Corrected curvature measures

A unified approach for the geometric analysis of discrete data

Jacques-Olivier Lachaud Shonan Village Center, Japan May 20th, 2025

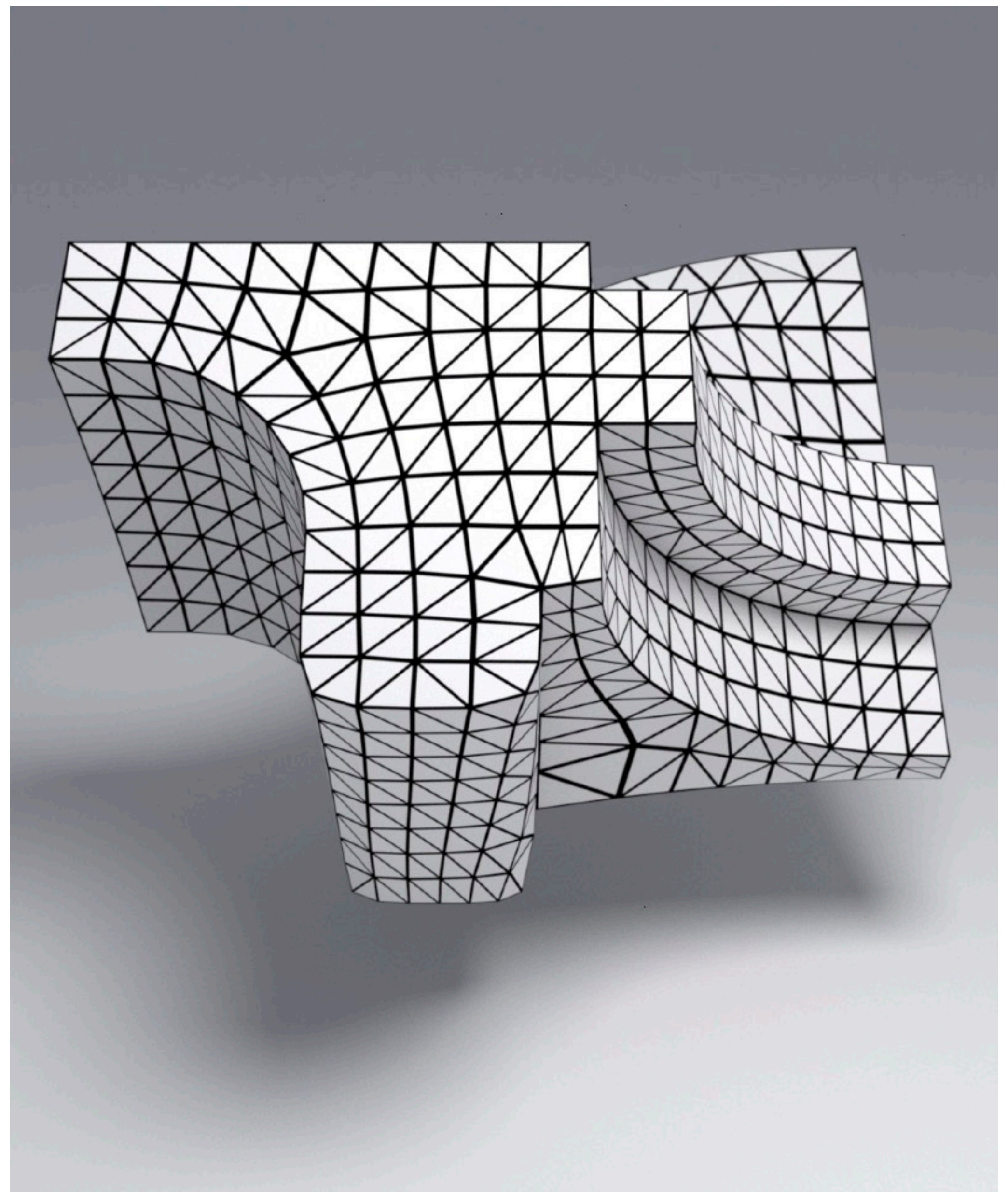
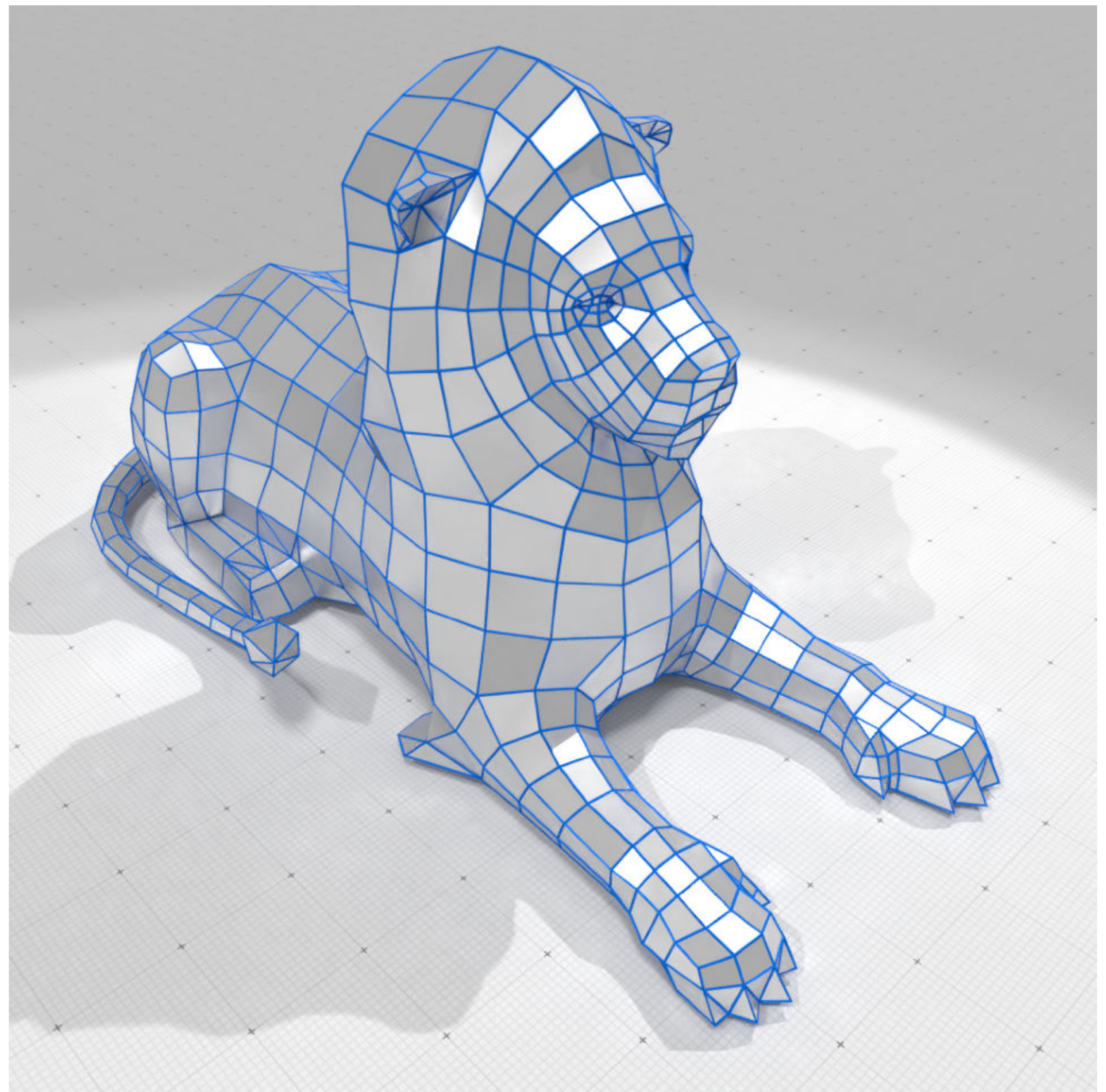
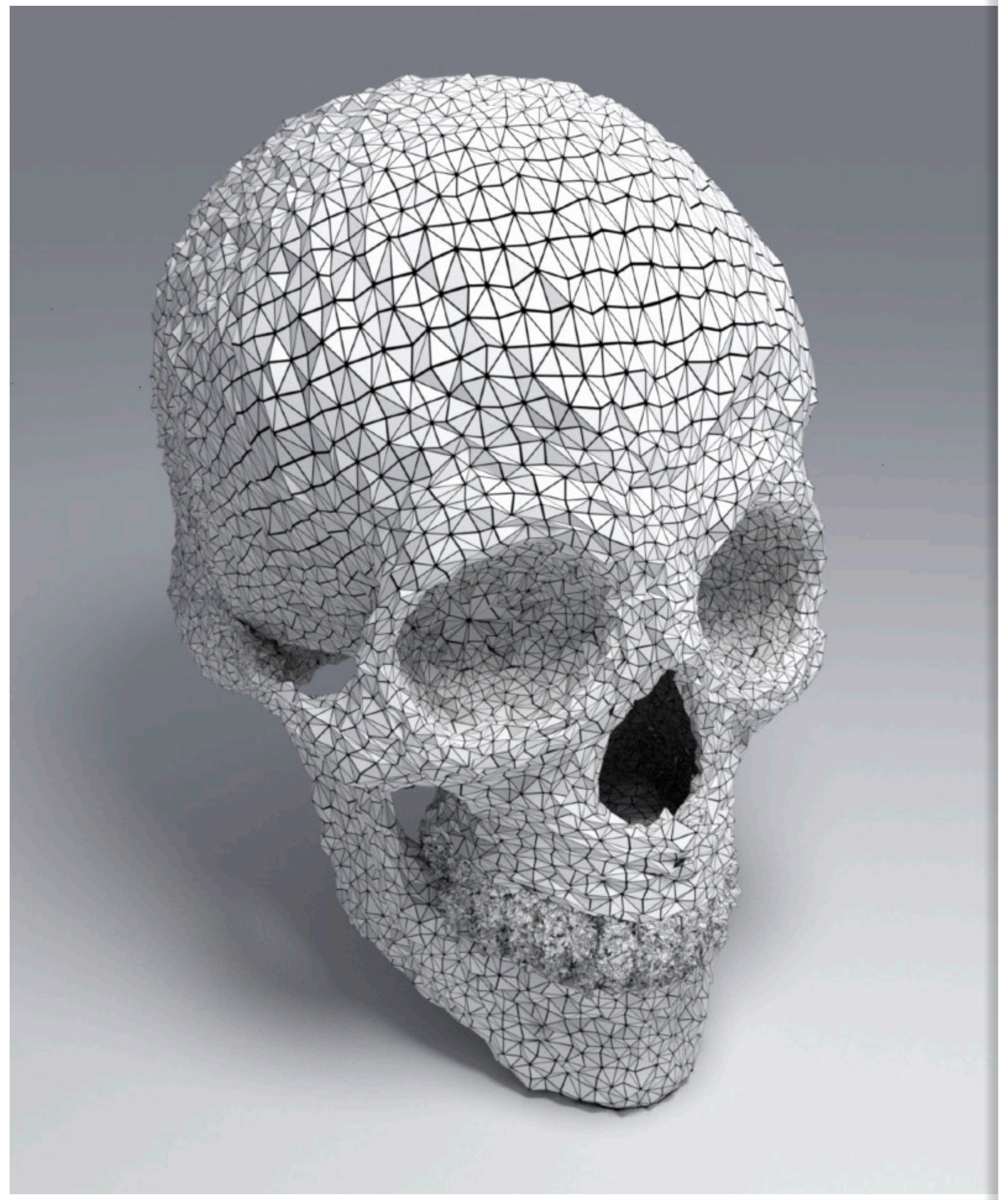
Jacques-Olivier LACHAUD
LAMA, Univ. Savoie Mont Blanc
/ LIRIS

