Full convexity: new characterisations and applications

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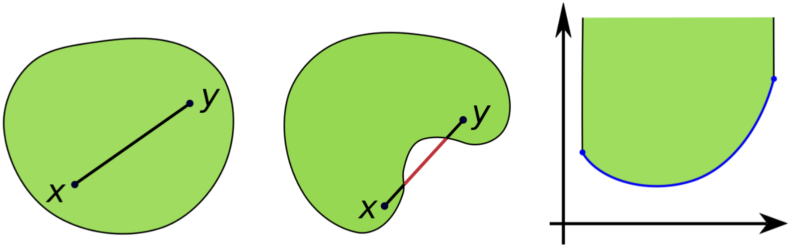
Politecnico di Milano

Convexity is a central tool in mathematics



convex set non convex set convex function

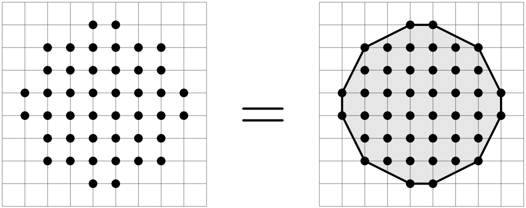
- convexity is a central tool in (continuous) mathematics
- study the geometry of shapes (not smooth everywhere)
- study the geometry of functions (not differentiable everywhere)
- allow convex analysis, convex optimization
- extensions to metric space, matrices, etc.

What about defining convexity in images, where data is discrete ?

Full convexity vs usual digital convexity

Definition (Usual digital convexity (or 0-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$



X = $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d$ $\Rightarrow$ convex

Full convexity vs usual digital convexity

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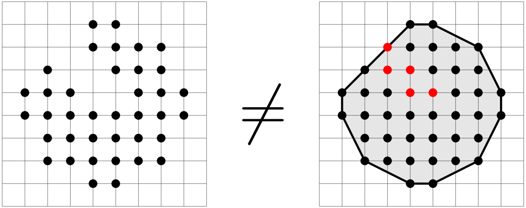

X $\neq$ $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d$

$\Rightarrow$ not convex

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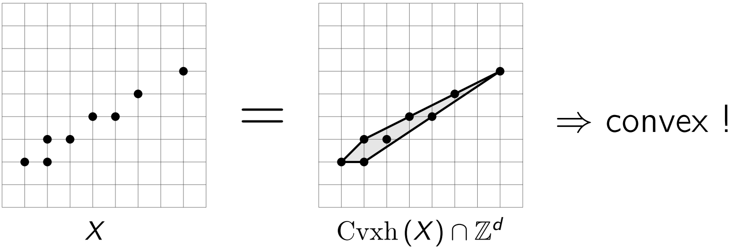

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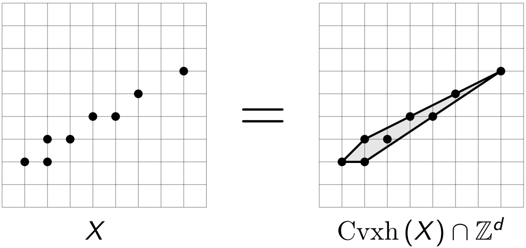
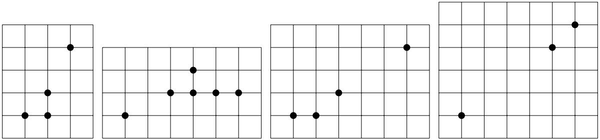


$\Rightarrow$ convex !

Full convexity is a specialization of digital convexity that guarantees (simple) connectedness in **arbitrary dimension**



digitally convex sets that are not fully convex

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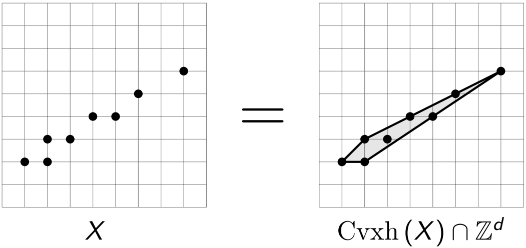

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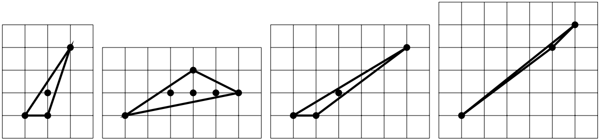

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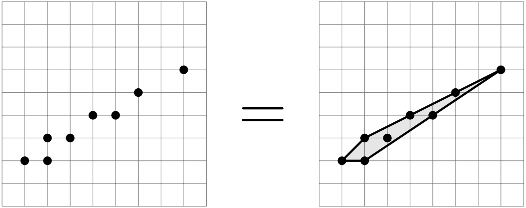

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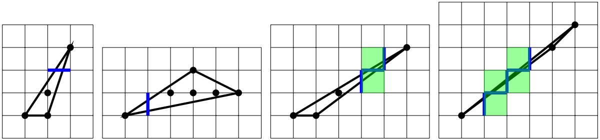
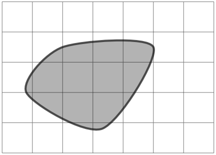


digitally convex sets that are not fully convex

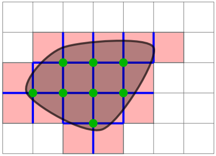
Cubical grid, intersection complex

- cubical grid complex $\mathscr{C}^d$
  - $\mathscr{C}_0^d$ vertices or 0-cells $= \mathbb{Z}^d$
  - $\mathscr{C}_1^d$ edges or 1-cells $=$ open unit segment joining 0-cells
  - $\mathscr{C}_2^d$ faces or 2-cells $=$ open unit square joining 1-cells
  - ...
- *intersection complex* of $Y \subset \mathbb{R}^d$

$$\bar{\mathscr{C}}_k^d[Y] := \{c \in \mathscr{C}_k^d, \bar{c} \cap Y \neq \emptyset\}$$



Y



cells $\bar{\mathscr{C}}_0^d[Y]$, $\bar{\mathscr{C}}_1^d[Y]$, $\bar{\mathscr{C}}_2^d[Y]$

Full convexity

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathscr{C}}_k^d[X] = \bar{\mathscr{C}}_k^d[\mathrm{Cvxh}\,(X)]. \quad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

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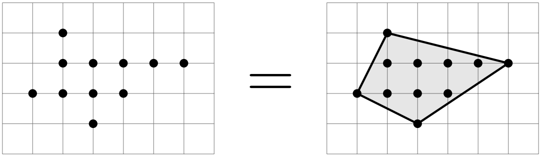

$\bar{\mathscr{C}}_0^d[X]$ $\qquad$ $\bar{\mathscr{C}}_0^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex

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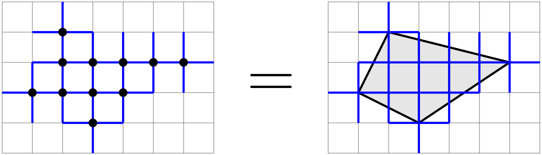

$\bar{\mathscr{C}}_1^d[X] = \bar{\mathscr{C}}_1^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex

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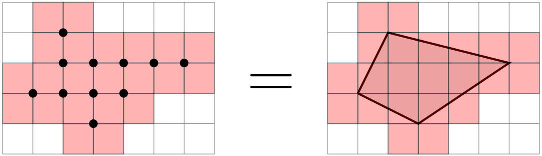
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathscr{C}}_2^d[X]$ $\qquad = \qquad$ $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex, and 2-convex, hence fully convex.

