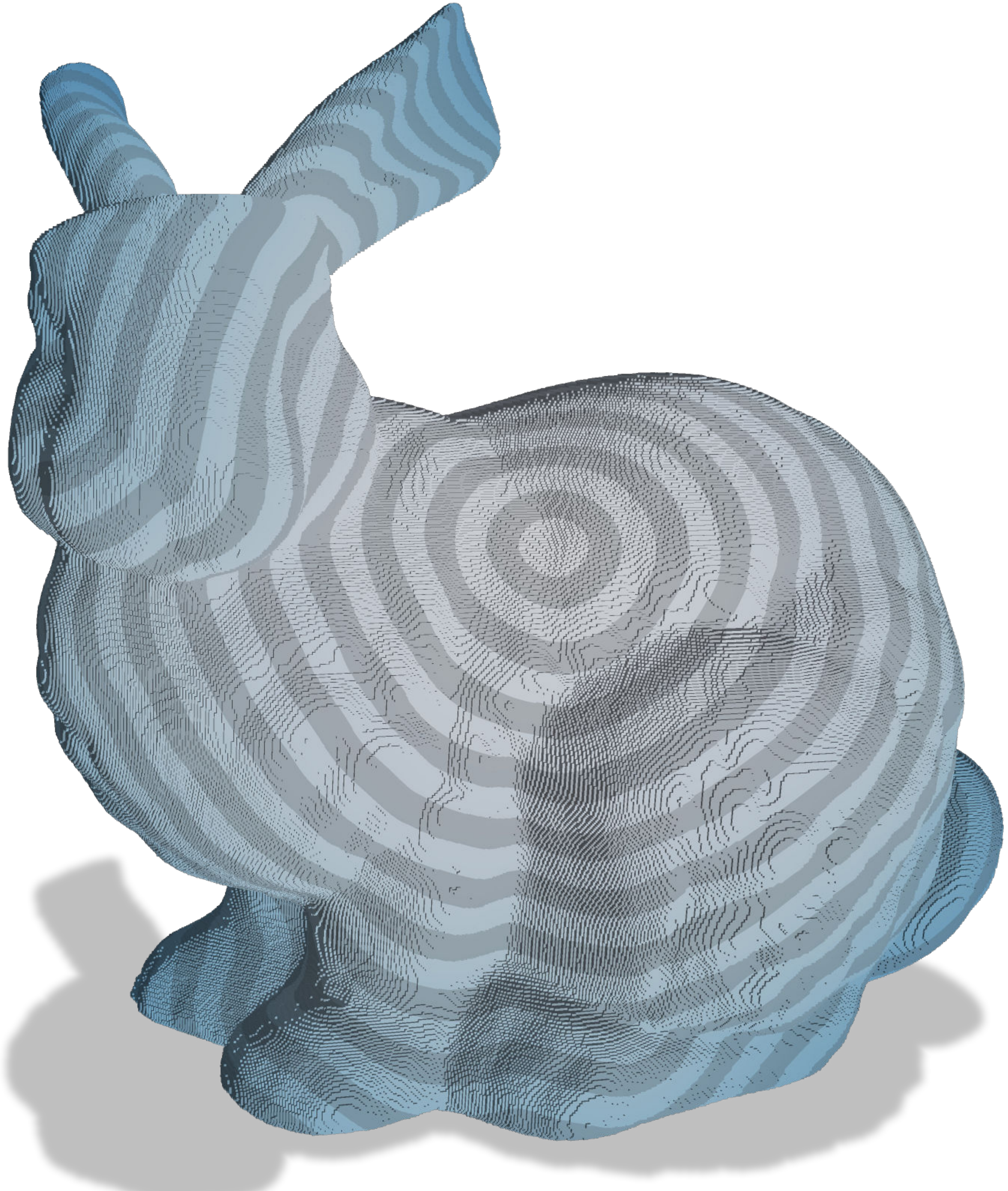
  std::vector<double> Gcurv(surfels.size()),k1(surfels.size()),k2(surfels.size());
  std::vector<RealVector> d1(surfels.size()),d2(surfels.size());
  auto i=0;
  for(auto &t: KTensor) //AOS->SOA
  {
    k1[i]    = std::get<0>(t);
    k2[i]    = std::get<1>(t);
    d1[i]    = std::get<2>(t);
    d2[i]    = std::get<3>(t);
    Gcurv[i] = k1[i]*k2[i];
    ++i;
  }