Pascal ROMON
LAMA, Univ. Gustave Eiffel

Boris THIBERT
LJK, Univ. Grenoble Alpes

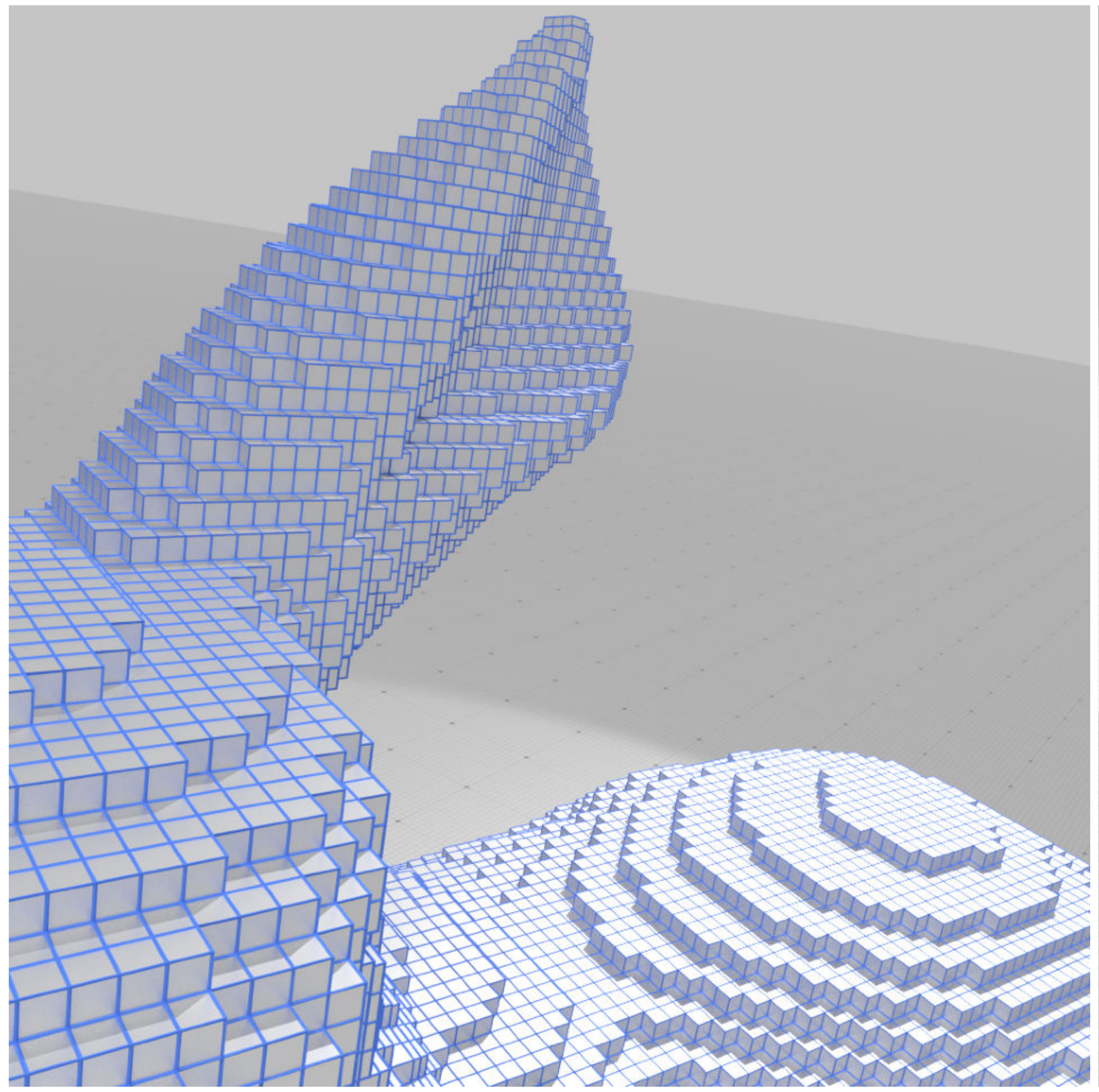
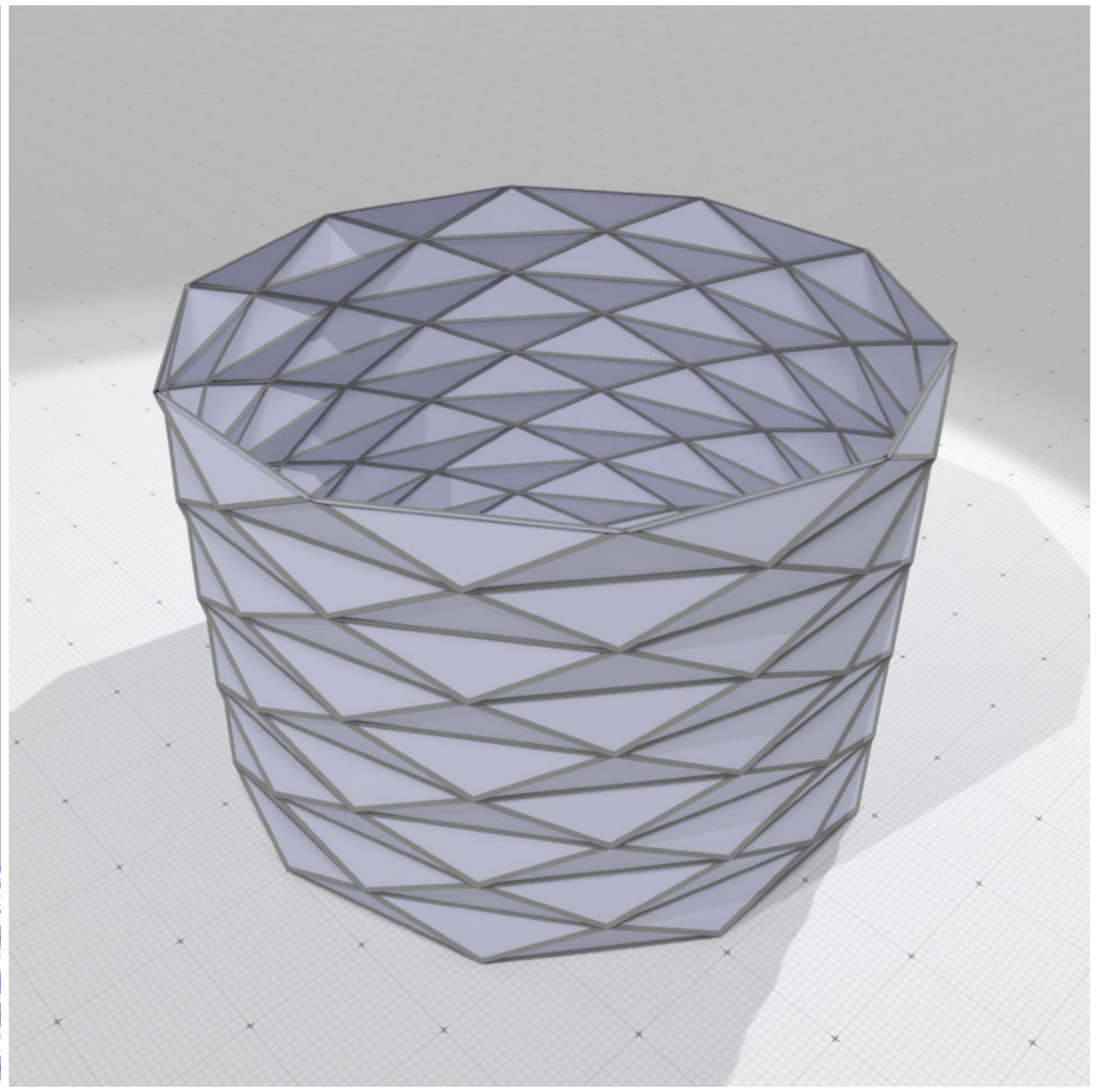
David COEURJOLLY
LIRIS / CNRS