Full convexity

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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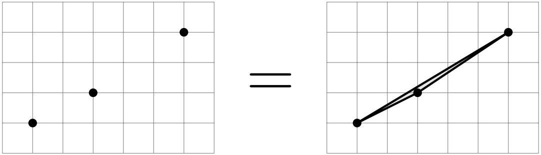
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathscr{C}}_0^d[X]$ $\quad$ $\bar{\mathscr{C}}_0^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex

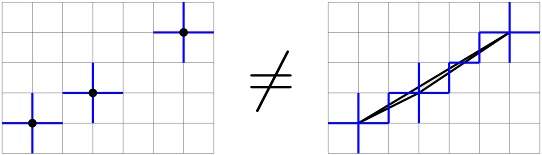
Full convexity

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{Cvxh}\,(X)]. \qquad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathscr{C}}^d_1[X] \neq \bar{\mathscr{C}}^d_1[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex

Full convexity

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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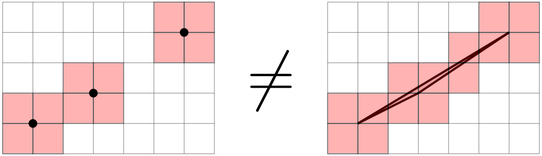

$\bar{\mathscr{C}}_2^d[X]$ $\neq$ $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex, nor 2-convex.

Full convexity

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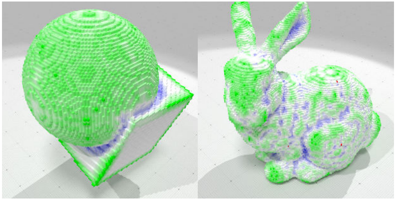
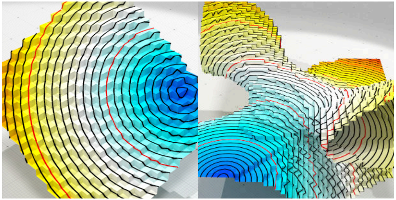
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

- full convexity eliminates too thin digital convex sets in arbitrary dimension
- fully convex sets are (simply) digitally connected
- digital lines and planes are fully convex
- connectedness allows *local geometric analysis* of digital shapes

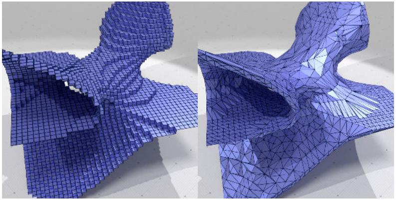
Applications of full convexity to digital shape analysis



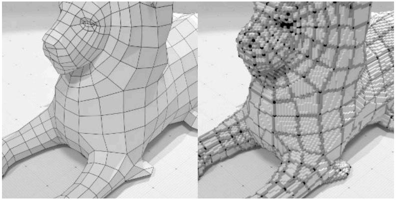
exact local shape analysis
(convex, concave, planar (white))



geodesics
(Euclidean distance in digital planes)



polyhedrization
(close and reversible)



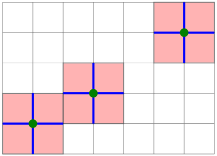
digital polyhedron
(cells are fully convex)

Full convexity: new characterizations and applications

Star of a set

For any $Y \subset \mathbb{R}^d$,

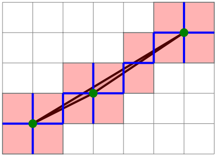
$$\mathrm{Star}\,(Y) = \{c \in \mathscr{C}^d, \bar{c} \cap Y\} = \bigcup_{0 \leqslant k \leqslant d} \bar{\mathscr{C}}^d_k[Y]$$



$X \subset \mathbb{Z}^d$, $\mathrm{Star}\,(X)$

Full convexity

$= ?$

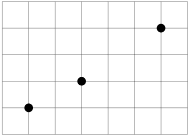


$\mathrm{Star}\,(\mathrm{Cvxh}\,(X))$

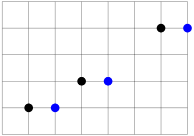
Computable characterization of full convexity

Discrete Minkowski sum U_α

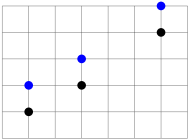
- let $X \subset \mathbb{Z}^d$, denote $\mathrm{e}_i(X)$ the translation of X with axis vector e_i
- let $I^d := \{1, \dots, d\}$ be the set of possible directions
- let $U_\emptyset(X) := X$, and, for $\alpha \subset I^d$ and $i \in \alpha$, recursively $U_\alpha(X) := U_{\alpha \setminus i}(X) \cup \mathrm{e}_i(U_{\alpha \setminus i}(X))$.



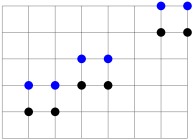
$U_\emptyset(X) = X$



$U_{\{1\}}(X) = U_\emptyset(X) \cup \mathrm{e}_1(U_\emptyset(X))$



$U_{\{2\}}(X) = U_\emptyset(X) \cup \mathrm{e}_2(U_\emptyset(X))$



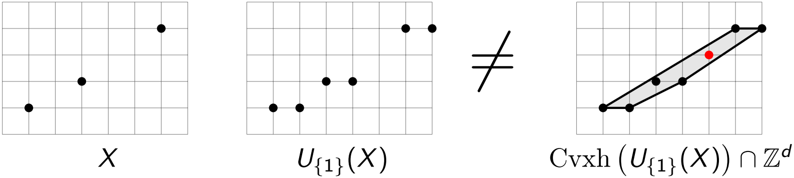
$U_{\{1,2\}}(X) = U_{\{1\}}(X) \cup \mathrm{e}_2(U_{\{1\}}(X))$

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{Cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{2}$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.



Computable characterization of full convexity

A morphological characterization

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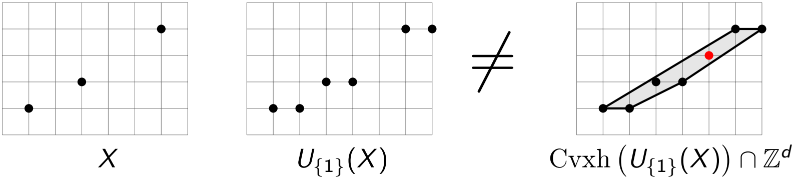

Algorithm:
$\forall k, 0 \leqslant k \leqslant d$,
$\forall \alpha \in I_k^d$

- compute $U_\alpha(X)$
- compute $\mathrm{Cvxh}\,(U_\alpha(X))$ and enumerate lattice points within

$= ?$

Computable characterization of full convexity

A morphological characterization

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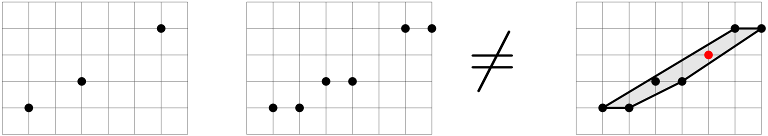


2^d convex hull computations and enumerations $\mathbb{Z}^d$

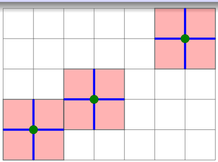
Algorithm:
$\forall k, 0 \leqslant k \leqslant d$,
$\forall \alpha \in I_k^d$

- compute $U_\alpha(X)$
- compute $\mathrm{Cvxh}\left(U_\alpha(X)\right)$ and enumerate lattice points within

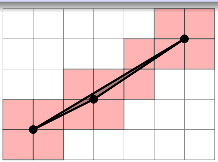
$= ?$

One convex hull computation is enough (2D illustration)