  //Attaching quantities
  digsurf->addFaceVectorQuantity("II normal vectors", normalsII, polyscope::VectorType::AMBIENT);
  digsurf->addFaceScalarQuantity("II mean curvature", Mcurv);
  digsurf->addFaceScalarQuantity("II Gaussian curvature", Gcurv);
  digsurf->addFaceScalarQuantity("II k1 curvature", k1);
  digsurf->addFaceScalarQuantity("II k2 curvature", k2);
  digsurf->addFaceVectorQuantity("II first principal direction", d1, polyscope::VectorType::AMBIENT);
  digsurf->addFaceVectorQuantity("II second principal direction", d2, polyscope::VectorType::AMBIENT);
}
```



digital surface geometry processing





$$\dot{u} = \Delta u \qquad u(0) = u_0$$

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u

$$\dot{u} = \Delta u \qquad u(0) = u_0$$

u

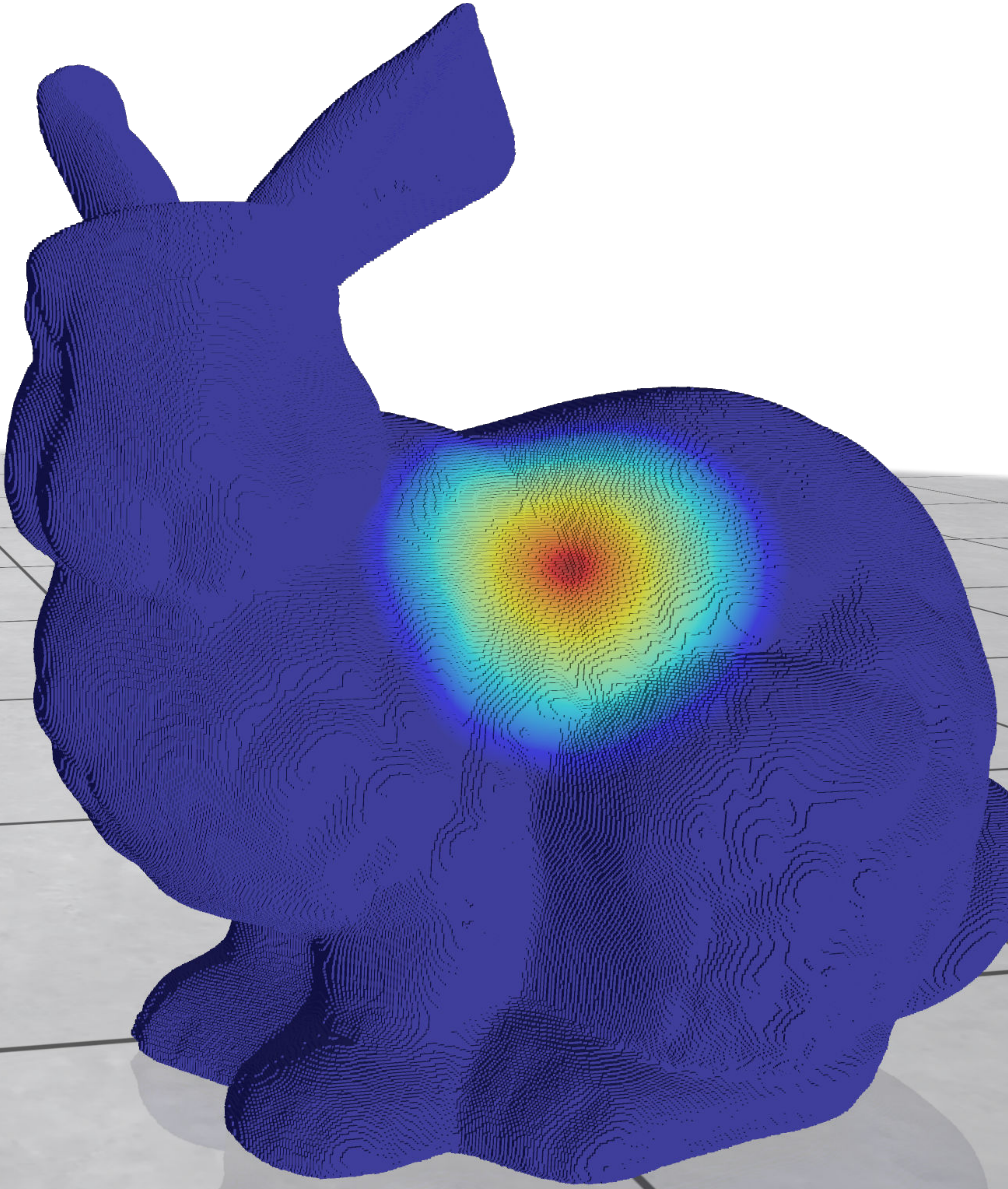
∇u

$\dot{u} = \Delta u \qquad u(0) = u_0$

u

∇u

$\operatorname{div} \overrightarrow{F}$



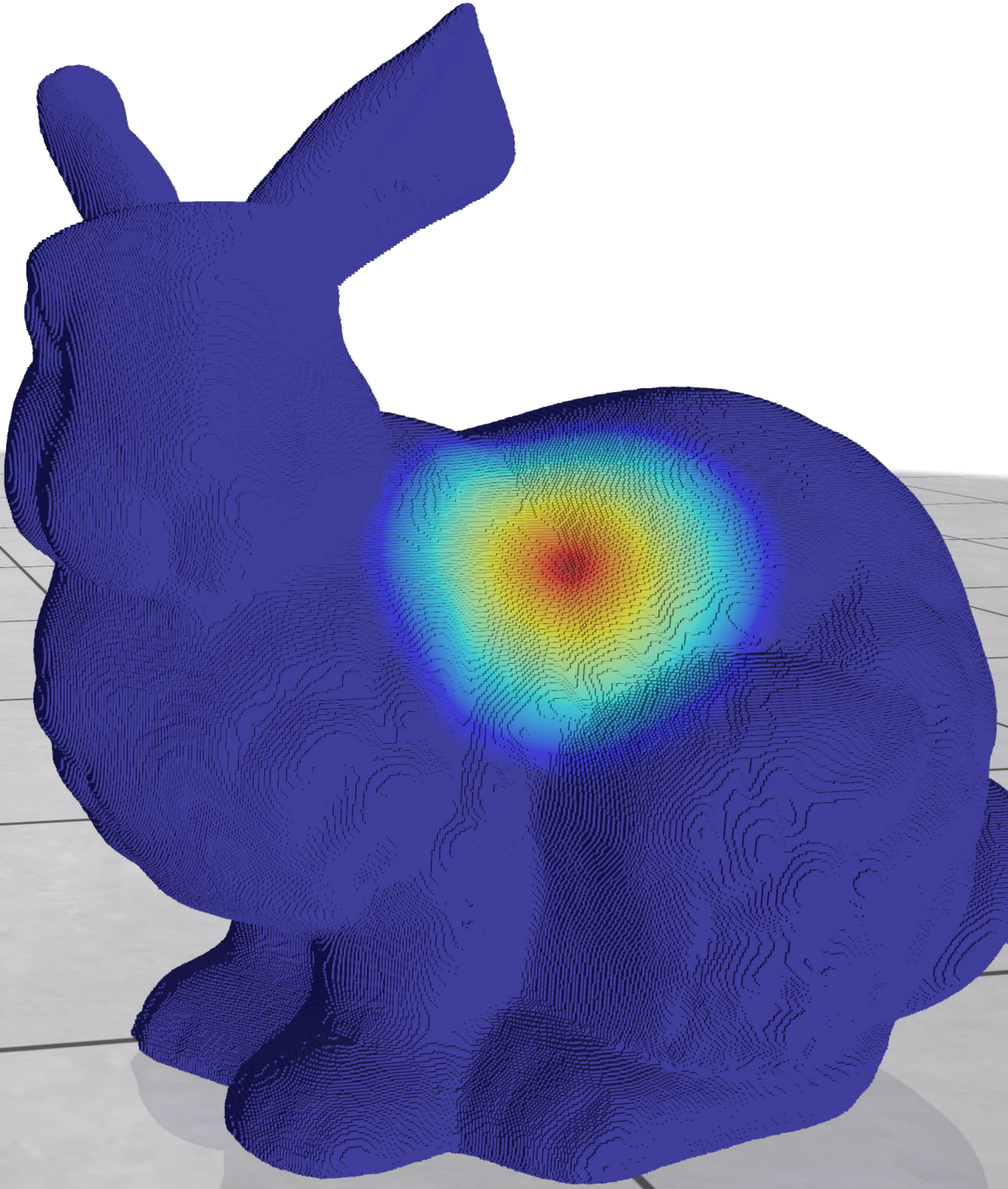
$$\dot{u} = \Delta u \qquad u(0) = u_0$$

u

∇u

$\text{div}\ \overrightarrow{F}$

$\text{curl}\ \overrightarrow{F}$



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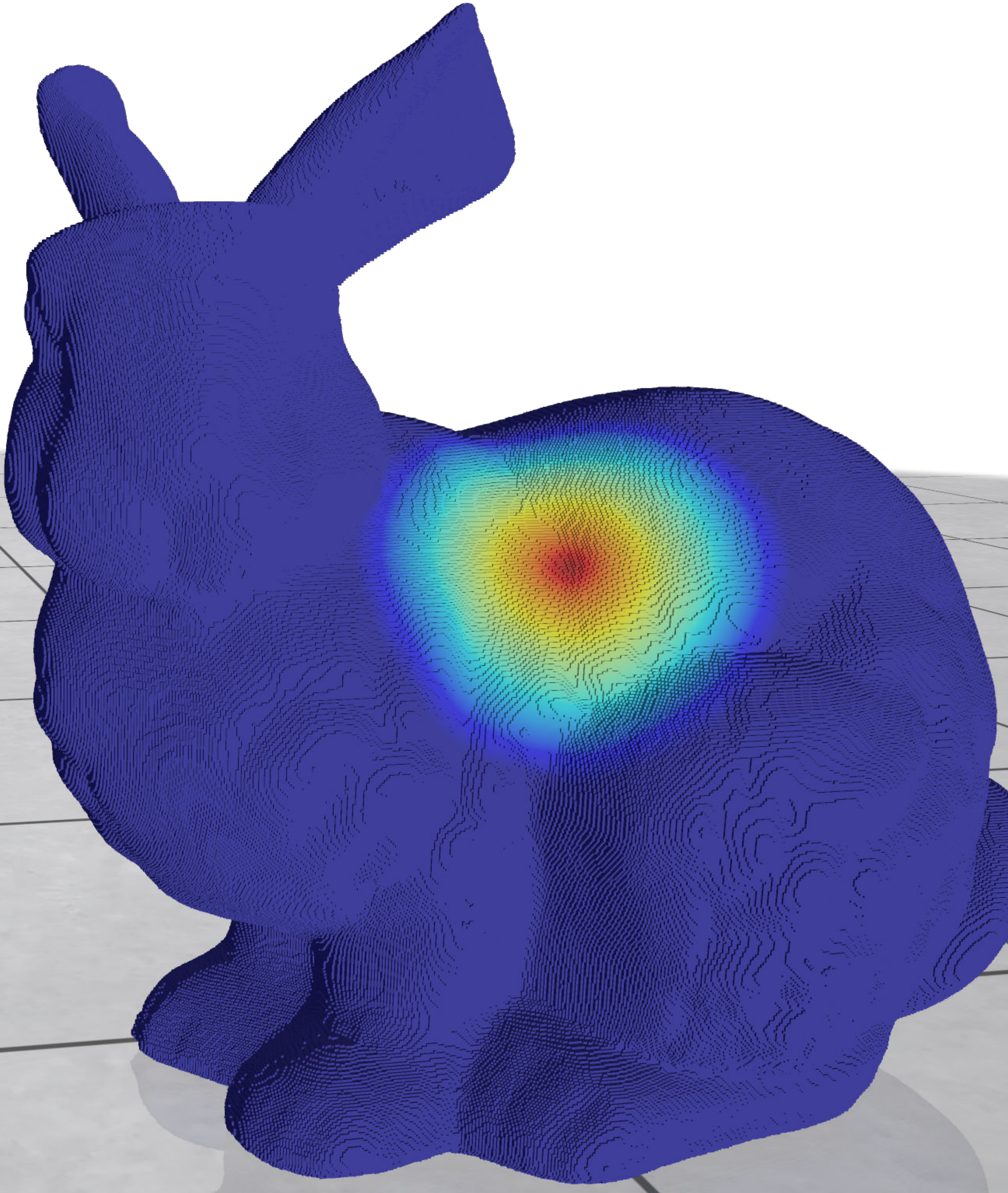
u

∇u

$\operatorname{div} \vec{F}$

$\operatorname{curl} \vec{F}$