Céline LABART
LAMA, Univ. Savoie Mont Blanc

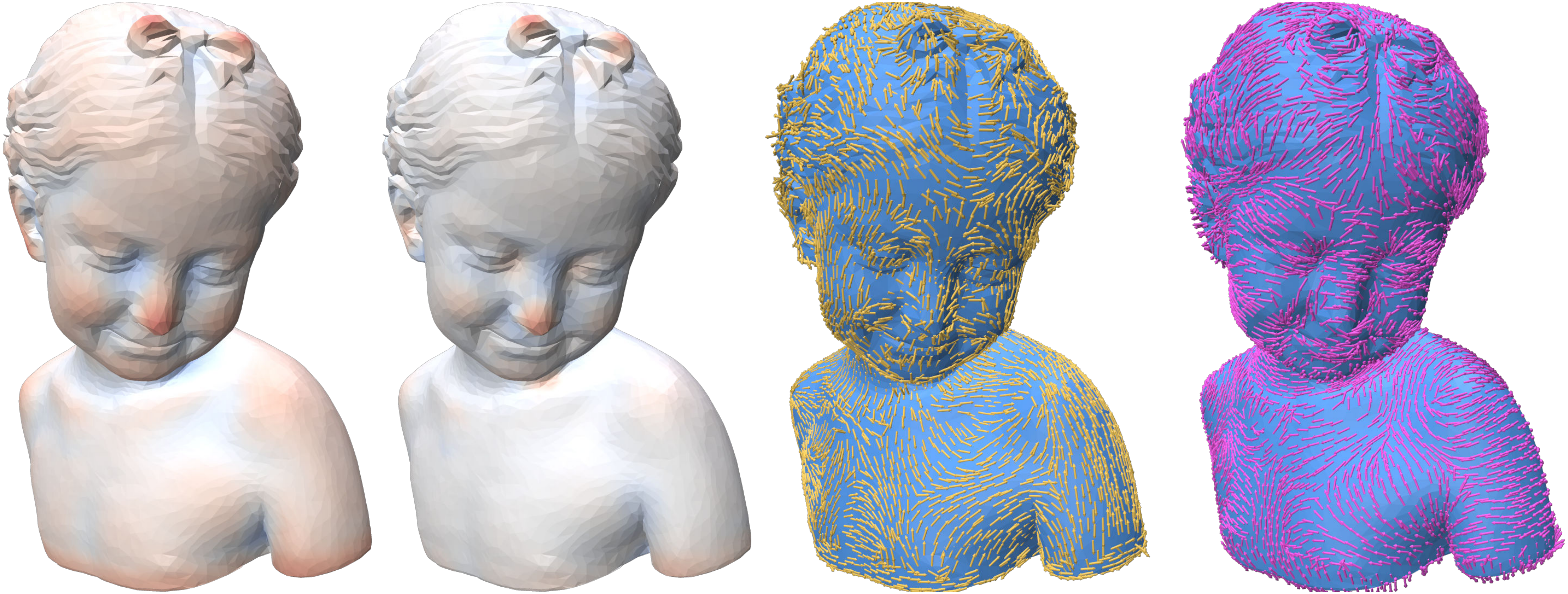
Geometry for discretised surfaces

- Triangulated mesh
- quadrangulated mesh
- noisy positions or normals
- digital surfaces
- Schwarz lantern
- Point clouds


How to define curvatures on these objects in a common way ?

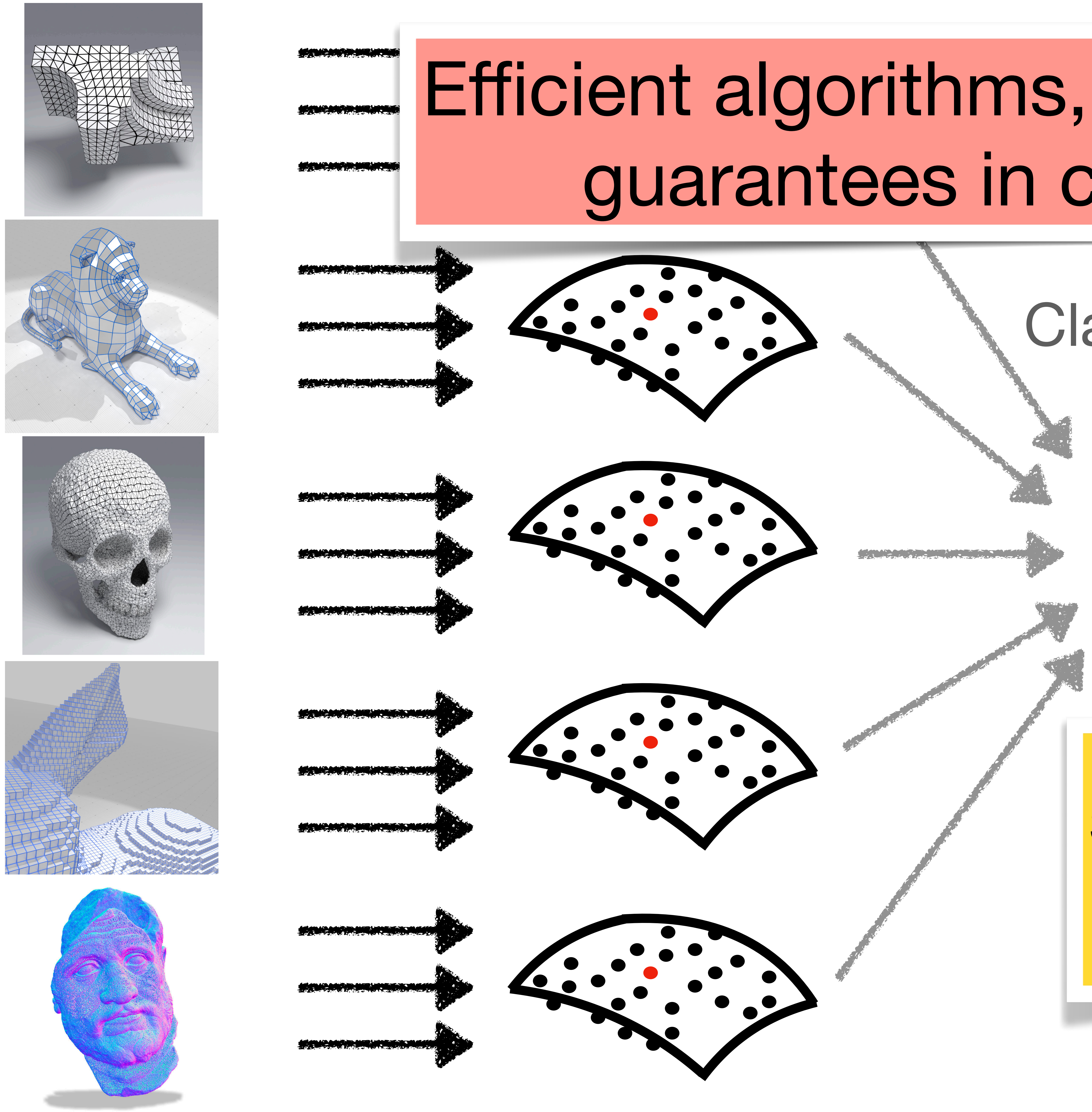
Principal curvatures and principal directions



CGAL

https://www.cgal.org/2023/11/20/curvature_and_remeshing/

Usual approaches to curvature on discrete data



Efficient algorithms, but challenging to have guarantees in case of noise in data

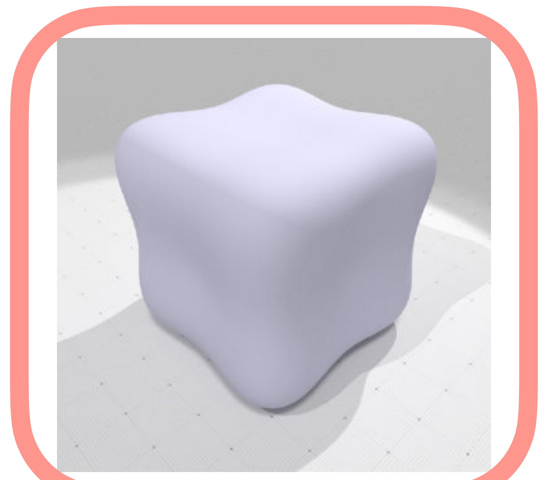
Classical **differential** geometry

Mean, Gaussian curvatures
Principal curvatures
and directions

Claim:
Smooth surface is close to data
$\Rightarrow$
Curvatures are close

Unified approach to curvatures

x : positions
n : normal vectors
u : vector field



(**x**, **n**)

(**x**, **u**)

Normal cycle [Wintgen 82][Fu92] [Cohen-Steiner, Morvan 03 and 06]

Lipschitz-Killing 2-forms

Grassmannian $\mathbb{R}^3 \times \mathbb{S}^2$

Geometric 2d « surface »

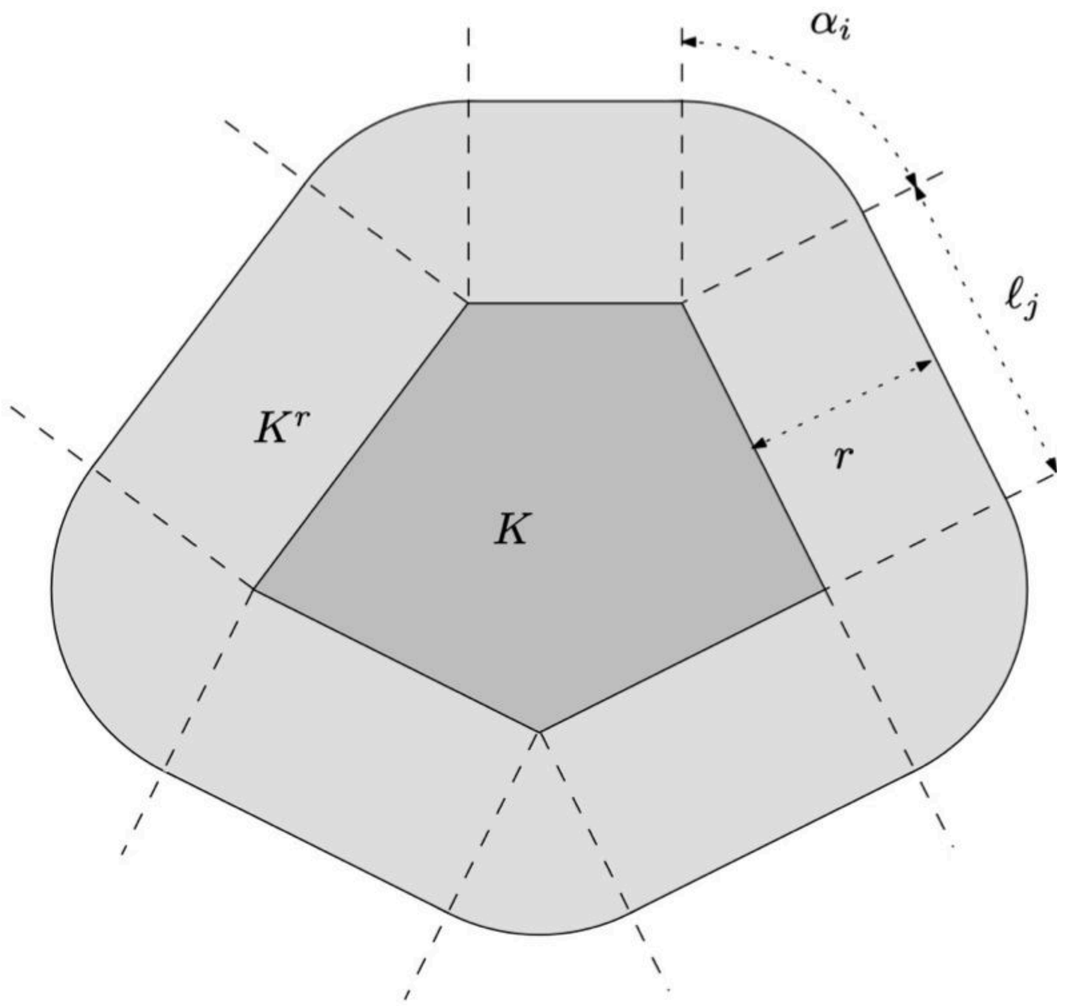
Curvature measures (Mean, Gaussian, Area, anisotropic)