Step 1: compute $\forall \alpha, \alpha \subset \{1,2\}, U_\alpha(X)$; compute $\mathrm{Cvxh}\left(U_{\{1,2\}}(X)\right) \cap \mathbb{Z}^2$



$X = \bar{\mathscr{C}}_0^d[X]$, $\bar{\mathscr{C}}_1^d[X]$, $\bar{\mathscr{C}}_2^d[X]$



$\mathrm{Cvxh}(X)$, $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}(X)]$

1-1 map $\updownarrow$

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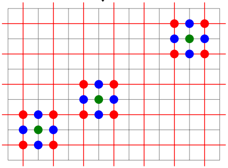
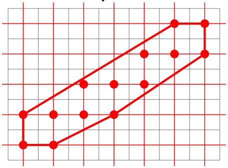


$U_\emptyset(X) + (\frac{1}{2}, \frac{1}{2})$, $U_{\{1\}}(X) + (0, \frac{1}{2})$

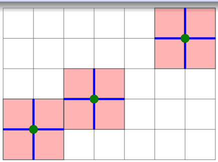
$U_{\{2\}}(X) + (\frac{1}{2}, 0)$, $U_{\{1,2\}}(X)$



$\mathrm{Cvxh}\left(U_{\{1,2\}}(X)\right) \cap \mathbb{Z}^2$

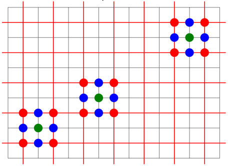
One convex hull computation is enough (2D illustration)

Step 2: compute intermediate points between two red points



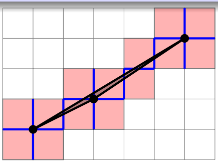
$X = \bar{\mathscr{C}}_0^d[X]$, $\bar{\mathscr{C}}_1^d[X]$, $\bar{\mathscr{C}}_2^d[X]$

1-1 map ↕



$U_\emptyset(X) + (\frac{1}{2}, \frac{1}{2})$, $U_{\{1\}}(X) + (0, \frac{1}{2})$

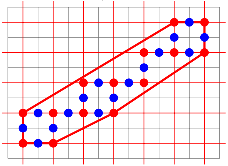
$U_{\{2\}}(X) + (\frac{1}{2}, 0)$, $U_{\{1,2\}}(X)$



$\mathrm{Cvxh}\,(X)$, $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}\,(X)]$

$\bar{\mathscr{C}}_1^d[\mathrm{Cvxh}\,(X)]$

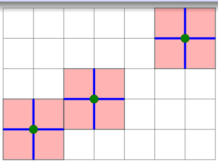
1-1 map ↕



$\mathrm{Cvxh}\,(U_{\{1,2\}}(X)) \cap \mathbb{Z}^2$

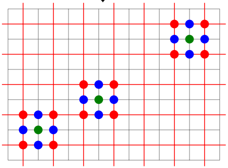
One convex hull computation is enough (2D illustration)

Step 3: compute intermediate points between four red points ...



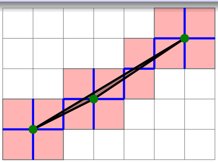
$X = \bar{\mathscr{C}}_0^d[X]$, $\bar{\mathscr{C}}_1^d[X]$, $\bar{\mathscr{C}}_2^d[X]$

1-1 map $\updownarrow$



$U_\emptyset(X) + (\frac{1}{2}, \frac{1}{2})$, $U_{\{1\}}(X) + (0, \frac{1}{2})$

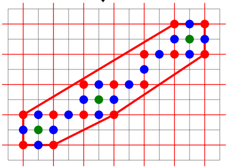
$U_{\{2\}}(X) + (\frac{1}{2}, 0)$, $U_{\{1,2\}}(X)$



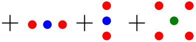
$\mathrm{Cvxh}\,(X)$, $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}\,(X)]$

$\bar{\mathscr{C}}_1^d[\mathrm{Cvxh}\,(X)]$, $\bar{\mathscr{C}}_0^d[\mathrm{Cvxh}\,(X)]$

1-1 map $\updownarrow$

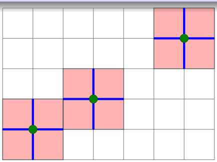


$\mathrm{Cvxh}\,\left(U_{\{1,2\}}(X)\right) \cap \mathbb{Z}^2$



One convex hull computation is enough (2D illustration)

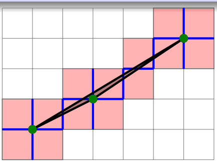
Step 4: check full convexity by counting points •, •, •.



$X = \bar{\mathscr{C}}_0^d[X]$, $\bar{\mathscr{C}}_1^d[X]$, $\bar{\mathscr{C}}_2^d[X]$

Full convexity

$= ?$



$\mathrm{Cvxh}\,(X)$, $\bar{\mathscr{C}}_2^d[\mathrm{Cvxh}\,(X)]$
$\bar{\mathscr{C}}_1^d[\mathrm{Cvxh}\,(X)]$, $\bar{\mathscr{C}}_0^d[\mathrm{Cvxh}\,(X)]$

1-1 map ↕

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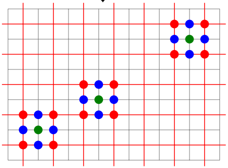

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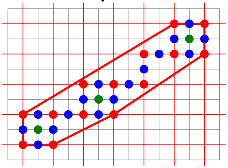

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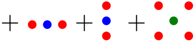
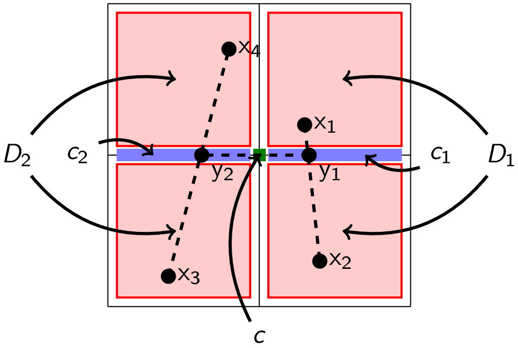


Main argument of the proof



Lemma

Let c be a k-cell of $\mathscr{C}^d$ and let $D = (\sigma_1, \ldots, \sigma_n)$ be the d-dimensional cells surrounding c (i.e., $\mathrm{Star}\,(c) \cap \mathscr{C}^d_d = D$), with $n = 2^{d-k}$. Picking one point x_i in each $\bar{\sigma}_i$, then it holds that there exists a point of $\bar{c}$ that belongs to $\mathrm{Cvxh}\,(\{x_i\}_{i=1,\ldots,n})$.

Looking for other characterizations of full convexity

1. characterization through “natural” segment convexity
2. characterization through projections

"Natural" segment convexity

Convexity in $\mathbb{R}^d$ $X \subset \mathbb{R}^d$ is convex iff
$\forall p, q \in X$, then $[pq]$ is a subset of X

"Natural" segment convexity

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[Minsky, Papert 88]

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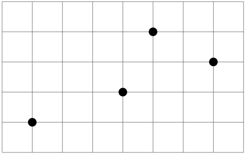
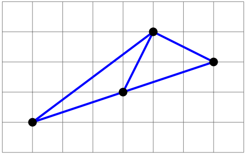


MP-convex !