$\Delta u := \operatorname{div} \nabla u$



$$\dot{u} = \Delta u \qquad u(0) = u_0$$

u

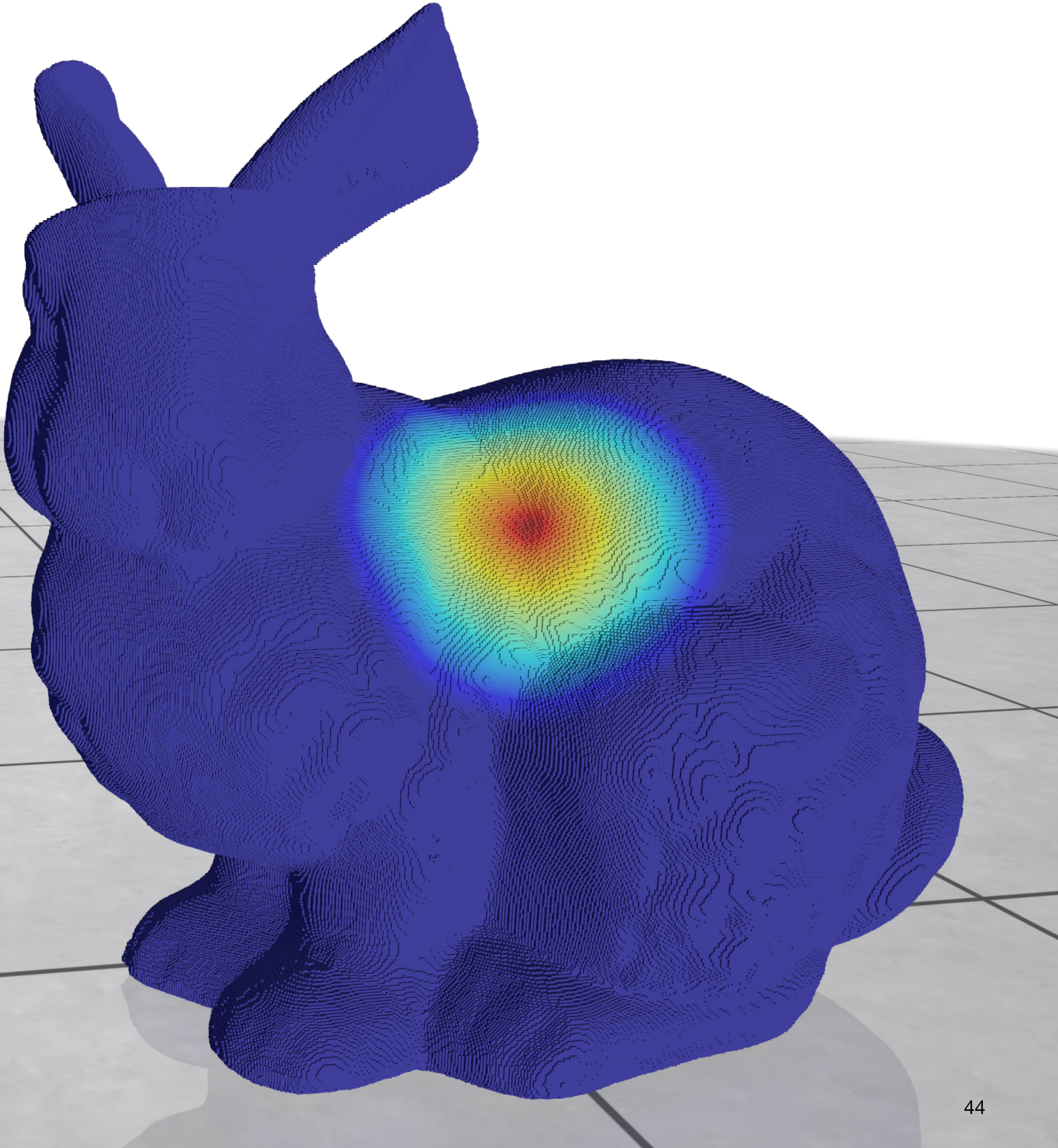
∇u

$\operatorname{div} \vec{F}$

$\operatorname{curl} \vec{F}$

$\Delta u := \operatorname{div} \nabla u$

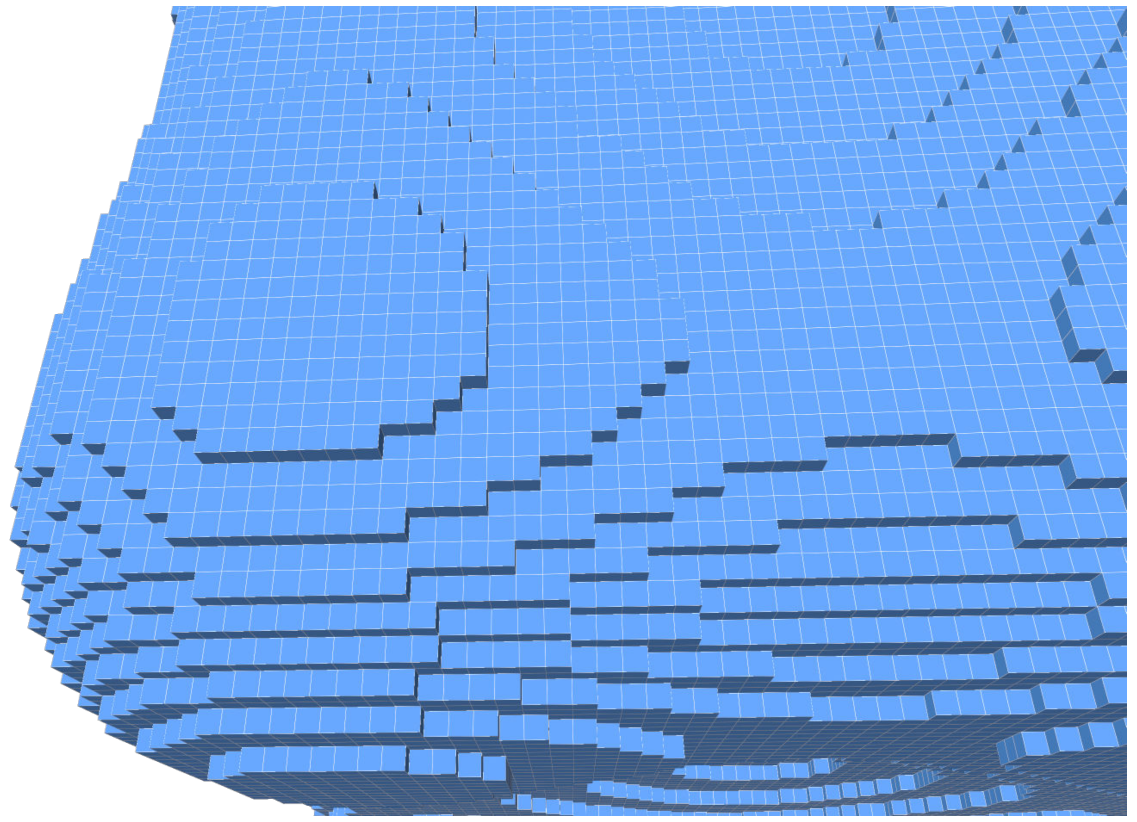
$\Delta u = g$



Discrete Differential Operators on Polygonal Meshes

[de Goes et al 20]

[C. & L. DGMM2022]



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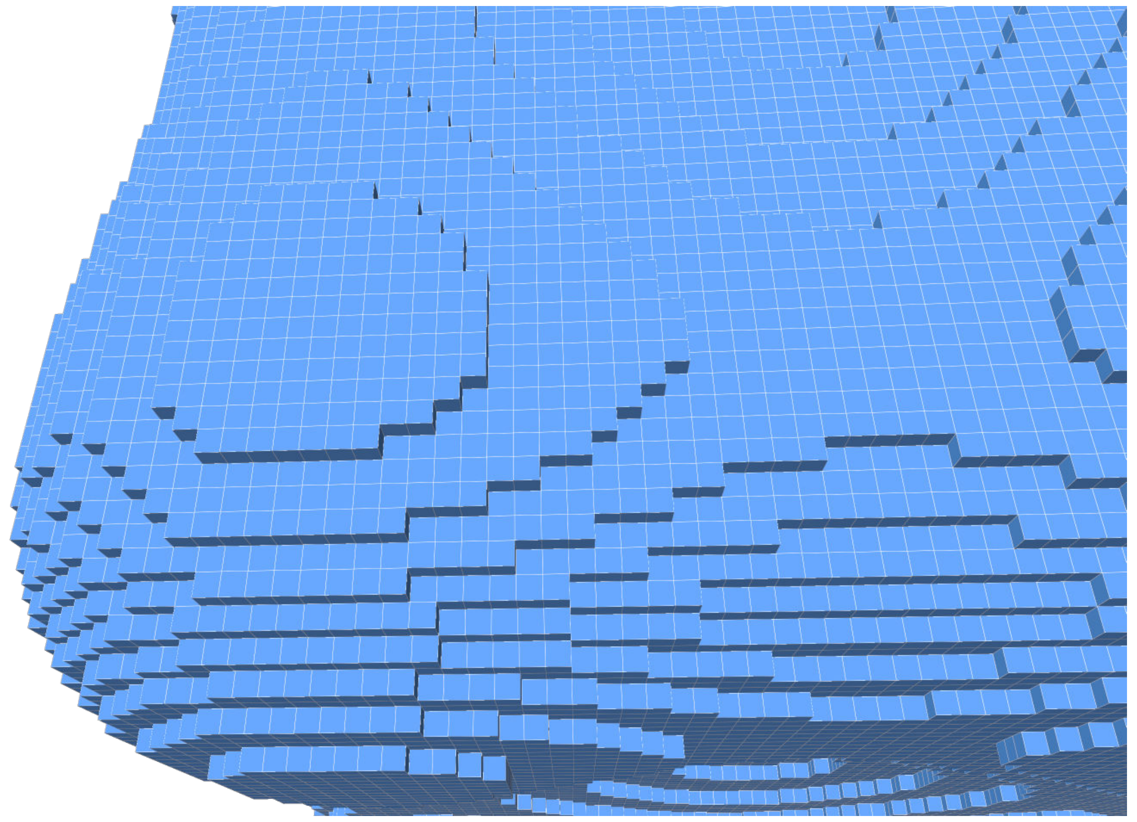
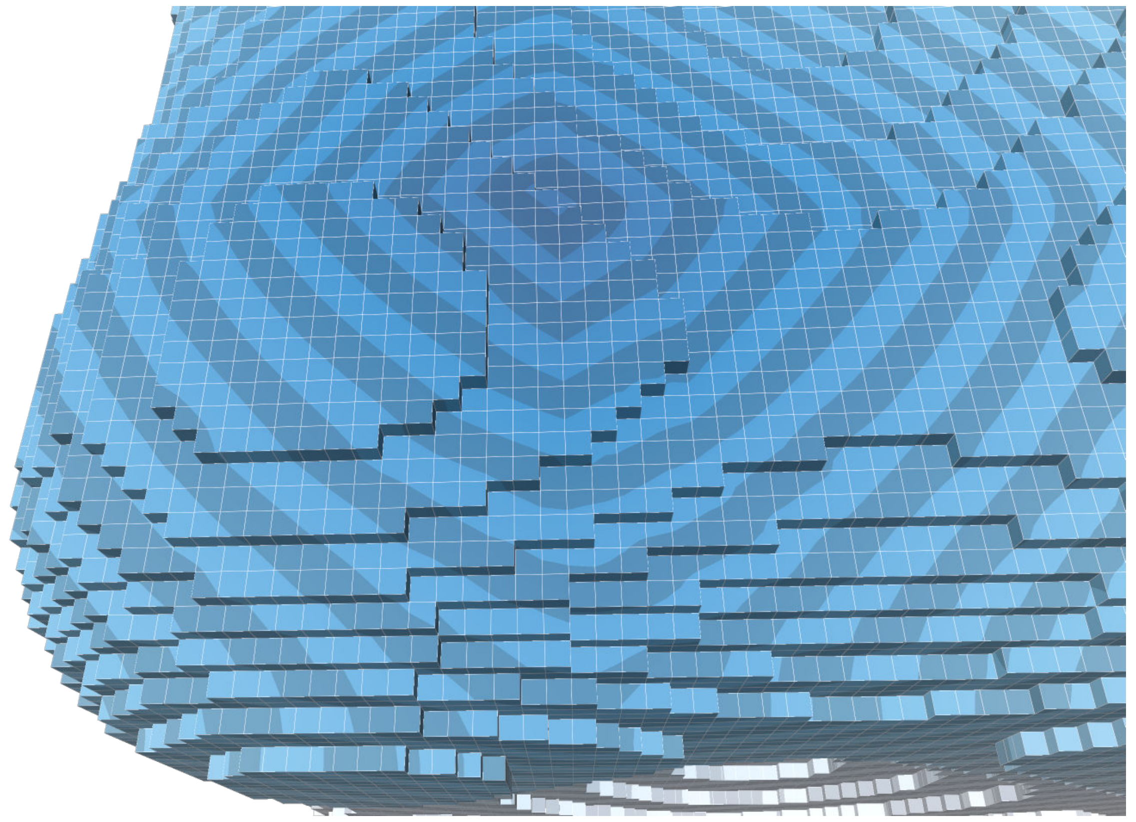

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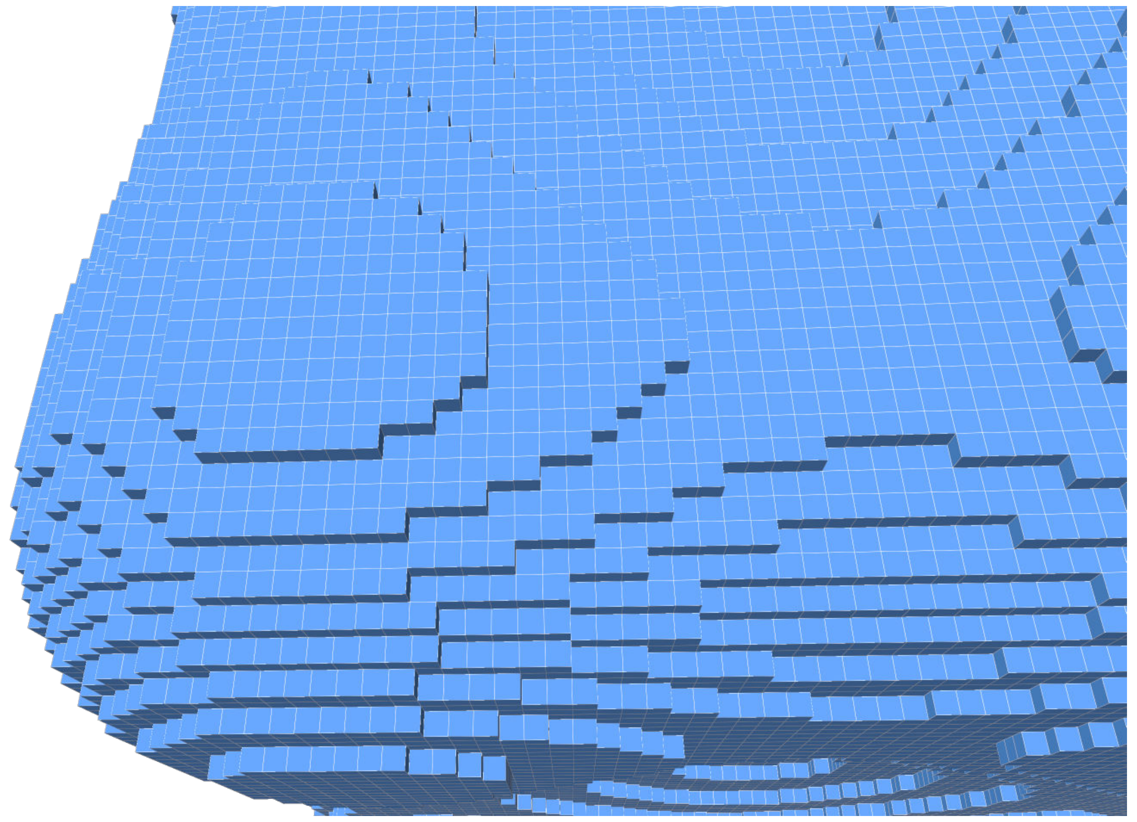