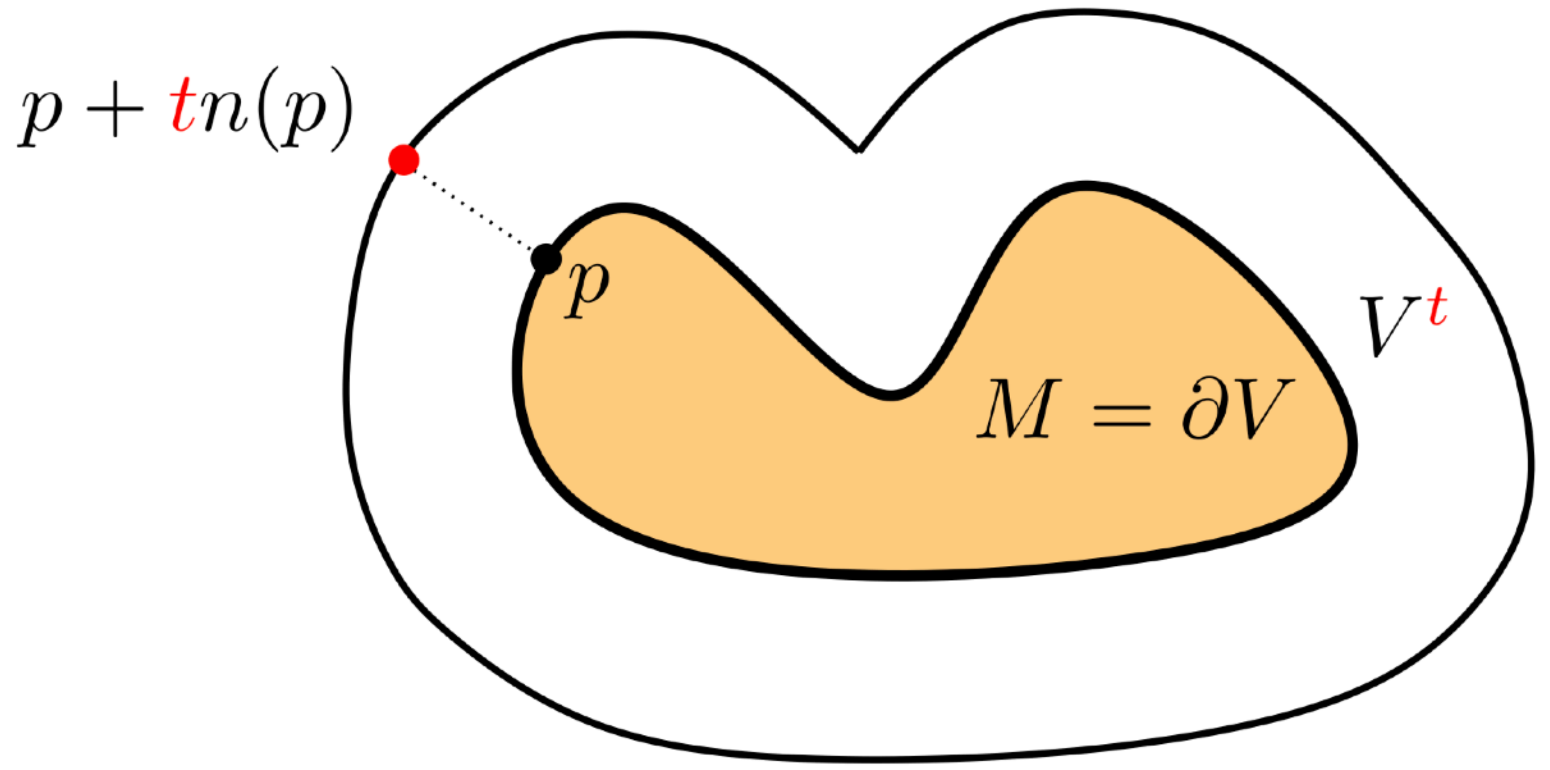
« surfaces » are close
ϵ positions error
η normals error

Theorem (stability): Curvatures measures $\epsilon + \eta$ close

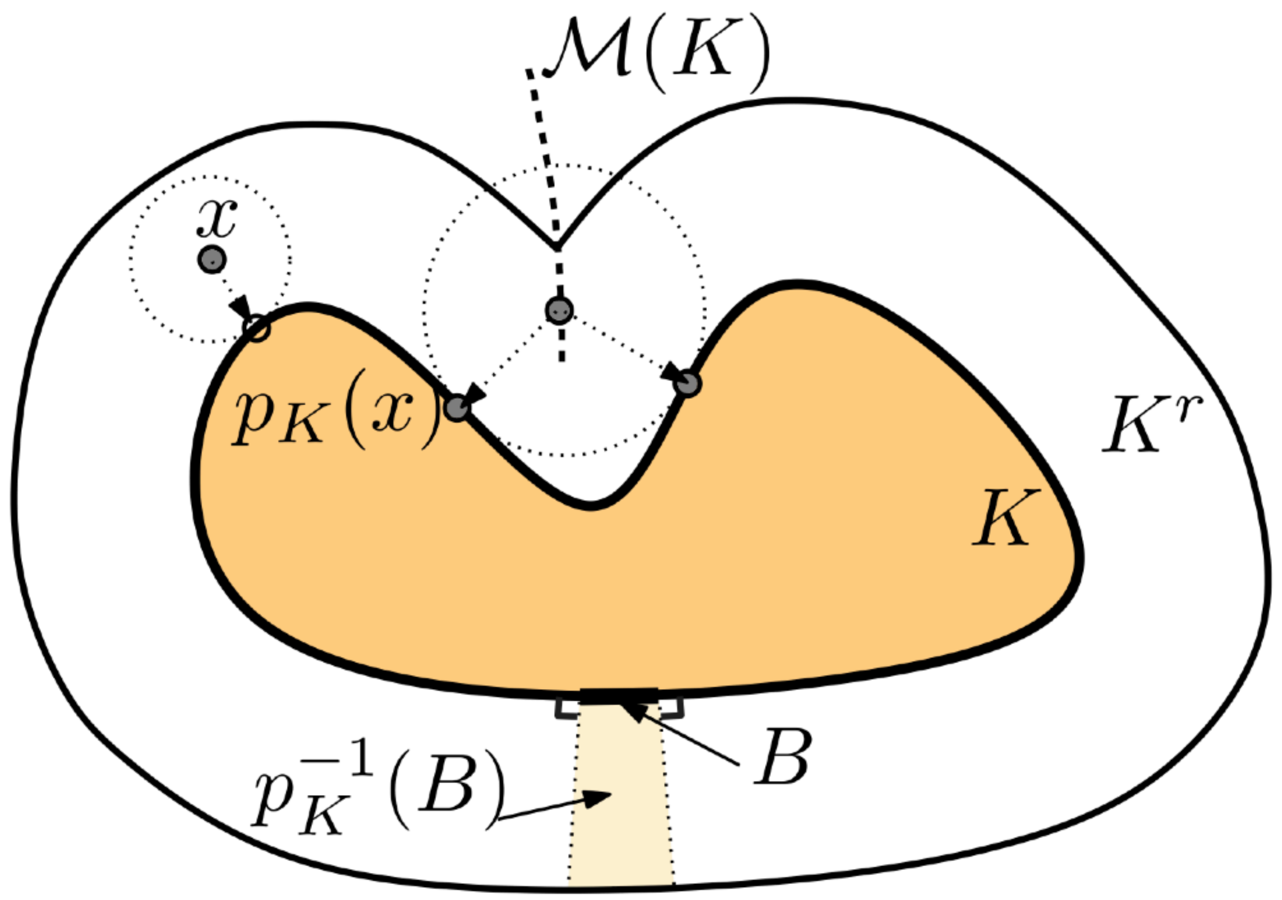
Short historical perspective (r-offsets)

$$\mathrm{Vol}(V^t) = \mathrm{Vol}(V) + \mathrm{Area}(M)t + \int_M H(p)\mathrm{d}p\ t^2 + \int_M G(p)\mathrm{d}p\ \frac{t^3}{3}$$