Each blue segment does not touch any other lattice point

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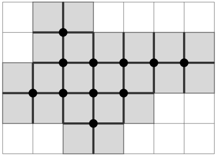
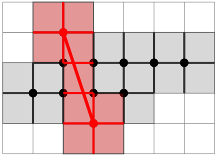
Segment convexity in $\mathbb{Z}^d$ $X \subset \mathbb{Z}^d$ is *segment convex* iff
$\forall p, q \in X$, then $\mathrm{Star}([pq])$ is a subset of $\mathrm{Star}(X)$



X segment convex



$\mathrm{Star}([pq]) \subset \mathrm{Star}(X)$

"Natural" segment convexity

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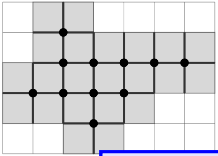

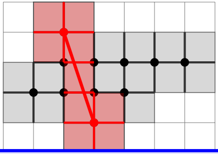

X segme

Full convexity $\Rightarrow$ Segment convexity
Full convexity $\Leftarrow$ Segment convexity ?

Projection convexity

Let $\mathscr{P}_j$ be the orthogonal projector associated to the j-th axis.

Lemma

If $X \subset \mathbb{Z}^d$ is fully convex, then $\forall j, 1 \leqslant j \leqslant d$, $\mathscr{P}_j(X)$ is fully convex (in $\mathbb{Z}^{d-1}$).

Definition (Projection convexity)

$X \subset \mathbb{Z}^d$ is P-convex iff:

(i) X is 0-convex,

(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\mathscr{P}_j(X)$ is P-convex.

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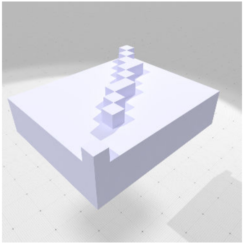

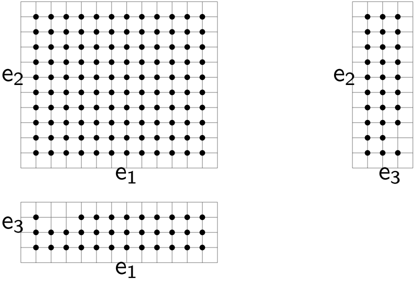

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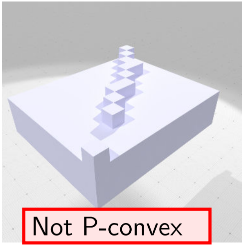

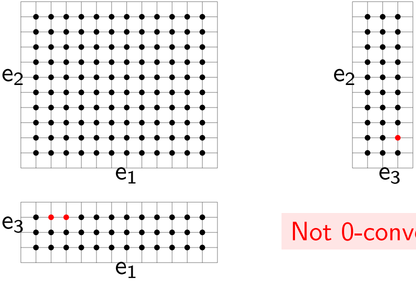

Not 0-convex

Projection convexity

Let $\mathscr{P}_j$ be the orthogonal projector associated to the j-th axis.

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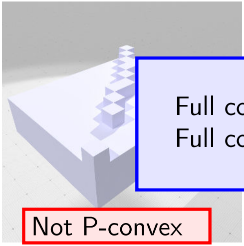
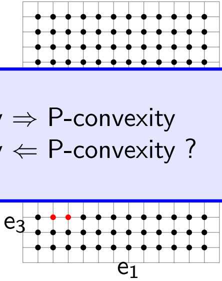
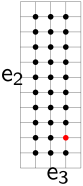




Full convexity $\Rightarrow$ P-convexity

Full convexity $\Leftarrow$ P-convexity ?



Not 0-convex

Full convexity: new characterizations and applications

Fully convex hulls ?

Let $X \subset \mathbb{Z}^d$. We wish to build a set $Z \subset \mathbb{Z}^d$ such that

- $X \subset Z$
- Z is fully convex
- Z is "close" geometrically to X

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1. fully convex enveloppe $\mathrm{FC}^*(X)$

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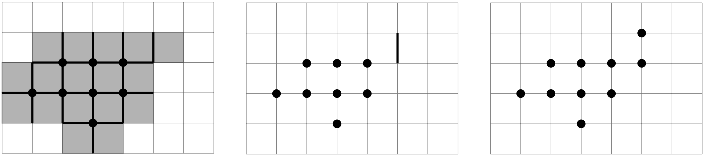
- Z is "close" geometrically to X

1. fully convex enveloppe $\mathrm{FC}^*(X)$
2. use Minkowski sums

Fully convex enveloppe $\mathrm{FC}^*(X)$

Local operators $\mathrm{Skel}(\cdot)$, $\mathrm{Extr}(\cdot)$

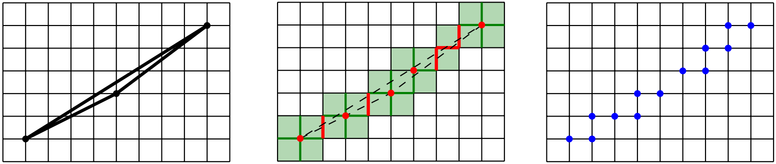


K — $K' = \mathrm{Skel}(K)$ (skeleton) — $\mathrm{Extr}(K')$ (extrema)

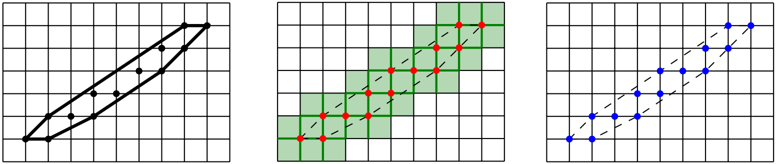
- for any complex $K \subset \mathscr{C}^d$, let $\mathrm{Skel}(K) := \bigcap_{K' \subset K \subset \mathrm{Star}(K')} K'$
- for any complex $K \subset \mathscr{C}^d$, let $\mathrm{Extr}(K) := \mathrm{Cl}(K) \cap \mathbb{Z}^d$

Fully convex enveloppe $\mathrm{FC}^*(X)$

- Iterative method for computing a fully convex enveloppe
- Let $\mathrm{FC}(X) := \mathrm{Extr}\left(\mathrm{Skel}\left(\mathrm{Star}\left(\mathrm{Cvxh}\left(X\right)\right)\right)\right)$
- Iterative composition $\mathrm{FC}^n(X) := \underbrace{\mathrm{FC} \circ \cdots \circ \mathrm{FC}}_{n \text{ times}}(X)$
- *Fully convex envelope* of X is $\mathrm{FC}^*(X) := \lim_{n \to \infty} \mathrm{FC}^n(X)$.



input X, $Y := \mathrm{Cvxh}\,(X)$ | $\mathrm{Star}\,(Y)$, $\mathrm{Skel}\,(\mathrm{Star}\,(Y))$ | $X' = \mathrm{FC}(X)$



input X', $Y' := \mathrm{Cvxh}\,(X')$ | $\mathrm{Star}\,(Y')$, $\mathrm{Skel}\,(\mathrm{Star}\,(Y'))$ | $X'' = \mathrm{FC}(X') = \mathrm{FC}^2(X)$

Fully convex enveloppe $\mathrm{FC}^*(X)$

Theorem

$X \subset \mathbb{Z}^d$ is fully convex if and only if $X = FC(X)$.

Theorem

For any finite $X \subset \mathbb{Z}^d$, $\mathrm{FC}^*(X)$ is fully convex.

Fully convex sets from Minkowski sums

- $H^+ := [0,1]^d$ (closed unit hypercube of positive orthant)
- $H := [-1,1]^d$ (closed hypercube of edge length 2)

Lemma

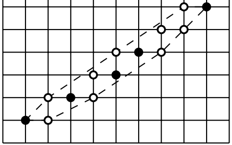
Let A and B be real closed convex sets, with $H^+ \subset B$, then $(A \oplus B) \cap \mathbb{Z}^d$ is a fully convex set.