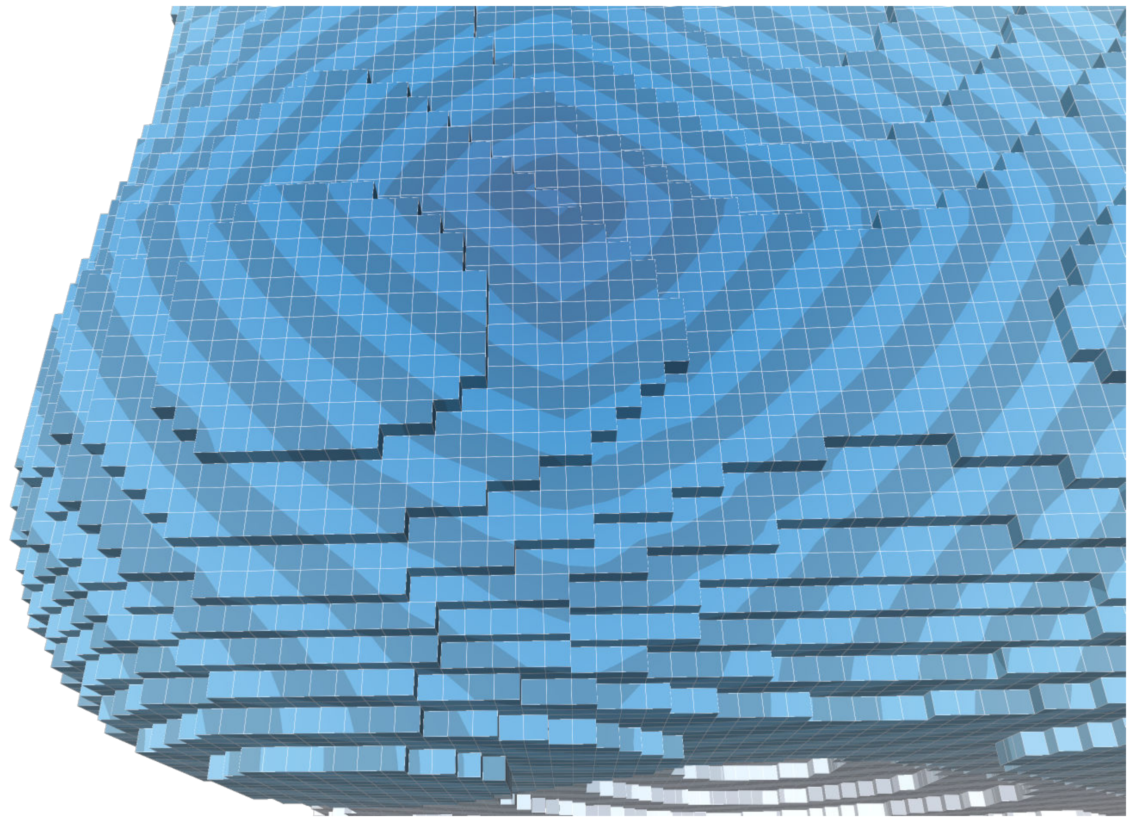
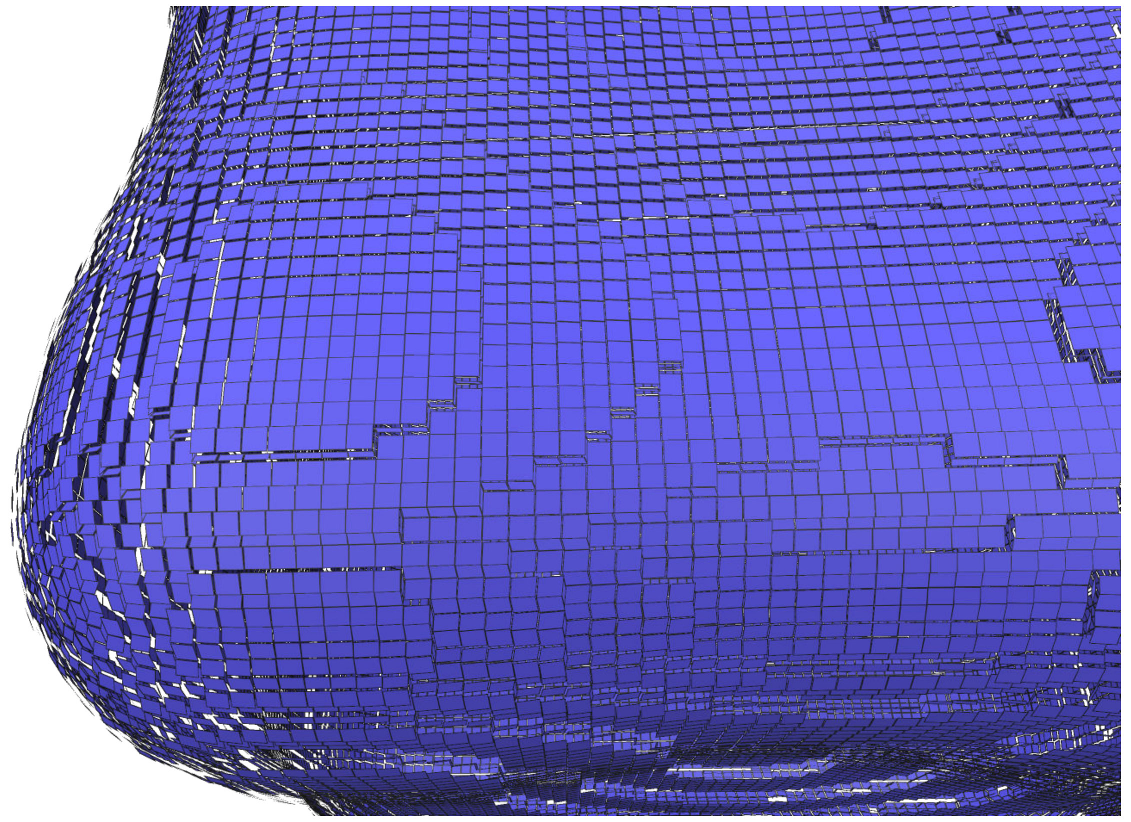
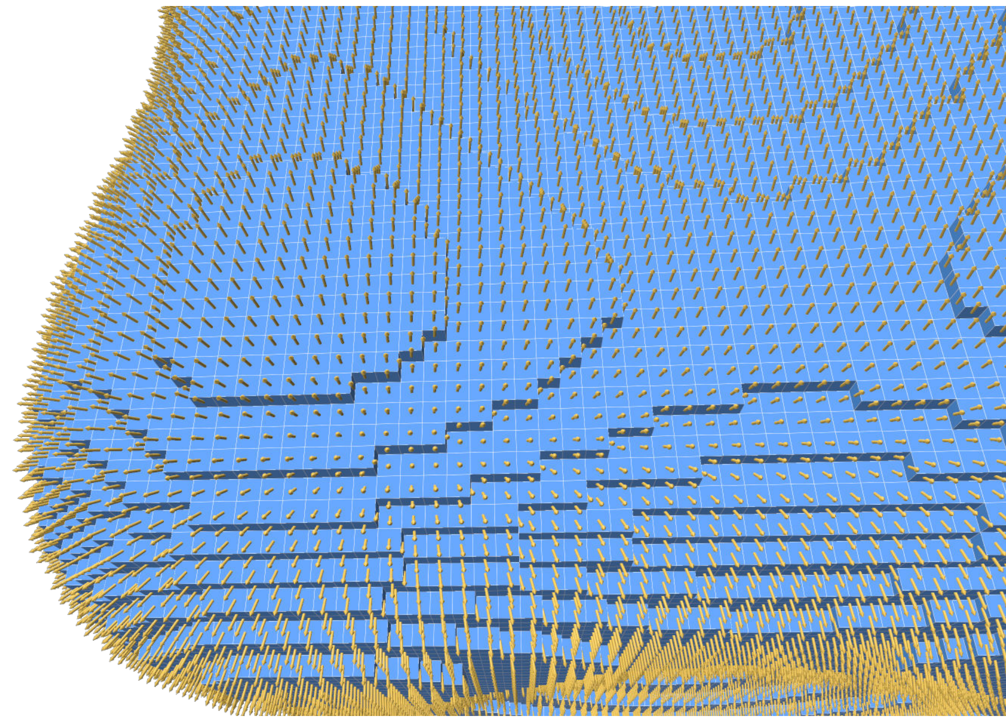




We can *correct* the face embedding using asymptotic convergence normal vector field

Challenges: advance corrections (e.g. on the Grassmanian, higher order schemes…) for asymptotic properties



Discrete Differential Operators on Polygonal Meshes

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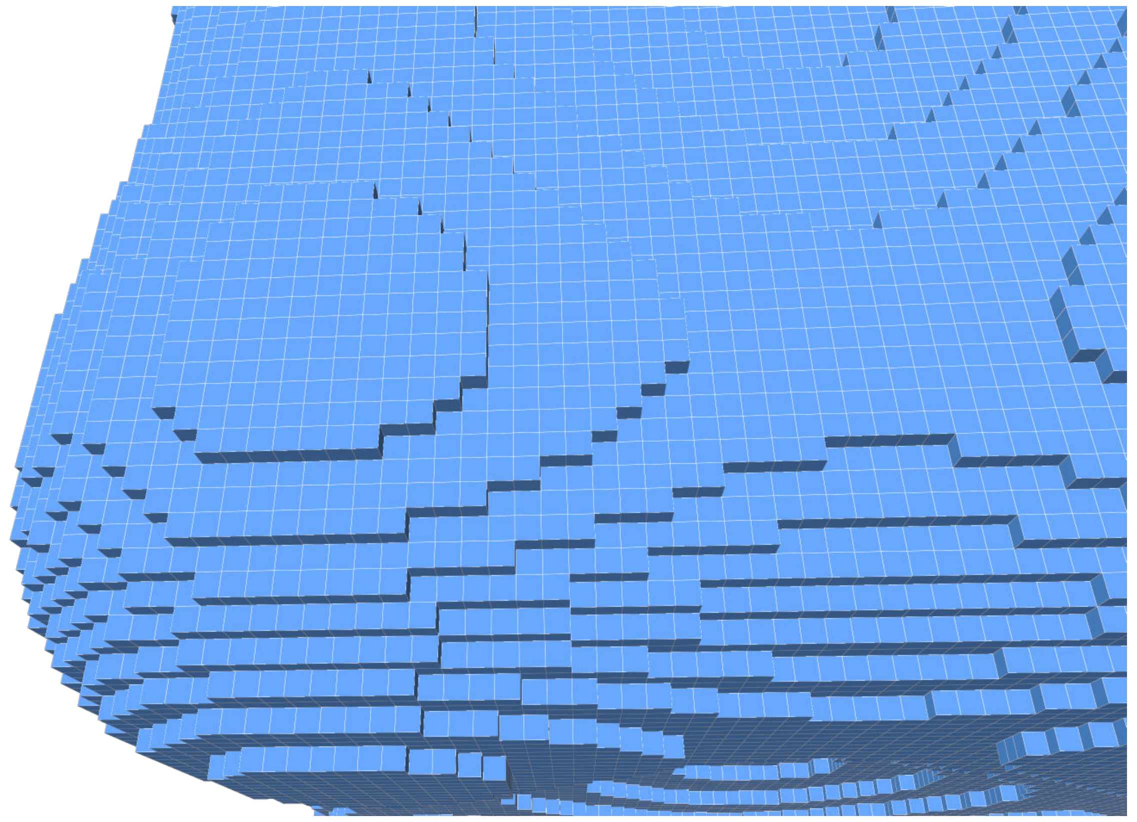
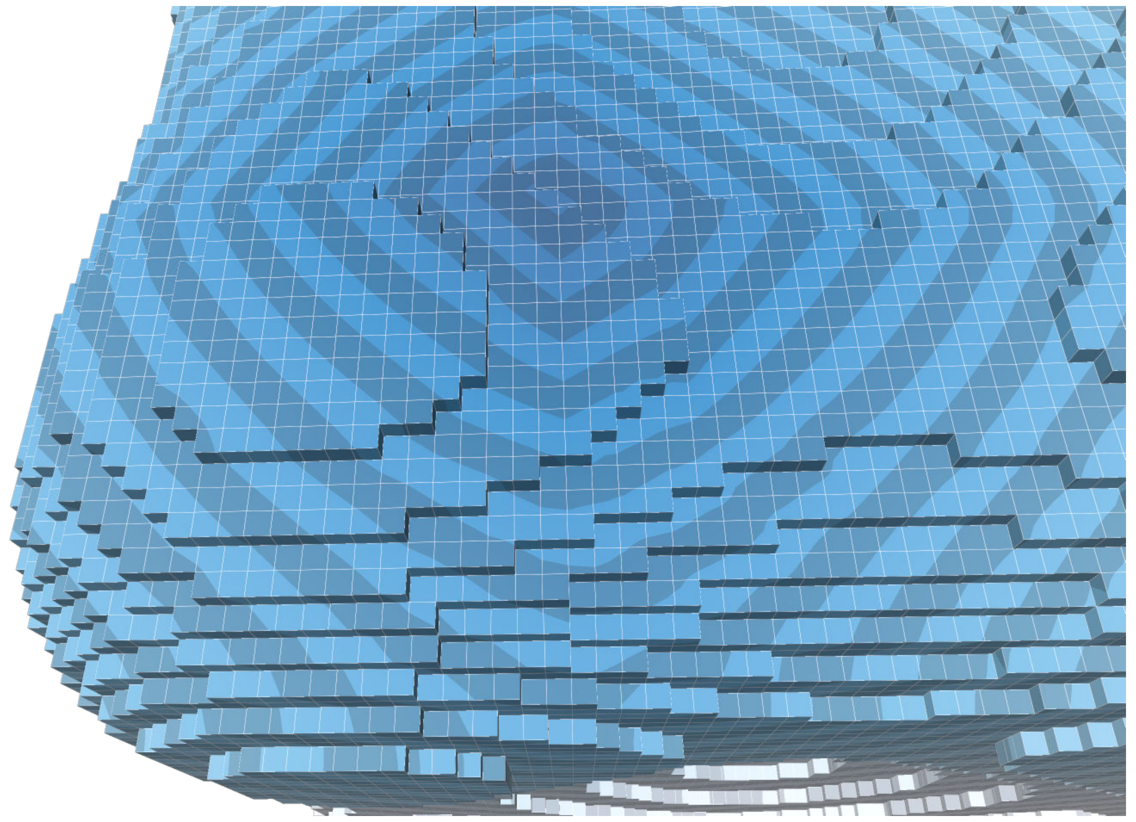
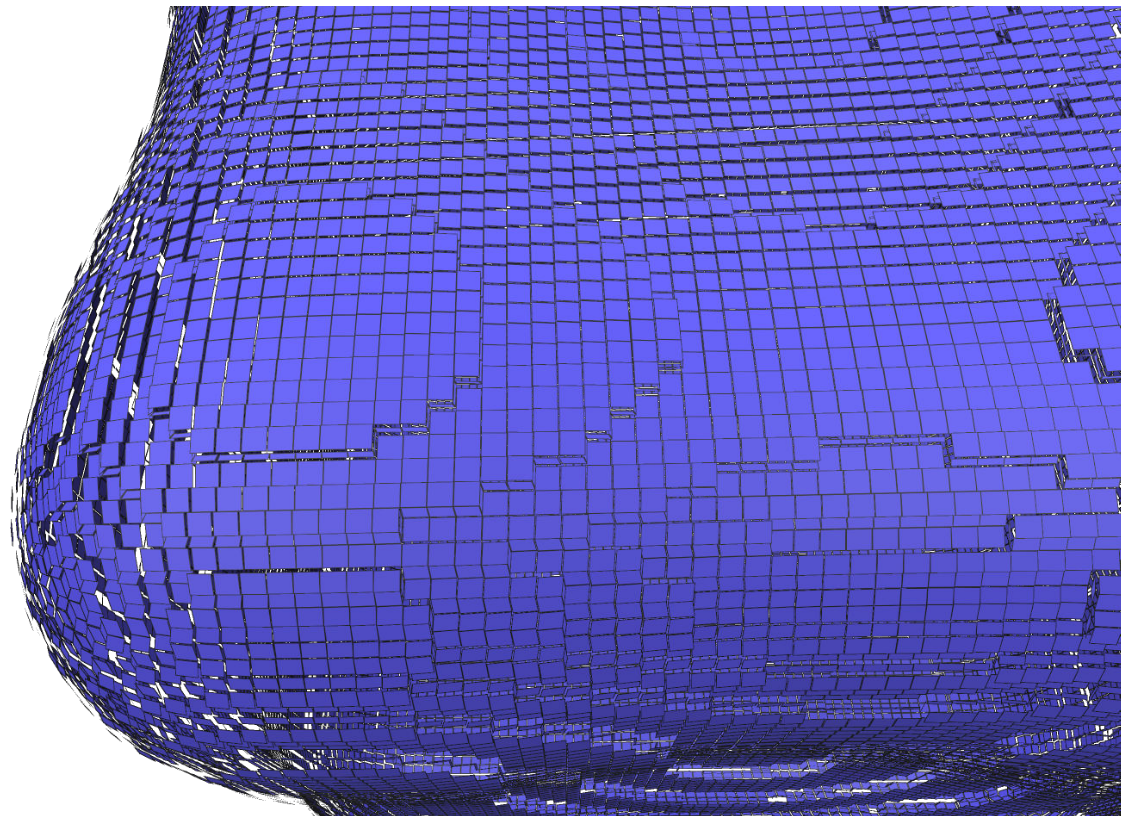
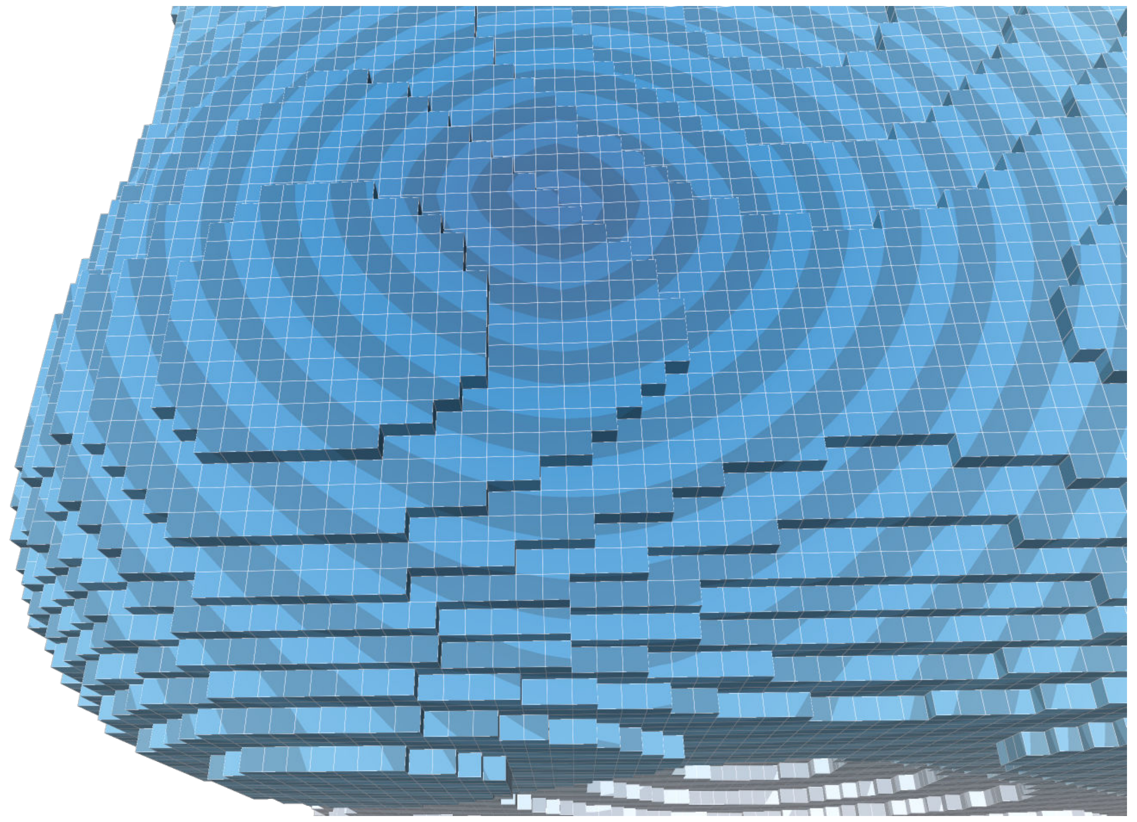