$$V^t = \{x \in \mathbb{R}^d,\ d(x, V) \leq t)\}$$

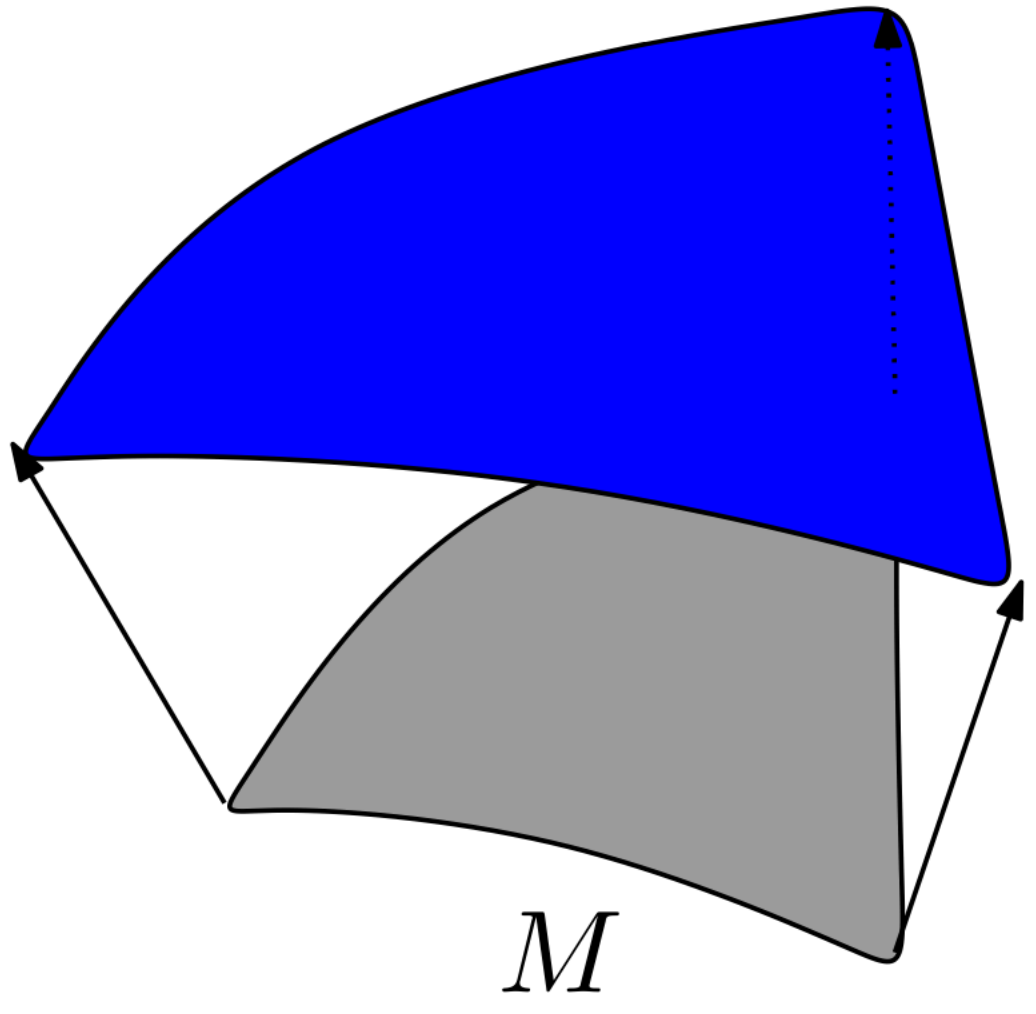


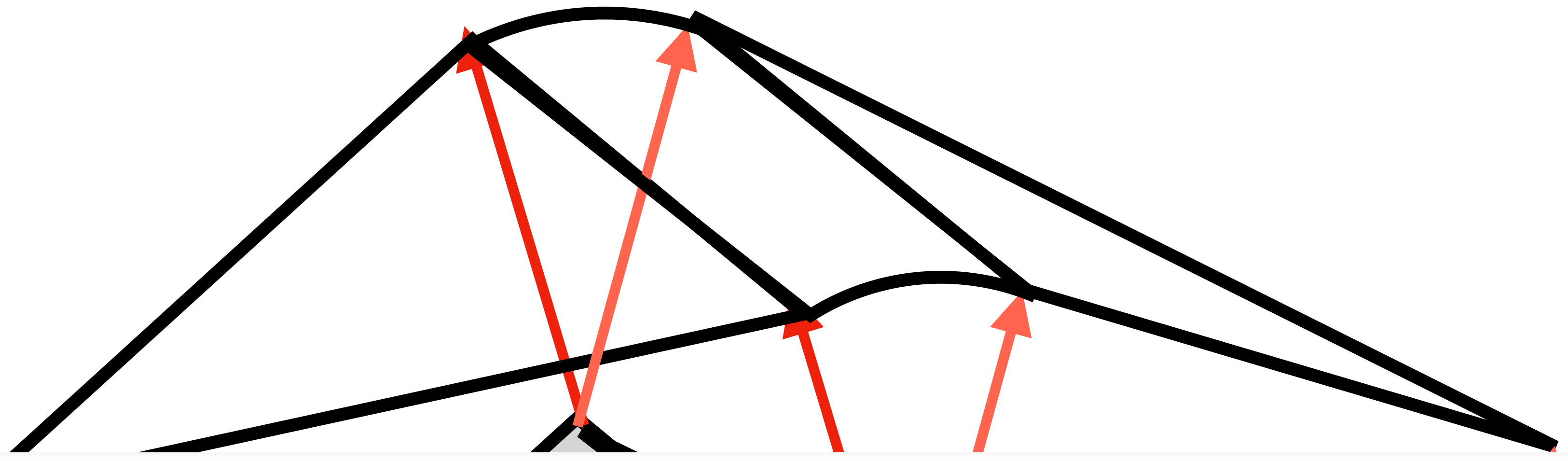
$\mathrm{Area}(K^r) = \mathrm{Area}(K) + \mathrm{leng}$ **What about discrete meshes ?** $K) + \mathrm{coef}_1\ t + \mathrm{coef}_2\ t^2 + \mathrm{coef}_3\ \frac{t^3}{3}$

$\mu_A(B)$ $\mu_H(B)$ $\mu_G(B)$

- Minkowski-Steiner formula (convex polygon)
- Tube's formula [Weyl 39] (smooth surface)
- Curvature measures [Federer 58, Federer 69] (C^2 smooth surface, convex sets)

Normal cycle and embedding into $\mathbb{R}^3 \times \mathbb{S}^2$





(M is smooth)

Proposition. Let $B \subset \mathbb{R}^3$ a ball

$$\int_{\mathrm{spt}(N(M))\cap(B\times\mathbb{S}^2)} \omega_A = \mathrm{Area}(M \cap B)$$

$$\int \ldots))\cap(B\times\mathbb{S}^2) \quad \omega_H = \int_{M\cap B} H(p)\,\mathrm{d}p$$

$$\ldots))\cap(B\times\mathbb{S}^2) \quad \omega_G = \int_{M\cap B} G(p)\,\mathrm{d}p$$

Area, mean and Gaussian curvature measures for smooth surfaces and (nice) meshes

- Area above triangles,
- Mean curvature above edges
- Gaussian curvature above vertices

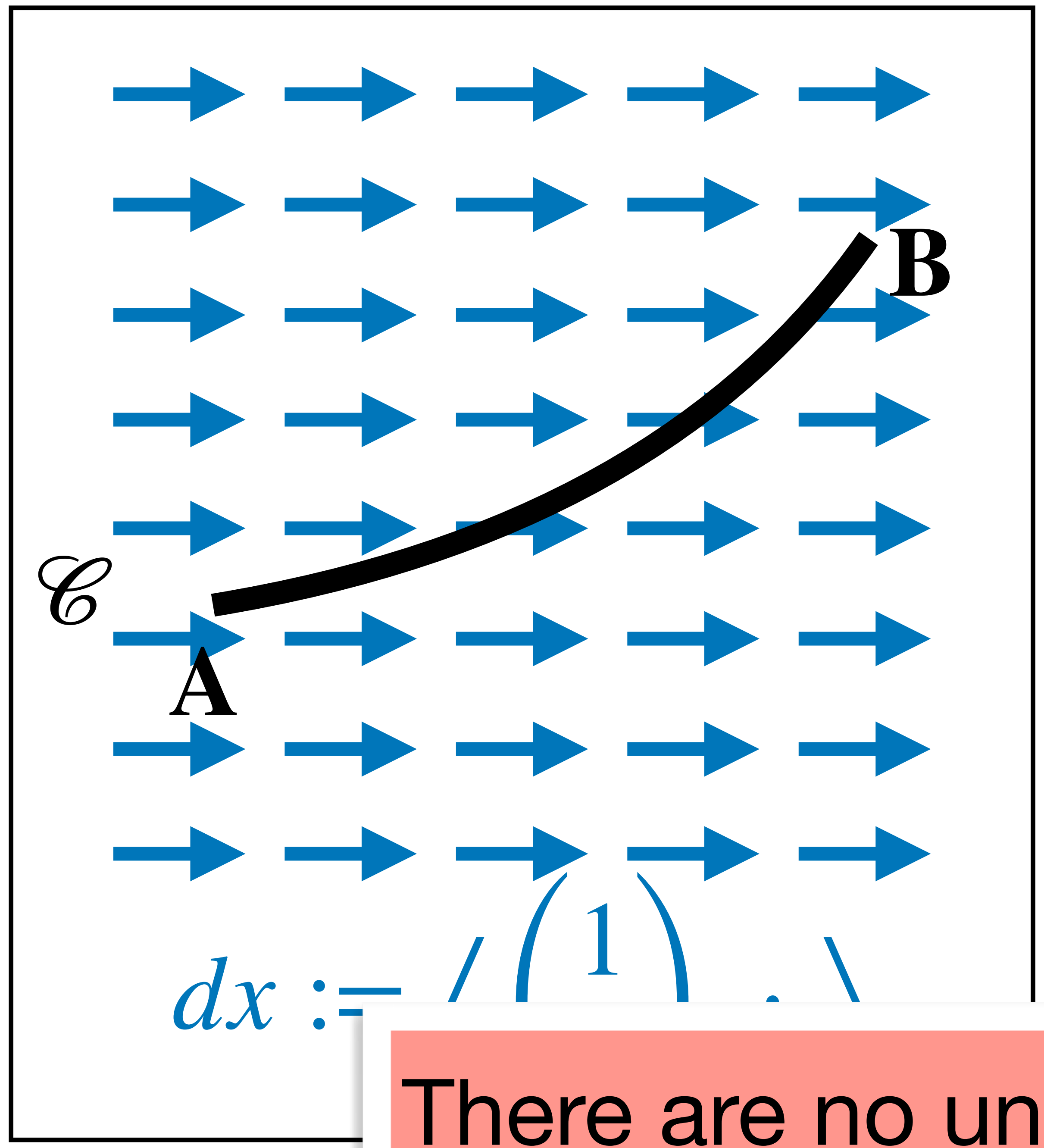
- Normal cycle [Wintge
- If M is a C^2 surface, t
- If M is a mesh, the no ... nder, piece of sphere), then an inclusion-exc
- N(M) is a 2-current (i.e. takes 2-form and returns a value)
- Integration with L