Theorem

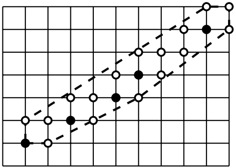
Let $X \subset \mathbb{Z}^d$, then

1. $(\mathrm{Cvxh}\,(X) \oplus H^+) \cap \mathbb{Z}^d$ *is fully convex,*
2. $\mathrm{Extr}\,(\mathrm{Star}\,(\mathrm{Cvxh}\,(X)))$ *is fully convex.*

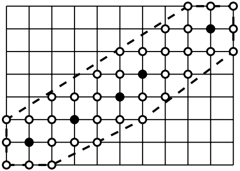
Comparison between hull operators



$\mathrm{FC}^*(X)$



$(\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$



$\mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$

operator	$\mathrm{FC}^*(X)$	$(\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$	$\mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$
Id. on fully cvx.	yes	no	no
idempotence	yes	no	no
symmetry	yes	no	yes
$\#(Out)/\#(In)$	low	medium	high
efficiency	iterative	yes	yes

Full convexity: new characterizations and applications

What is full convexity ?

Characterizations of full convexity

Fully convex hulls

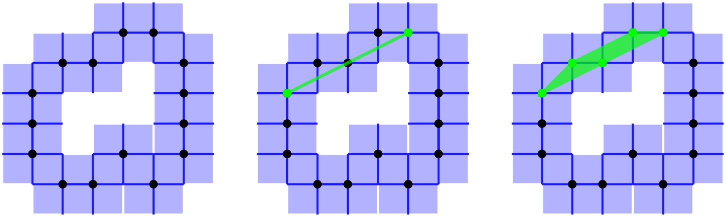
Polyhedrization

Conclusion

Full subconvexity / tangency

Definition

The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *fully subconvex to X* whenever $\mathrm{Star}\,(\mathrm{Cvxh}\,(A)) \subset \mathrm{Star}\,(X)$.



X and $\bar{\mathscr{C}}^d[X]$ | fully subconvex A | fully subconvex A

Full subconvexity / tangency

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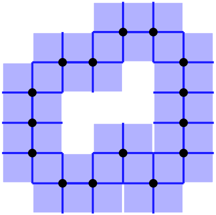
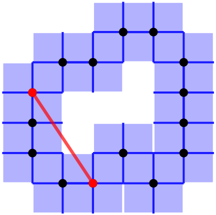
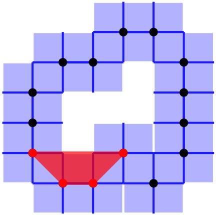
The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *fully subconvex to* X whenever $\mathrm{Star}\,(\mathrm{Cvxh}\,(A)) \subset \mathrm{Star}\,(X)$.



X and $\bar{\mathscr{C}}^d[X]$

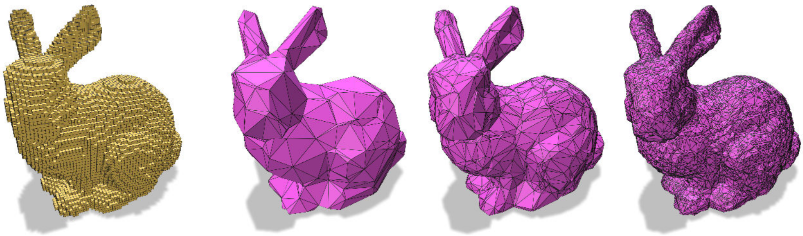


not fully subconvex
A



not fully subconvex
A

Build a polyhedral model from a digital set



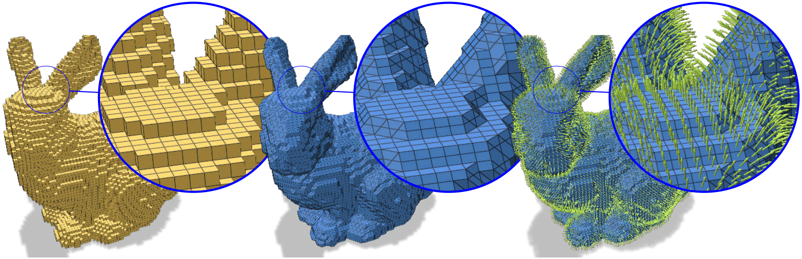
- **Input**: digital set $X \subset \mathbb{Z}^d$, its digital boundary $B := \partial X$
- **Output**: a polyhedral surface P approaching ∂X
- ideally, edges and faces of P should be fully subconvex to ∂X, i.e.

$$\forall \text{edge}(p,q) \in P, \mathrm{Star}\left(\mathrm{Cvxh}\left(\{p,q\}\right)\right) \subset \mathrm{Star}\left(\partial X\right)$$
$$\forall \text{face}(p,q,r) \in P, \mathrm{Star}\left(\mathrm{Cvxh}\left(\{p,q,r\}\right)\right) \subset \mathrm{Star}\left(\partial X\right)$$

- faces of P should align with pieces of digital planes of ∂X

Mixed variational and digital method

Initialization

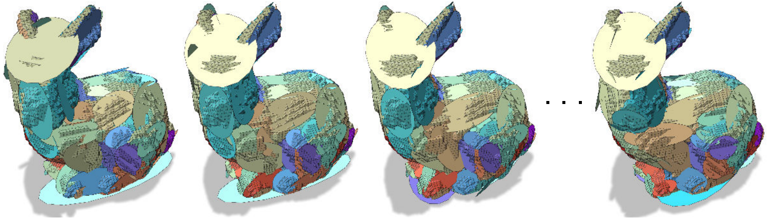


points in $\mathbb{Z}^3$ vertices in $\frac{1}{2}\mathbb{Z}^3$

1. compute dual surface S to digital surface ∂X
   $\Rightarrow$ a combinatorial 2-manifold
2. estimate normal vector field u to X using for instance integral invariant normal estimator

Mixed variational and digital method

Progressive proxy fitting, similar to "Variational shape approximation" [Alliez et al. 2004]



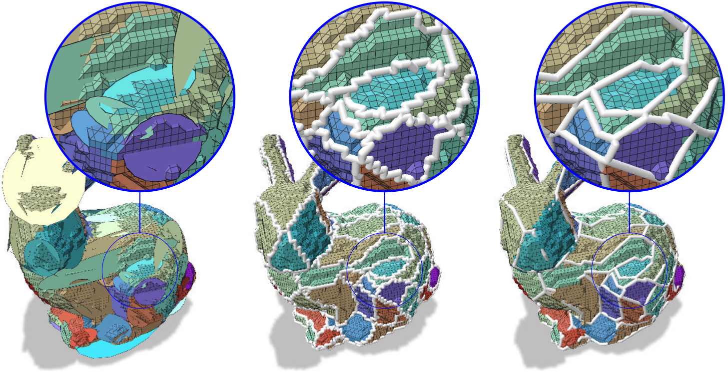
1. Proxies: choose K initial facets among N facets randomly, $i_1, \ldots, i_k$

$$E(\mathrm{label}, i_1, \ldots, i_k) := \sum_{k=1}^{K} \sum_{\substack{i=1 \\ \mathrm{label}(i)=k}}^{N} \mathrm{Area}(f_i) \|\mathbf{u}_i - \mathbf{u}_{i_k}\|^2$$