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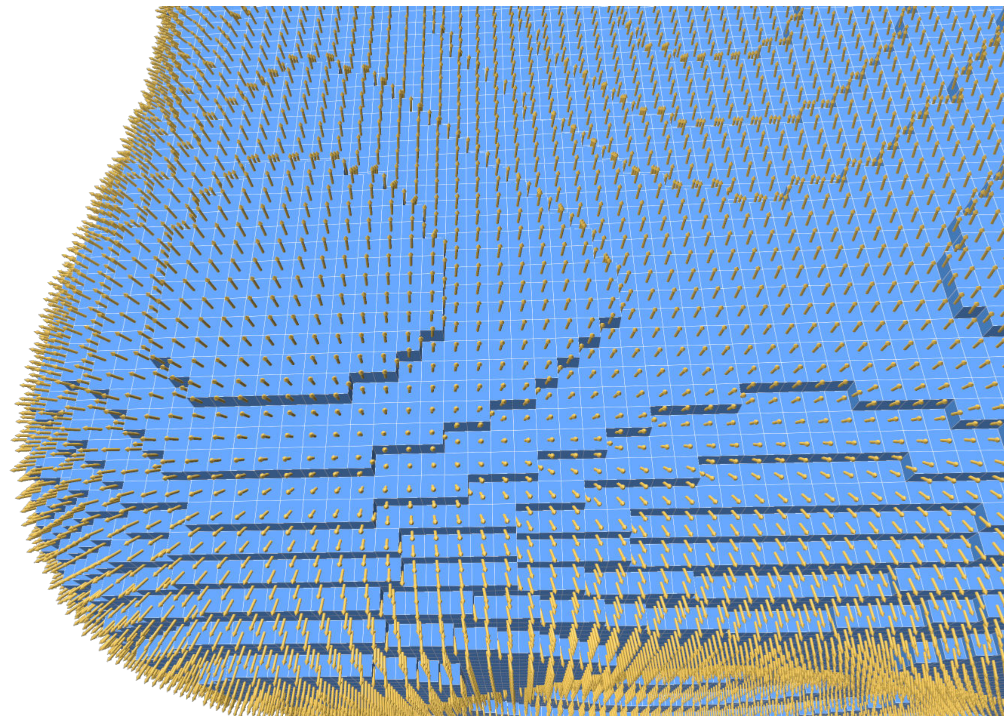
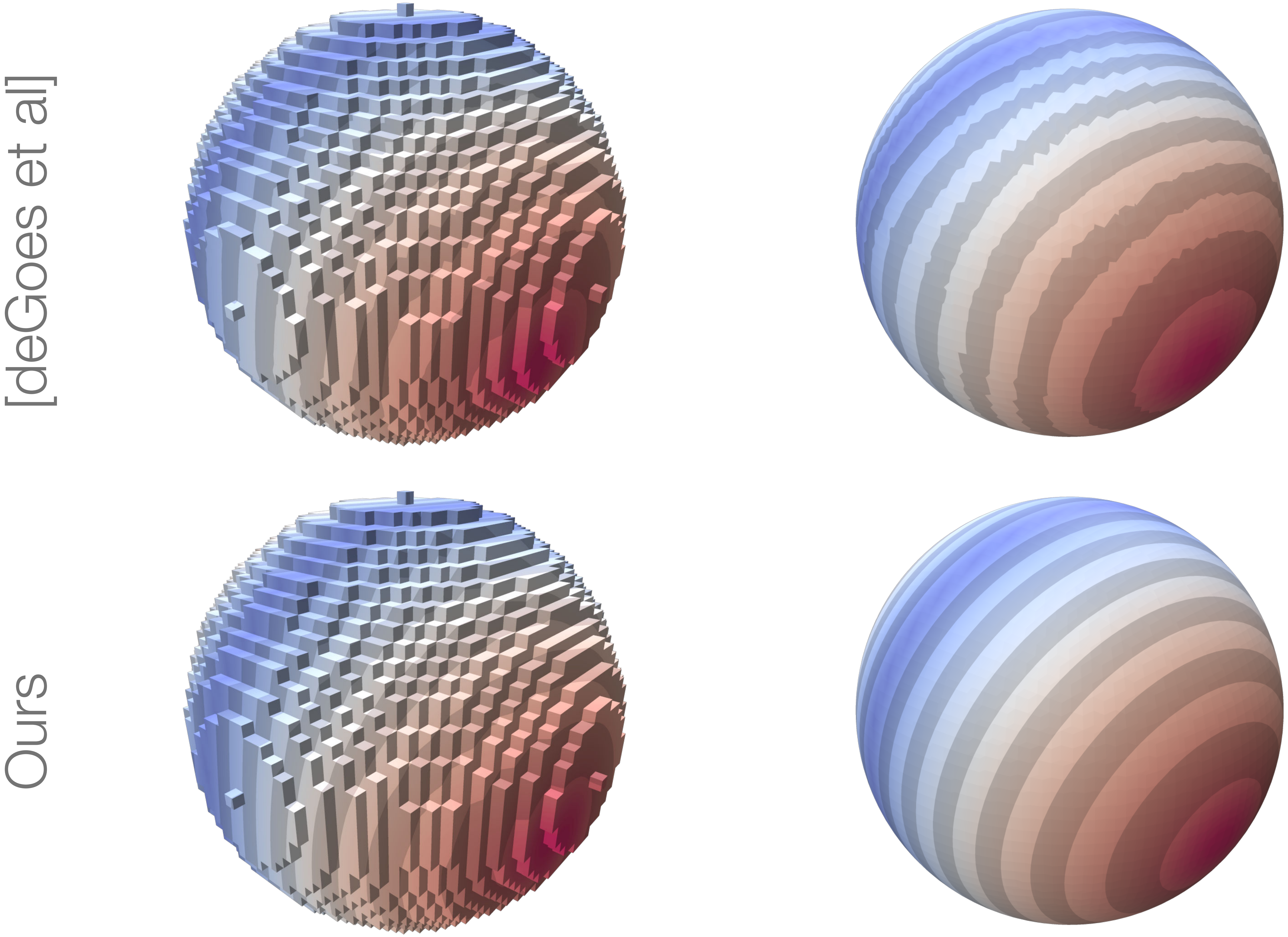
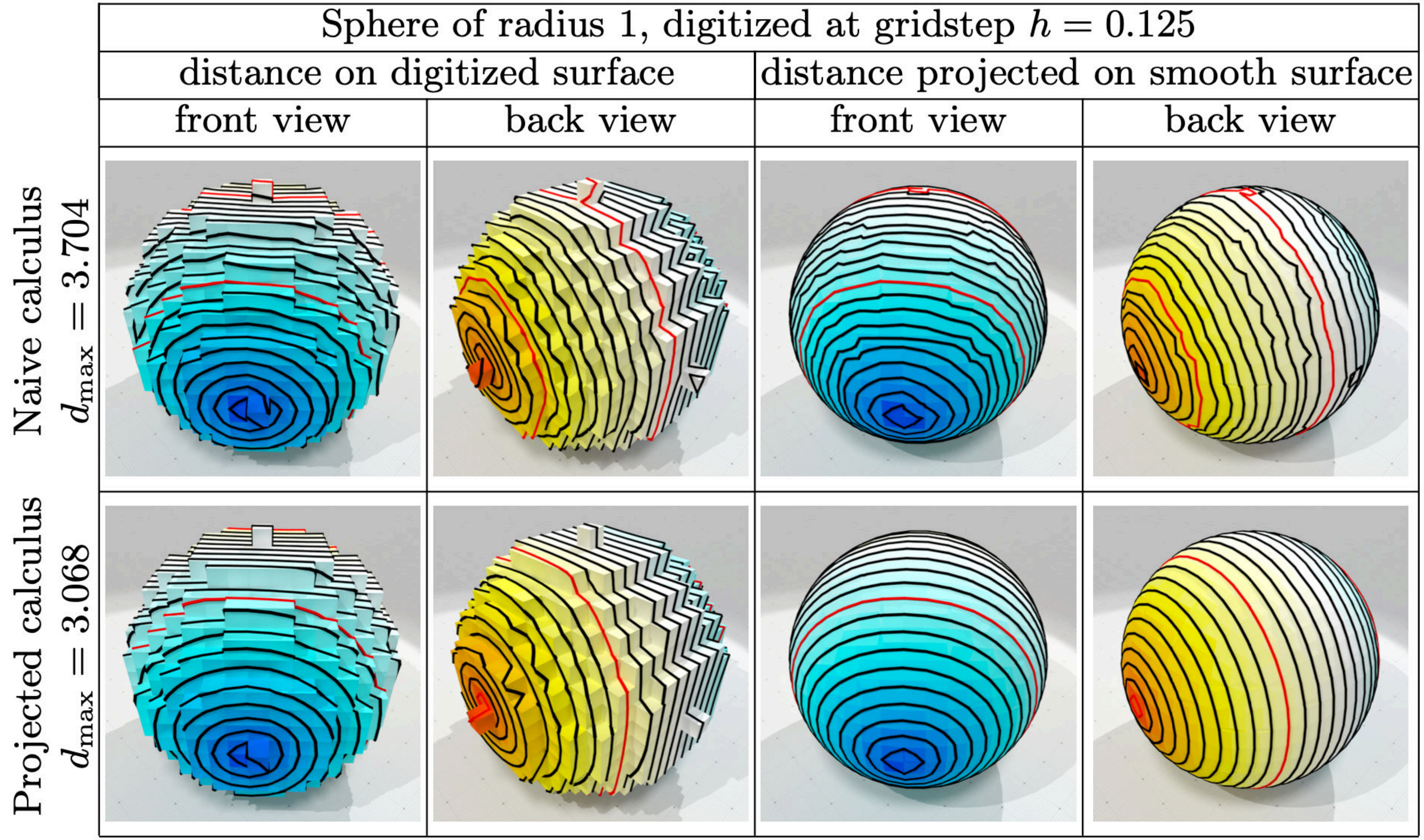


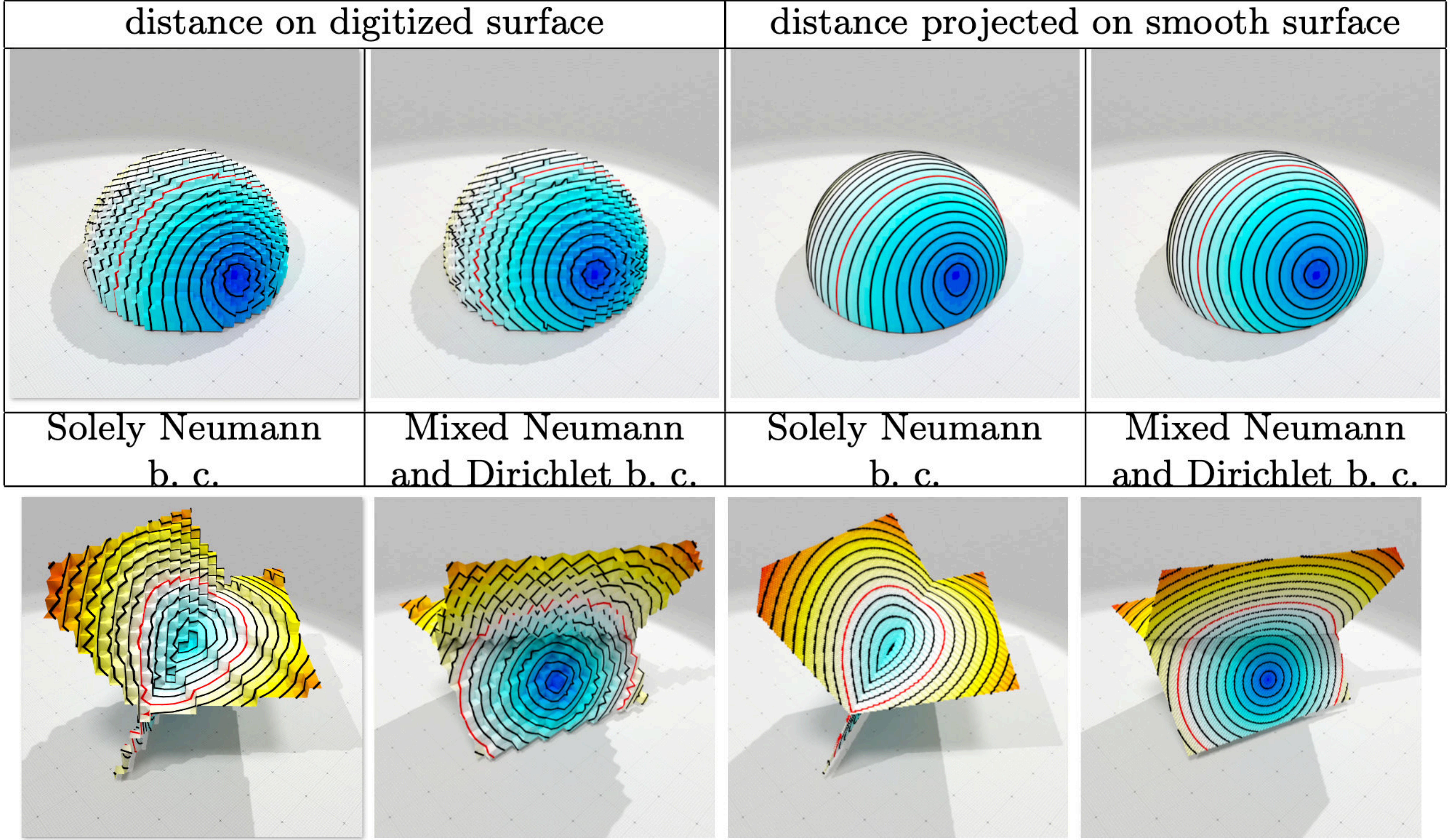
Experimental validation: stability of Laplace-Beltrami eigenvectors

Experimental validation: stability of Laplace-Beltrami eigenvectors



Experimental validation: Geodesics using the heat method




distance on digitized surface
distance projected on smooth surface
Solely Neumann b. c.
Mixed Neumann and Dirichlet b. c.
Solely Neumann b. c.
Mixed Neumann and Dirichlet b. c.

hands on…