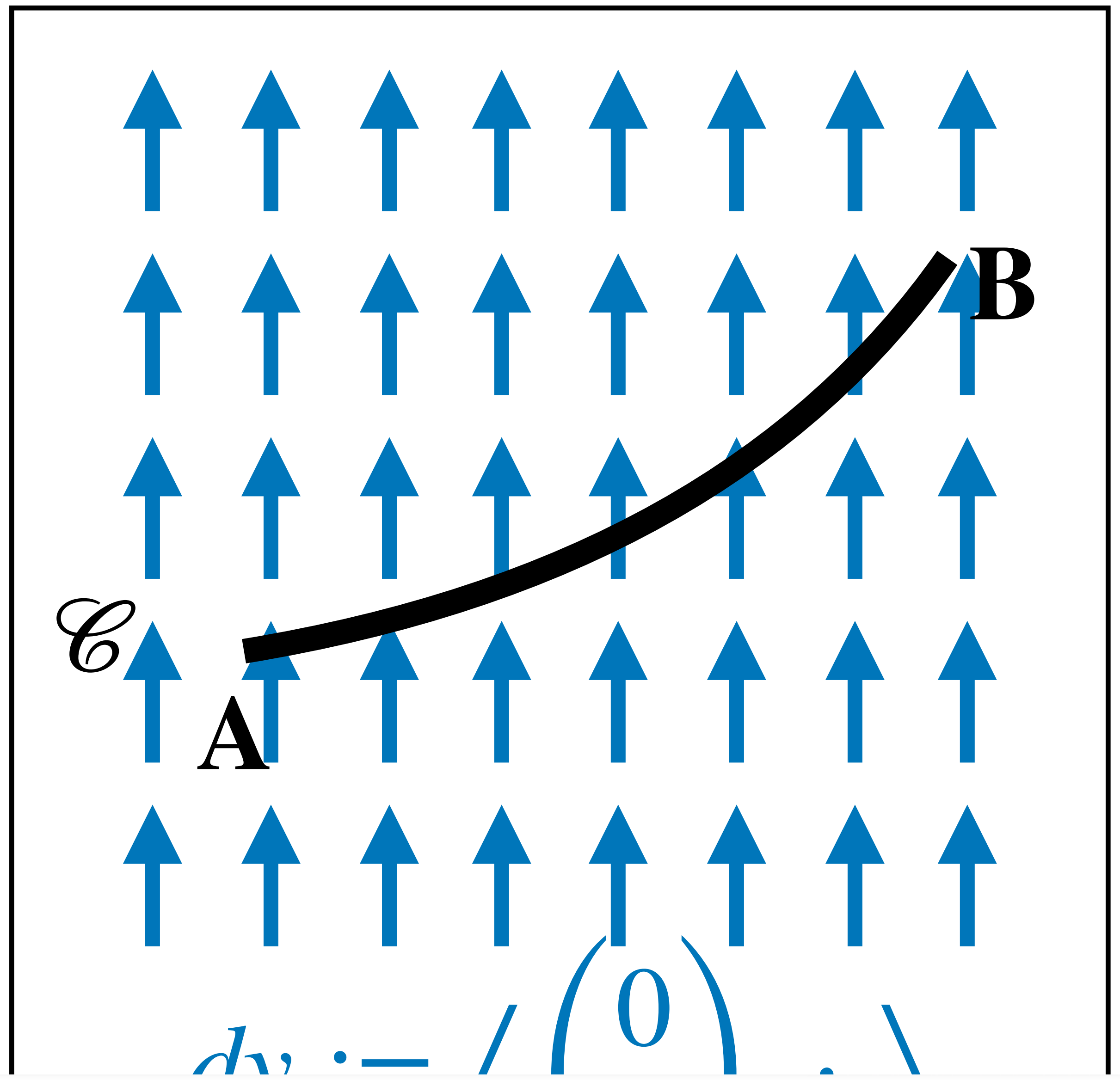
What about noisy meshes,
digital surfaces, or point clouds ?
And this is not point wise estimation of curvatures !

But first what are these Lipschitz-Killing differential forms ?

Differential forms

Differential 1-forms in $\mathbb{R}^2$





Tangent vector

$$\int_{\mathscr{C}} dx = \int_0^L \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dot{\mathscr{C}}(s) \right\rangle ds = x_B - x_A$$

$$\int_{\mathscr{C}} dx = \lim_{n->+\infty} \sum_{i=0}^{n} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{A}_{i+1} - \mathbf{A}_i \right\rangle = x_B - x_A$$

For $\mathbf{A} = \mathbf{A}_0, \ldots, \mathbf{A}_i, \ldots, \mathbf{A}_n = B$

An ordered sequence of points along $\mathscr{C}$

$$\int dy = \int^L \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dot{\mathscr{C}}(s) \right\rangle ds = y_B - y_A$$

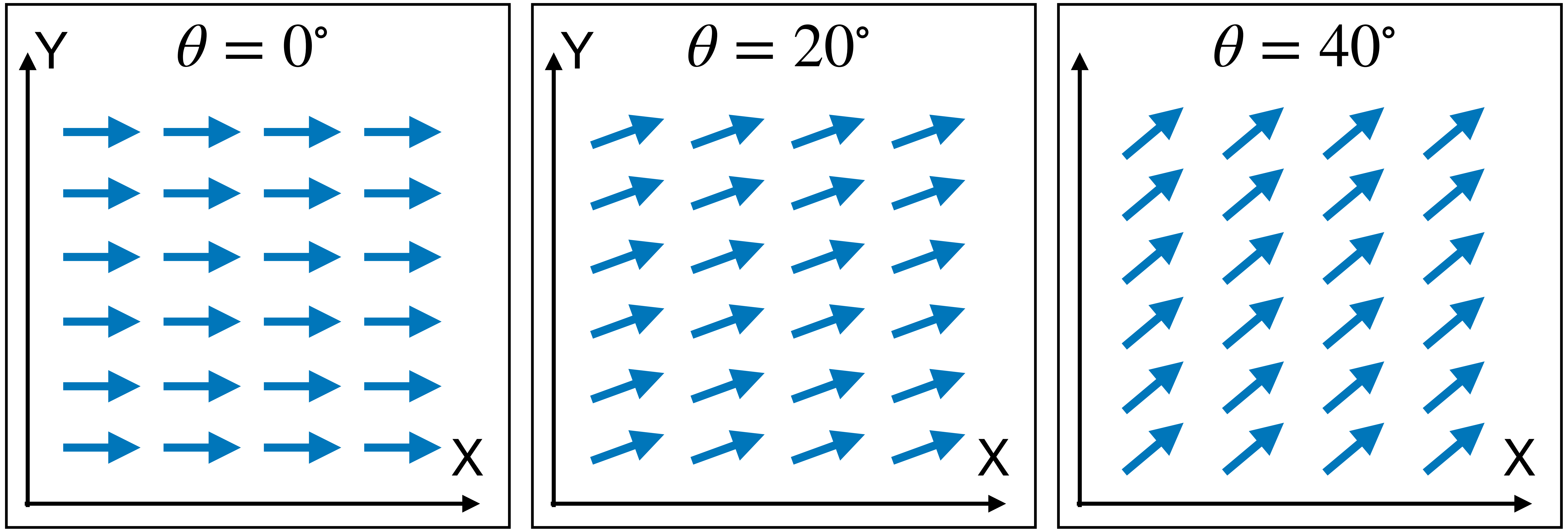
There are no universal differential 1-form in $\mathbb{R}^2$ that compute the length !

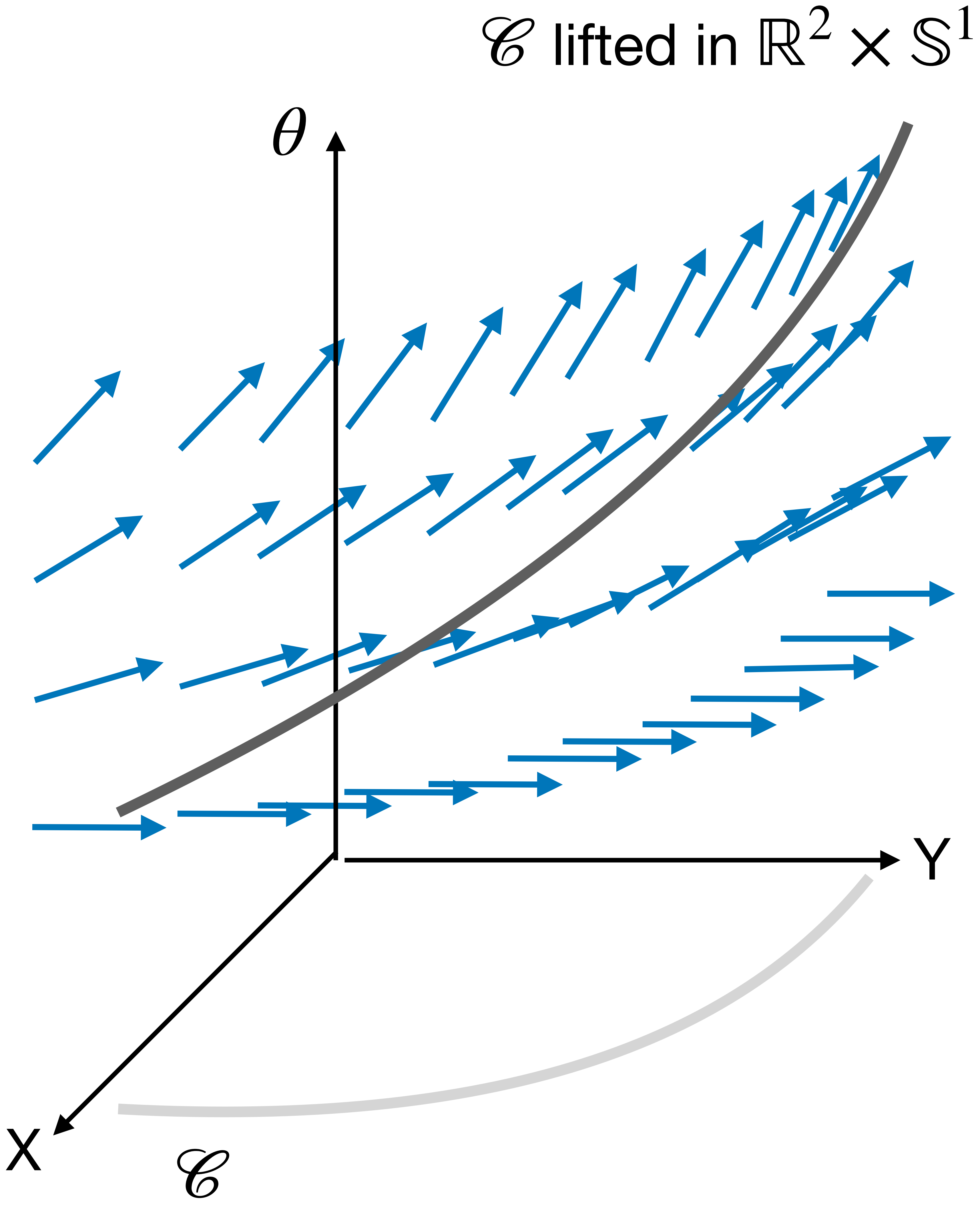
$dx + dy$ computes $x_B + y_B - (x_A + y_A)$