2. Label the $N - K$ remaining facets to one proxy by progressive aggregation to minimize E (with $i_1, \ldots, i_k$ fixed).
3. For each proxy k, determine the new best representant i_k to minimize E (label is fixed).
4. Loop back to 2 as long as E decreases

Mixed variational and digital method

Split region boundaries into tangent paths

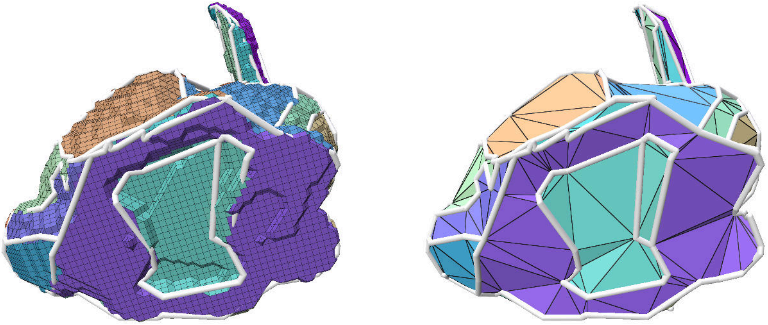


- boundaries between regions i and j are polylines with vertex set $P_{i,j}$ in $\frac{1}{2}\mathbb{Z}^3$
- $D_{i,j} := \mathrm{Extr}\left(\mathrm{Star}\left(P_{i,j}\right)\right)$ defines the constraint domain in $\frac{1}{2}\mathbb{Z}^3$
- simplified boundaries $B_{i,j}$ are polylines in $\frac{1}{2}\mathbb{Z}^3$ that are fully subconvex to the constraint domain, i.e. for each segment S of $B_{i,j}$:

$$\mathrm{Star}\left(\mathrm{Cvxh}\left(S\right)\right) \subset \mathrm{Star}\left(D_{i,j}\right) \subset \mathrm{Star}\left(\partial X\right)$$

Mixed variational and digital method

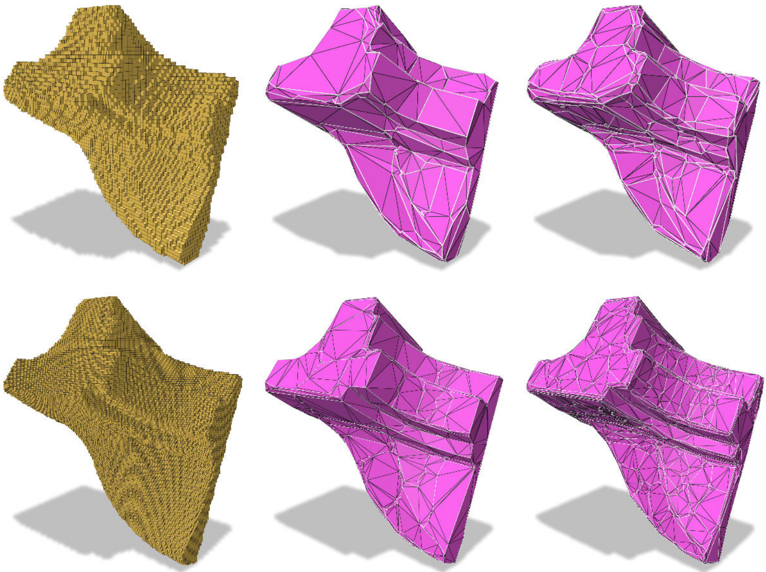
Triangulate regions with constrained Delaunay triangulation



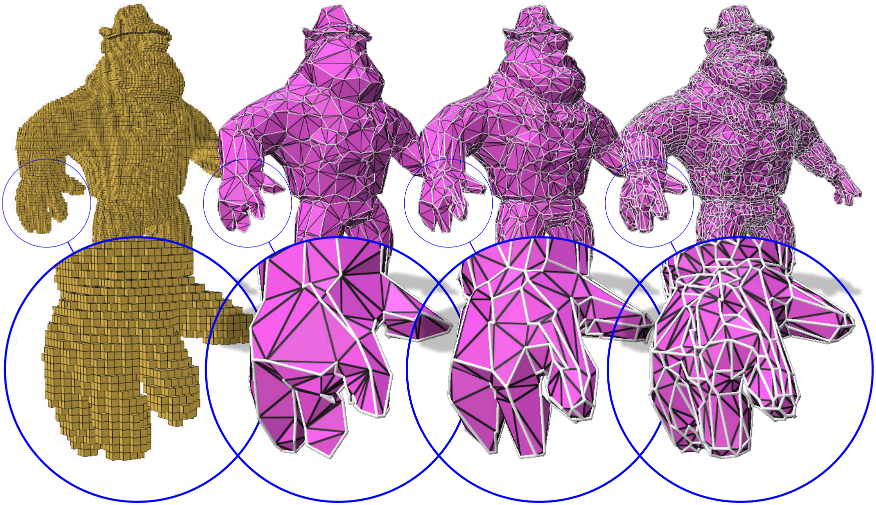
For each region i:

- vertices of simplified boundaries $B_{i,j}$ are projected onto proxy plane
- projected points triangulated using Delaunay triangulation, constrained with the projected edges of $B_{i,j}$
- triangles are projected back in 3D to get final triangulation

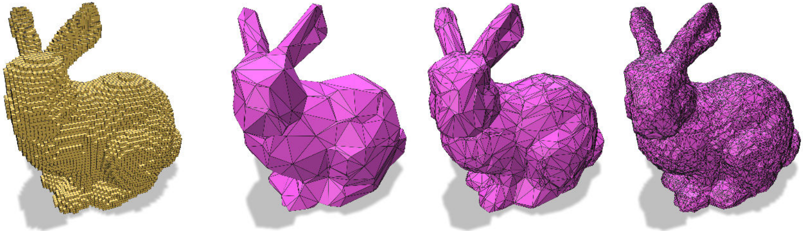
Some results (computation time 1-5s)



Some results (computation time 1-3s)



Build a polyhedral model from a digital set

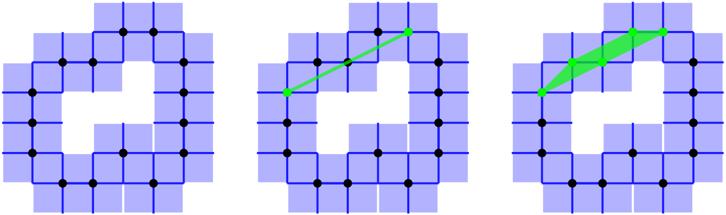


- **Input**: digital set $Z \subset \mathbb{Z}^d$, its digital boundary $X := \partial Z$
- **Output**: a polyhedral surface P approaching X
- edges and faces of P should be "close" to X

Full subconvexity / tangency

Definition

The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *fully subconvex to X* whenever $\mathrm{Star}\,(\mathrm{Cvxh}\,(A)) \subset \mathrm{Star}\,(X)$.



X and $\bar{\mathscr{C}}^d[X]$ — fully subconvex A — fully subconvex A

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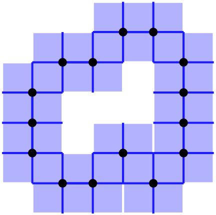

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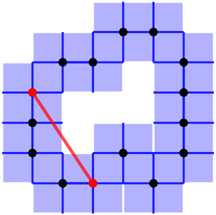
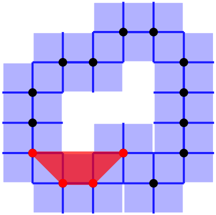


not fully subconvex

A



not fully subconvex

A

Formalization of polyhedrization problem

- a k-**simplex** is a $(k+1)$-tuple of lattice points, called its *vertices*. Its *faces* are exactly its non-empty proper subsets.
- a **polyhedron** P is a collection of k-simplices (σ_i^k), $0 \leqslant k \leqslant d-1$, such that any simplex $\sigma \in P$ must have its faces also in P.
- the **body** of P is $\|P\| := \cup_{\sigma \in P} \mathrm{Cvxh}\,(\sigma)$.