```
void initQuantities()
{
  PolygonalCalculus<SH3::RealPoint,SH3::RealVector> calculus(surfmesh);

  std::vector<PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector> gradients;
  std::vector<PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector> cogradients;
  std::vector<PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Real3dVector> normals;
  std::vector<PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Real3dVector> vectorArea;
  std::vector<PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Real3dPoint> centroids;
  std::vector<double> faceArea;

  for(auto f=0; f < surfmesh.nbFaces(); ++f)
  {
    PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector ph = phiFace(f);
    PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector grad = calculus.gradient(f) * ph;
    gradients.push_back( grad );
    PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector cograd =  calculus.coGradient(f) * ph;
    cogradients.push_back( cograd );
    normals.push_back(calculus.faceNormalAsDGtalVector(f));

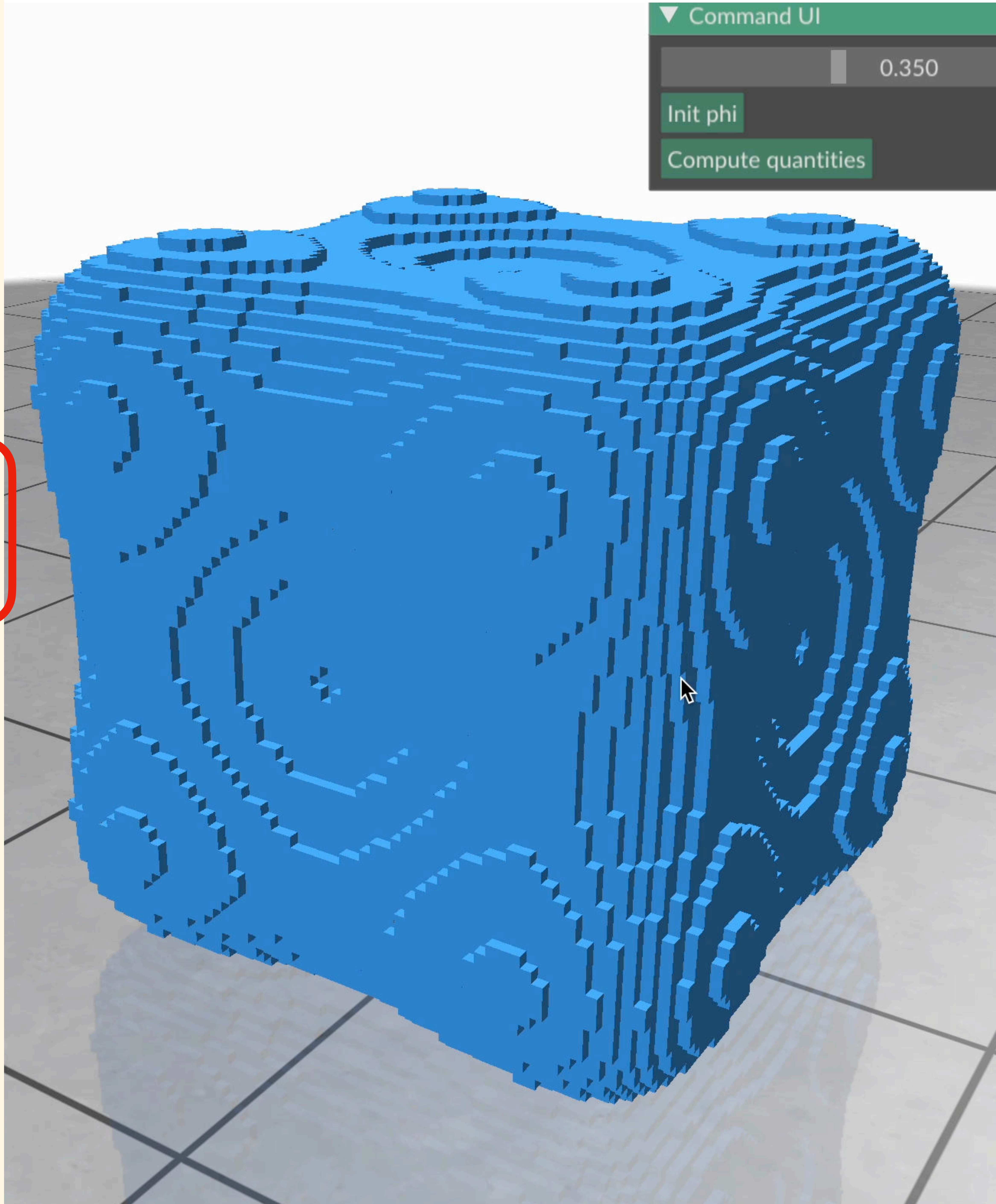
    auto vA = calculus.vectorArea(f);
    vectorArea.push_back({vA(0) , vA(1), vA(2)});

    faceArea.push_back( calculus.faceArea(f));

    centroids.push_back( calculus.centroidAsDGtalPoint(f) );
  }

  psMesh->addFaceVectorQuantity("Gradients", gradients);
  psMesh->addFaceVectorQuantity("co-Gradients", cogradients);
  psMesh->addFaceVectorQuantity("Normals", normals);
  psMesh->addFaceScalarQuantity("Face area", faceArea);
  psMesh->addFaceVectorQuantity("Vector area", vectorArea);

  polyscope::registerPointCloud("Centroids", centroids);
}
```



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    PolygonalCalculus<SH3::RealPoint,SH3::RealVector>::Vector grad = calculus.gradient(f) * ph;
    gradients.push_back( grad );
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    cogradients.push_back( cograd );
    normals.push_back(calculus.faceNormalAsDGtalVector(f));

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    vectorArea.push_back({vA(0) , vA(1), vA(2)});

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    centroids.push_back( calculus.centroidAsDGtalPoint(f) );
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  psMesh->addFaceVectorQuantity("Gradients", gradients);
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  psMesh->addFaceScalarQuantity("Face area", faceArea);
  psMesh->addFaceVectorQuantity("Vector area", vectorArea);

  polyscope::registerPointCloud("Centroids", centroids);
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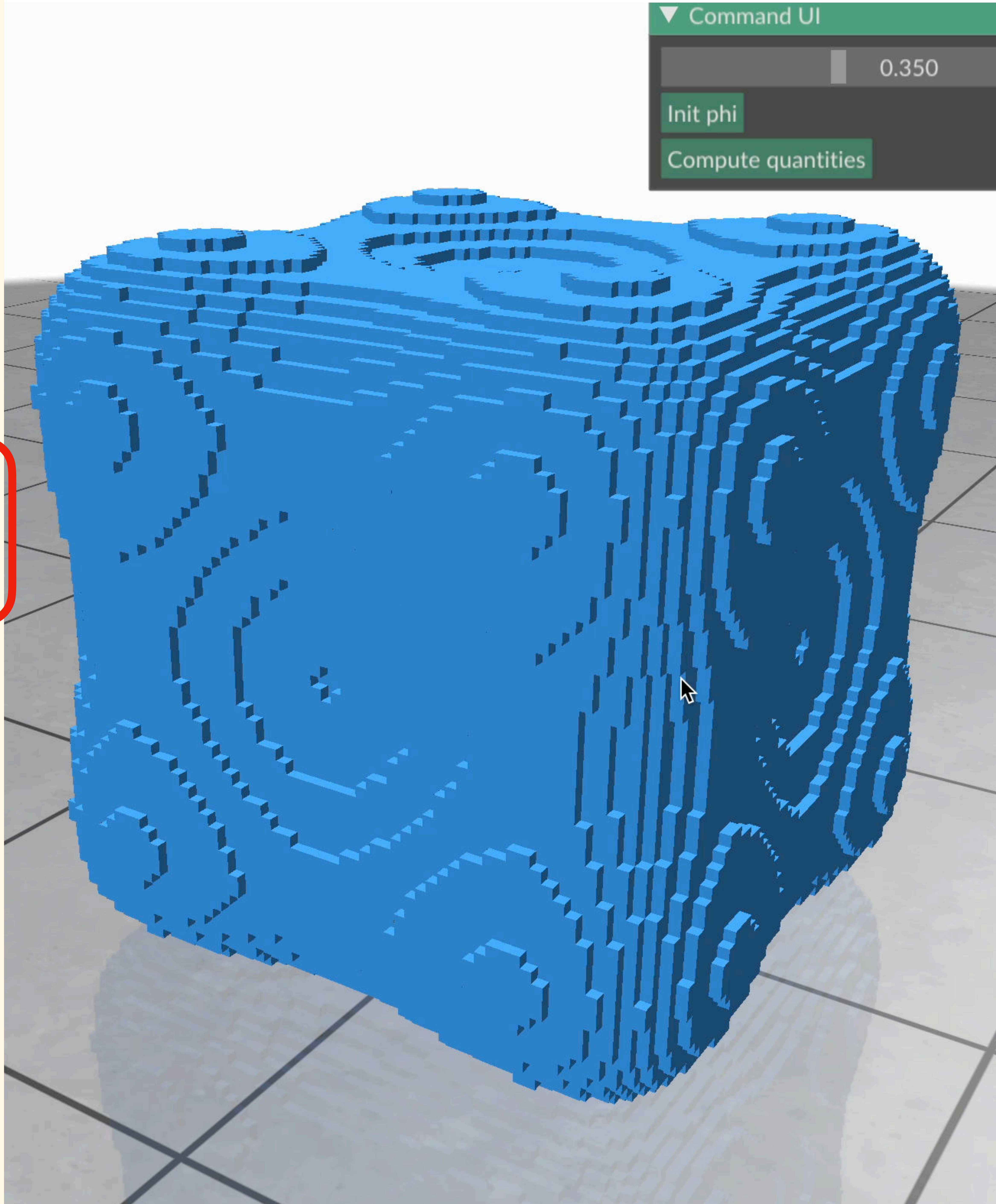
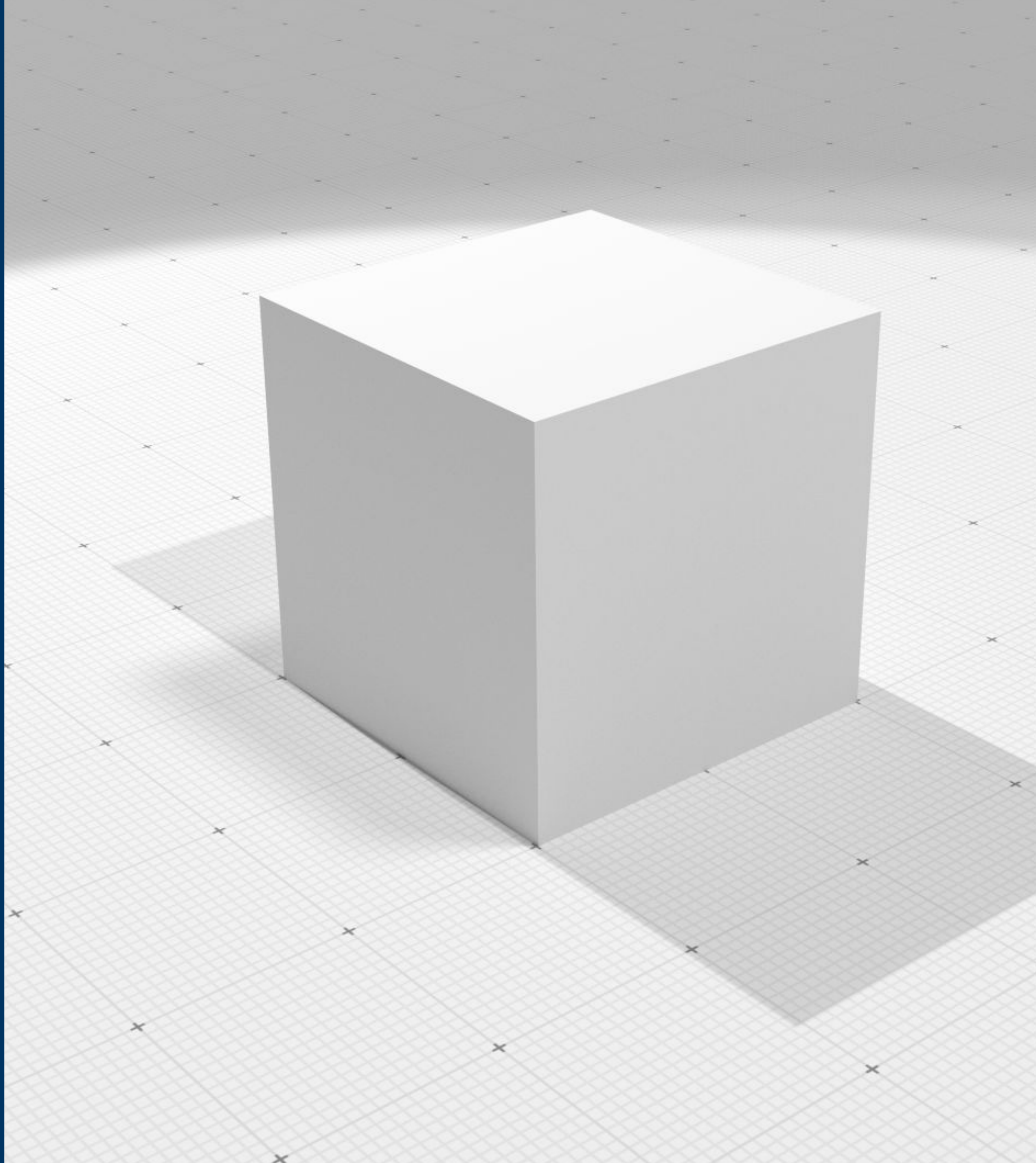
```



conclusion



Conclusion

Topology and geometry processing on regular data:

- fast algorithms thanks to the regularity of the data
- simple topological structure
- integer based computations
- advanced surface based geometry processing

… in $\mathbb{Z}^d$

dgtal.org

Discord

https://discord.gg/PtCk6vtS

Challenges

- **Foundation of Digital Geometry**
  - Objects (hyperplane, spheres..): arithmetical properties,
  - Digital convexity
  - Bijective transformations
  - Alternative pavings
- **Discrete <-> Continuous**
  - Digitization: stable properties (topology, geometric quantities…)
  - Unified model
  - Reconstruction (2d, 3d…)
- **Applications**
  - Material sciences
  - Image processing

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