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Input: digital boundary $X \subset \mathbb{Z}^d$
Output: a polyhedron P such that:
(P covers X) $X \subset \mathrm{Extr}\,(\mathrm{Star}\,(\|P\|))$
(P fully subconvex to X) $\mathrm{Extr}\,(\mathrm{Star}\,(\|P\|)) \subset \mathrm{Extr}\,(\mathrm{Star}\,(X))$
(Geometric opt.) P minimizes its area, its number of faces, etc.

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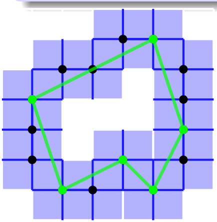
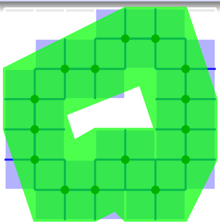




Theorem
$\|P\|$ and X are Hausdorff close by 1, i.e. $d_\infty^H(\|P\|, X) \leq 1$.

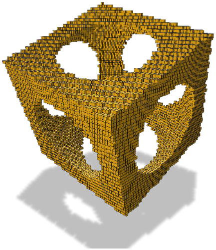
Simple greedy algorithm in 3D

- initial polyhedron P : triangulated digital surface X
- Let $L(i) = i$ be the initial labeling of vertices $X = (\mathrm{x}_i)$
- **foreach** initial edge (i, j) of P taken in random number
    1. **if** $L(i) = L(j)$ **then** **continue**
    2. $m_1 \leftarrow \text{mergeScore}(L(i), L(j))$
    3. $m_2 \leftarrow \text{mergeScore}(L(j), L(i))$
    4. **if** $\min(m_1, m_2) = +\infty$ **then** **continue**
    5. **if** $m_1 < m_2$ **then** merge $L(j) \leftarrow L(i)$
    6. **else** merge $L(i) \leftarrow L(j)$

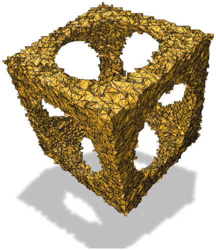
mergeScore(k, l) test the edge merge (k, l) by identifying vertex l to vertex k. Returns either $+\infty$ if the new faces are not fully subconvex or covering, or returns the difference of area induced by the merge.

Invariant After each merge, P still covers X and P is still fully subconvex to X.

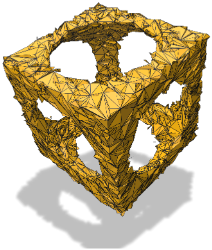
Simple greedy algorithm in 3D



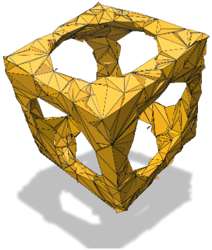
20924 quads



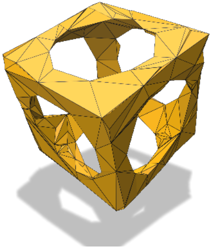
22028 triangles



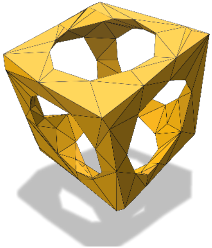
8070 triangles



1886 triangles



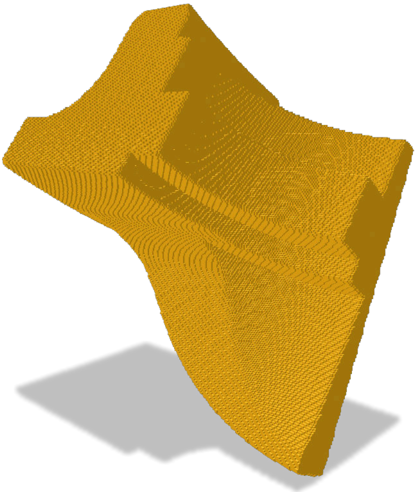
460 triangles



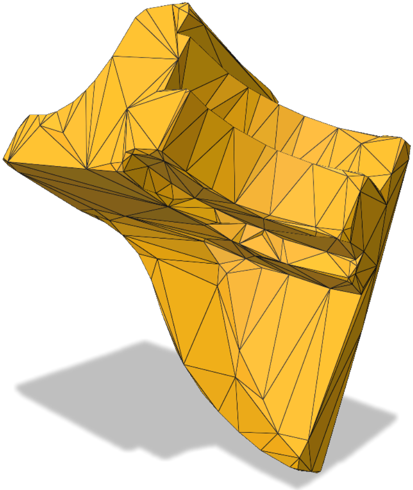
250 triangles

Computation time is 28s, area decreases from 20924 to 13723.1

Some results

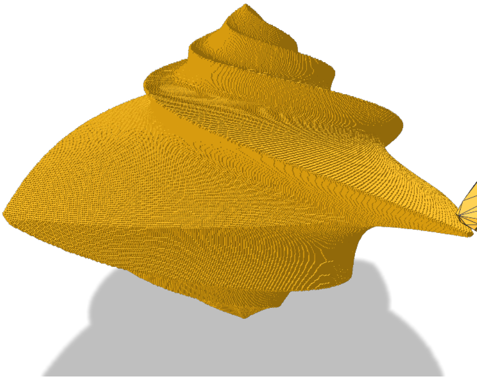


186760 quads

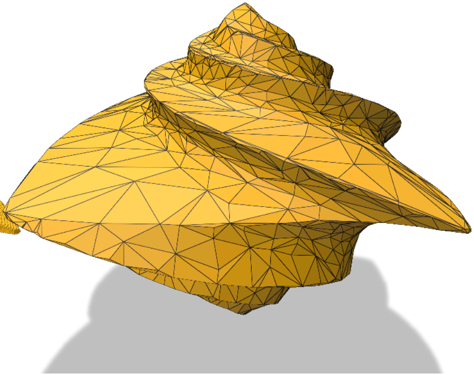


542 triangles

Some results

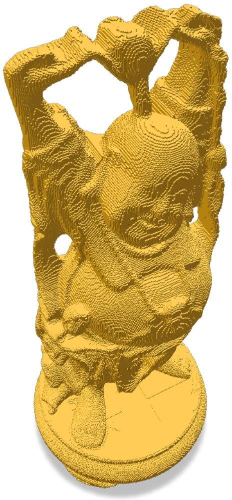


692916 quads
Computation time is 1504s

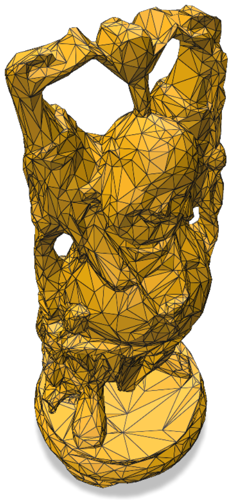


2510 triangles

Some results



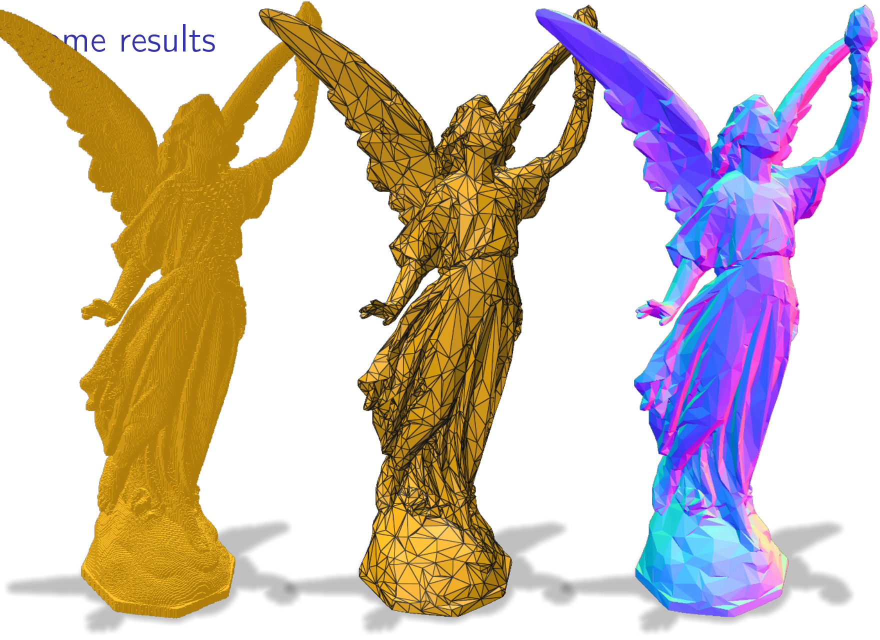
520816 quads
Computation time is 723s



7956 triangles

(color = normal vector)

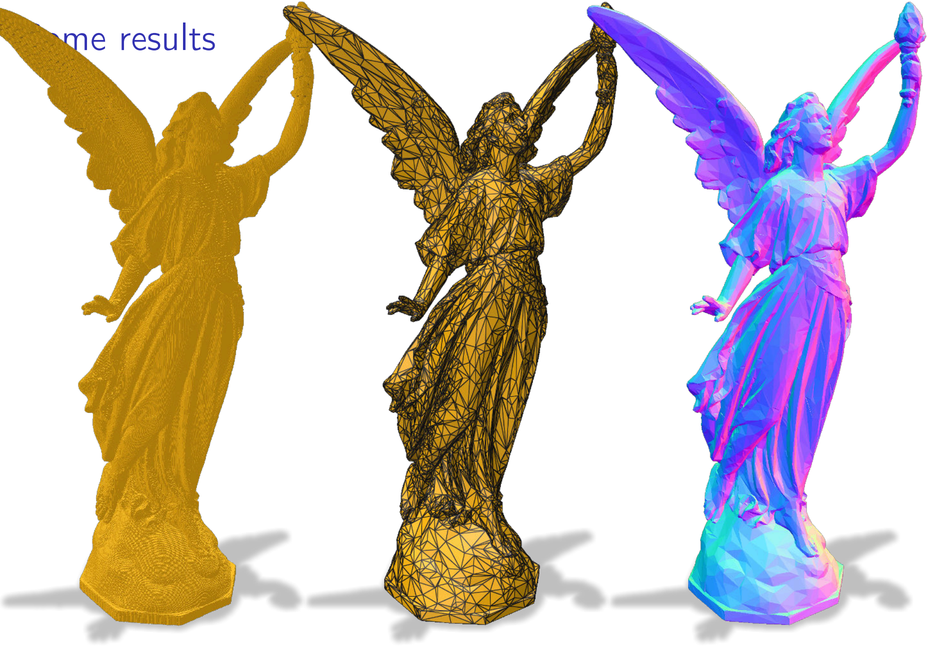
me results



384624 quads 7457 triangles (color = normal vector)

Computation time is 504s

me results

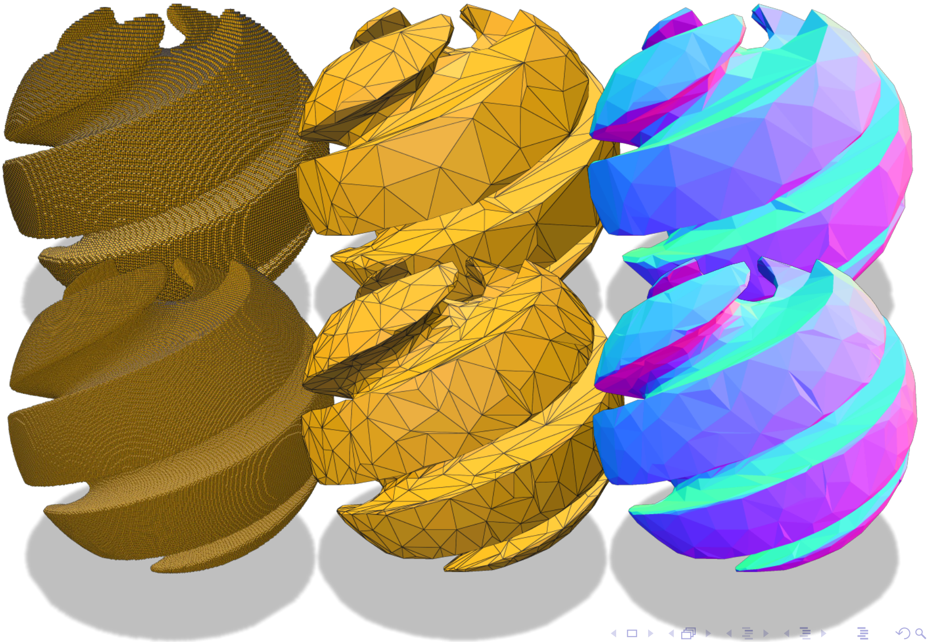


1543692 quads
Computation time is 2416s

15695 triangles

(color = normal vector)

Some results



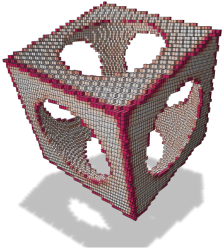
Full convexity: new characterizations and applications

Conclusion and future works

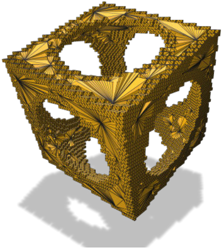
- new characterizations of full convexity
- complexity of full convexity check reduced by factor 2^d
- several methods to build fully convex "hulls"
- polyhedrization covering and fully subconvex to input data
- d-D C++ implementation in DGtal `dgtal.org`

- prove remaining characterizations
- determine number of iterations of $\mathrm{FC}^*(\cdot)$
- speed-up polyhedrization
- smarter optimizations for polyhedrization ?

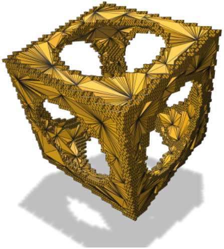
Smarter optimization following curvature information



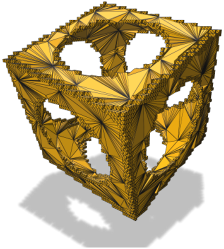
20924 quads



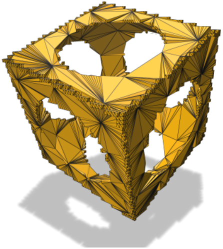
30916 triangles



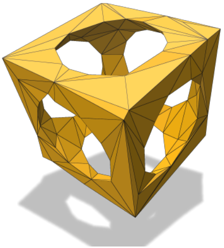
21812 triangles



14152 triangles



6550 triangles



236 triangles

Computation time is 57s, area decreases from 20924 to 14229.6