$\sqrt{dx^2 + dy^2}$ is not a 1-form

- a 1-form
- Differenti

Differential 1-forms in $\mathbb{R}^2 \times \mathbb{S}^1$

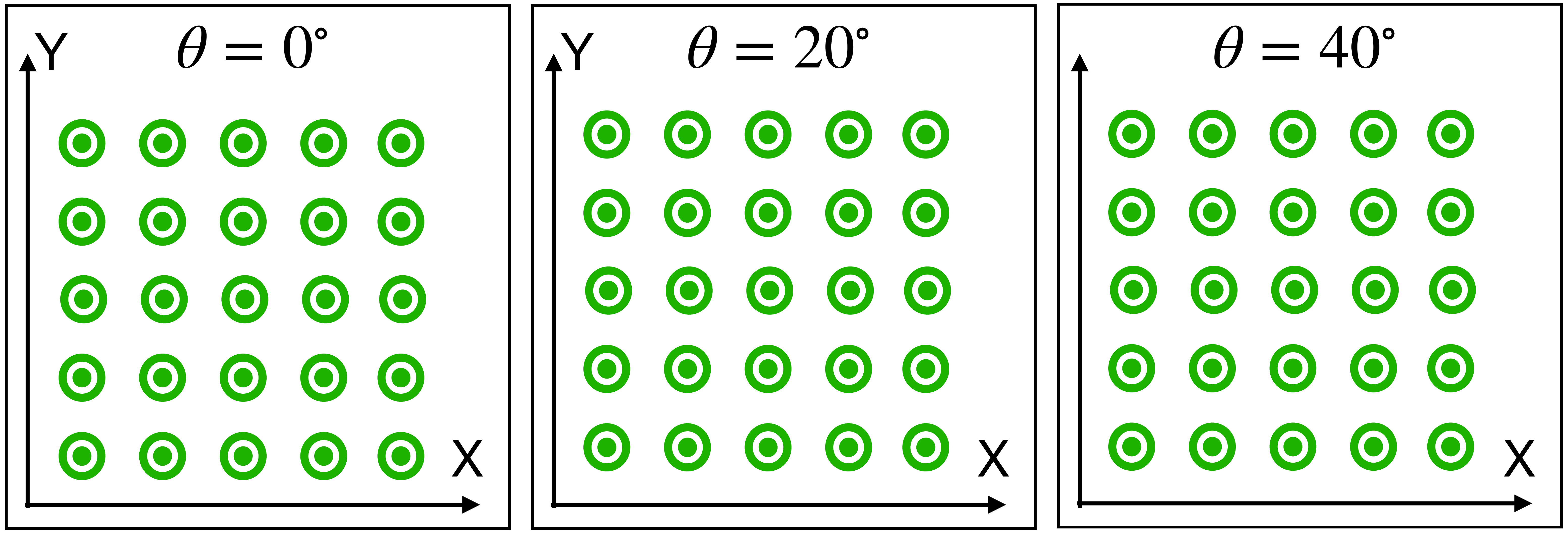


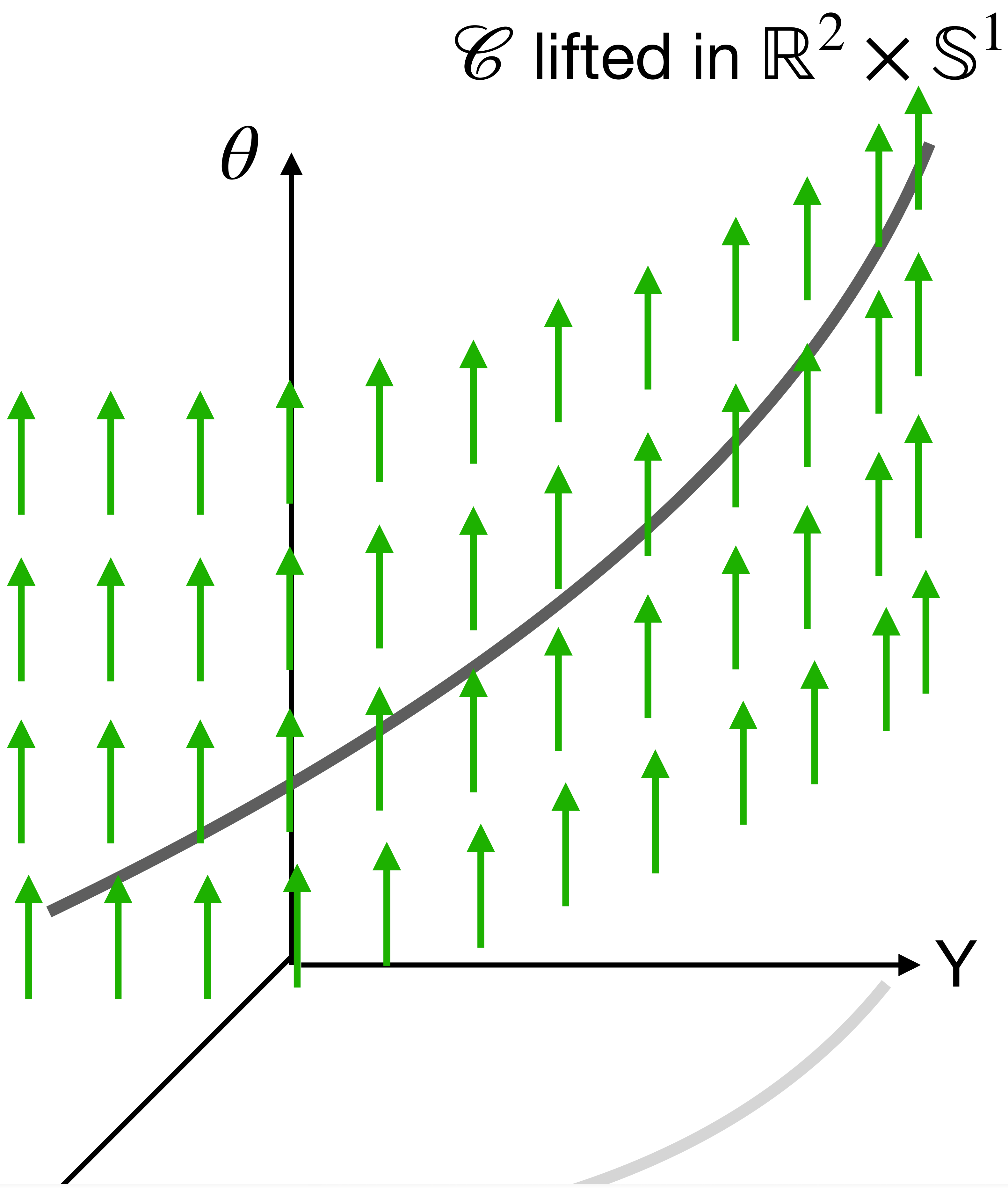


$$\omega^{\text{length}}_{\mathbf{x},\theta} := \left\langle \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \cdot \right\rangle_{T_p\mathbf{R}^2\times\mathbf{S}^1}$$

- $\omega^{\text{length}}_{\mathbf{x},\theta}$ measures lengths !
- $\mathscr{C}$ lifted in $\mathbb{R}^2 \times \mathbb{S}^1$ as $\Gamma(\mathscr{C}) = \{(\mathbf{x}, \angle\mathbf{n}^{\perp}(\mathbf{x})), \mathbf{x} \in \mathscr{C}\}$
- (Here $\mathbb{S}^1$ identified with angles)

Differential 1-forms in $\mathbb{R}^2 \times \mathbb{S}^1$





$$\omega^{\text{curv}}_{\mathbf{x},\theta} := \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \cdot \right\rangle_{T_p\mathbf{R}^2\times\mathbf{S}^1}$$

- $\omega^{\text{curv}}_{\mathbf{x},\theta}$ measures curvatures !
- $\mathscr{C}$ lifted in $\mathbb{R}^2 \times \mathbb{S}^1$ as $\Gamma(\mathscr{C}) = \{(\mathbf{x}, \angle \mathbf{n}^{\perp}(\mathbf{x})), \mathbf{x} \in \mathscr{C}\}$
- (Here $\mathbb{S}^1$ identified with angles)

$\omega^{\text{length}}_{\mathbf{x},\theta}$ and $\omega^{\text{curv}}_{\mathbf{x},\theta}$ are the Lipschitz-Killing 1-forms (or invariant forms)

Invariant forms in 3D

Differential forms in Grassmannian $\mathbb{R}^3 \times \mathbb{S}^2$

For any point $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^3 \times \mathbb{S}^2$ and tangent vectors $\boldsymbol{\xi}, \boldsymbol{\nu} \in T_{(\mathbf{x},\mathbf{u})}(\mathbb{R}^3 \times \mathbb{S}^2)$

Area form $\omega^{(0)}_{(\mathbf{x},\mathbf{u})}(\boldsymbol{\xi}, \boldsymbol{\nu}) = \det(\mathbf{u}, \boldsymbol{\xi}_p, \boldsymbol{\nu}_p)$

Mean curvature form $\omega^{(1)}_{(\mathbf{x},\mathbf{u})}(\boldsymbol{\xi}, \boldsymbol{\nu}) = \det(\mathbf{u}, \boldsymbol{\xi}_p, \boldsymbol{\nu}_n) + \det(\mathbf{u}, \boldsymbol{\xi}_n, \boldsymbol{\nu}_p)$

Gaussian curvature form $\omega^{(2)}_{(\mathbf{x},\mathbf{u})}(\boldsymbol{\xi}, \boldsymbol{\nu}) = \det(\mathbf{u}, \boldsymbol{\xi}_n, \boldsymbol{\nu}_n)$

Anisotropic form analog to “2nd Fundamental form”

Curvatures through local integration of Lipschitz-Killing forms

Normal cycle lifting (with $\mathbf{n}(\mathbf{x})$ normal at $\mathbf{x}$)

$\Gamma : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{S}, \mathbf{x} \mapsto (\mathbf{x}, \mathbf{n}(\mathbf{x}))$

Given any ball $B(\mathbf{y}, r)$, we compute:

Length meas

$\mu^{\text{length}}(B \cap \mathscr{C}$

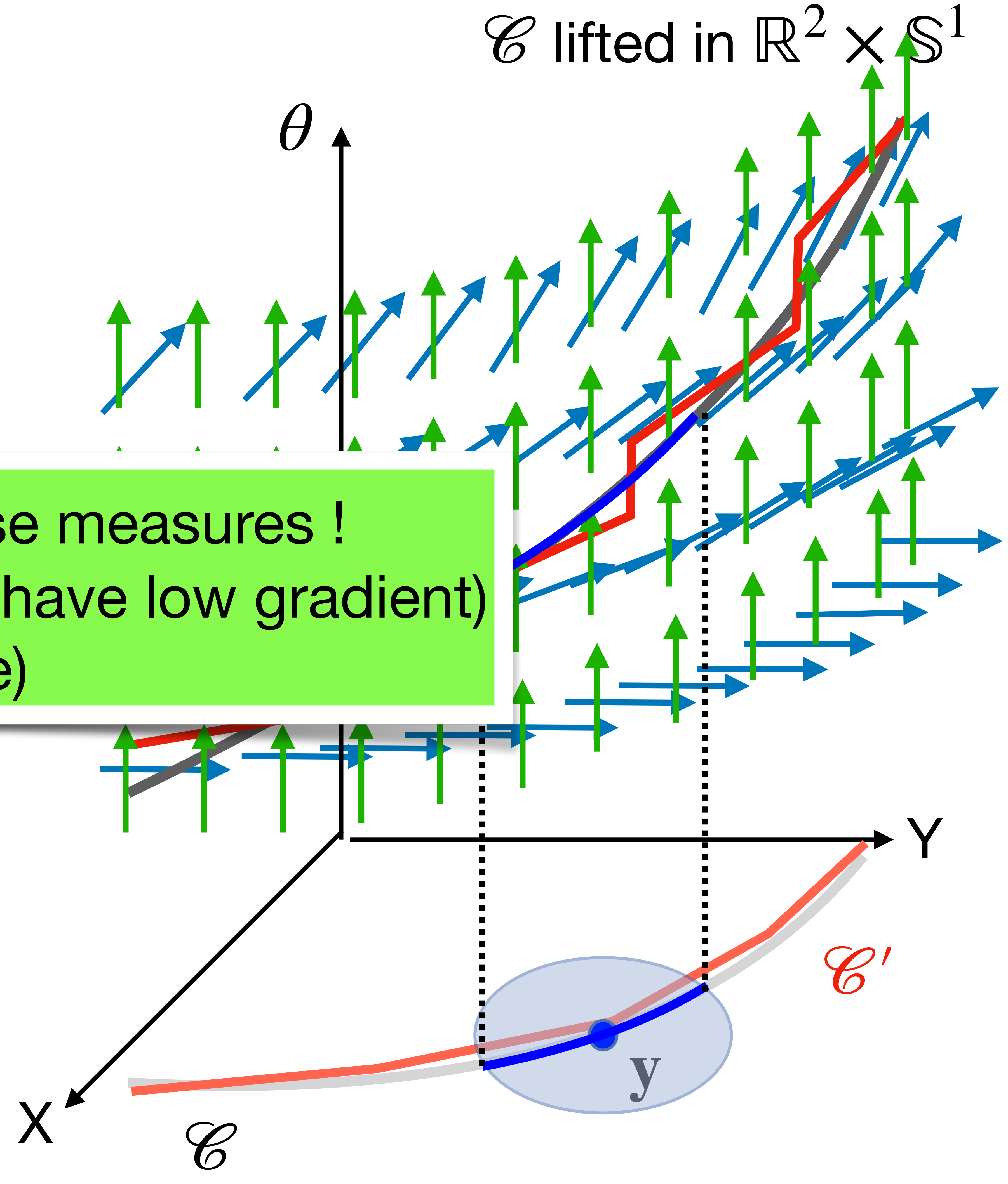
Curves that are close in $\mathbb{R}^2 \times \mathbb{S}^1$ have close measures !
(Since differential forms apply continuously and have low gradient)
All results hold in 3D (and more)

Curvature measure

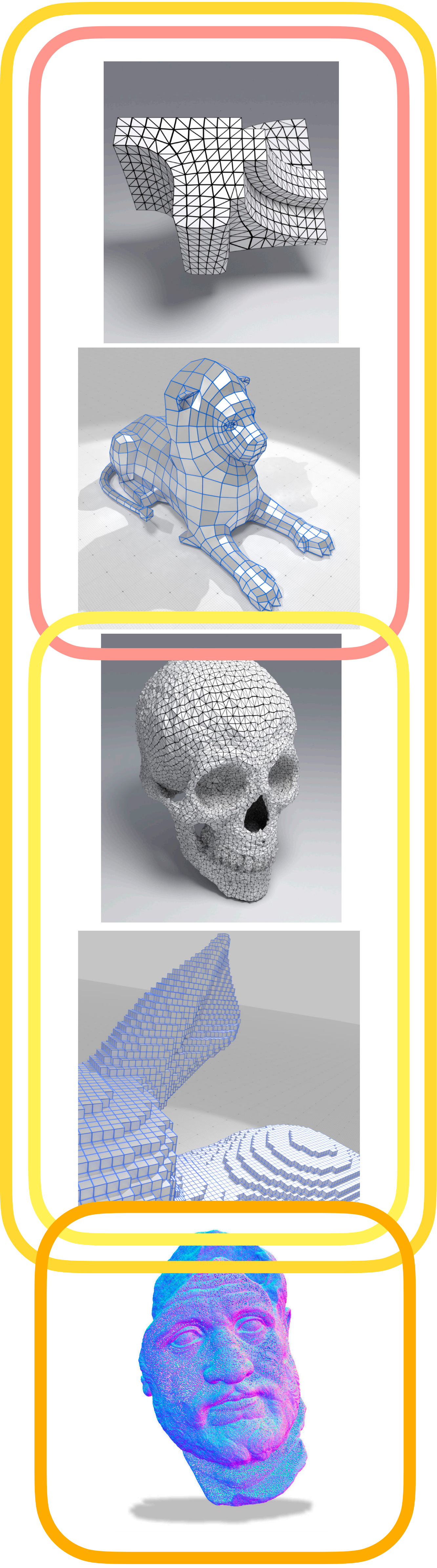
$$\mu^{\text{curv}}(B \cap \mathscr{C}) = \int_{\Gamma(B \cap \mathscr{C})} \omega^{\text{curv}} = \int_{B \cap \mathscr{C}} \Gamma^* \omega^{\text{curv}}$$

Curvature

$$\kappa(\mathbf{y}, r) := \frac{\mu^{\text{curv}}(B \cap \mathscr{C})}{\mu^{\text{length}}(B \cap \mathscr{C})}$$



Our main contributions



Is the Normal Cycle the definitive solution to curvature estimation ?

1. The Normal Cycle performs poorly on perturbed meshes
And even worse on digital surfaces.
We propose to **correct the measures** using an external vector field $\mathbf{u}$

2. The Normal Cycle requires a radius parameter, which may be difficult to set.
Moreover we can exhibit simpler and more accurate formula
using an **interpolation trick**

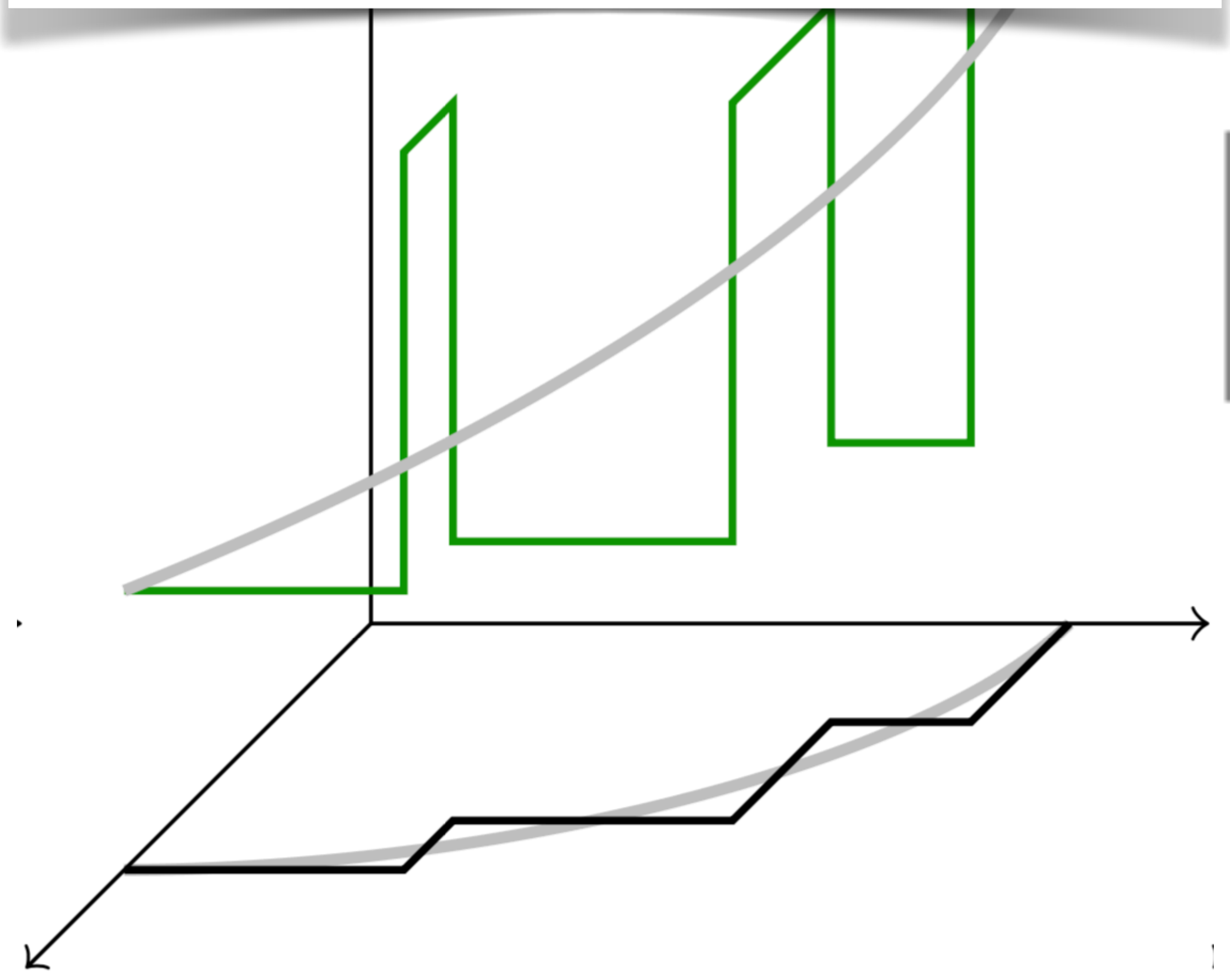
3. The Normal Cycle is unable to handle point clouds.
We use a **local randomization and superposition of measures**
to guarantee stable curvature estimations

1. Correcting curvature measures

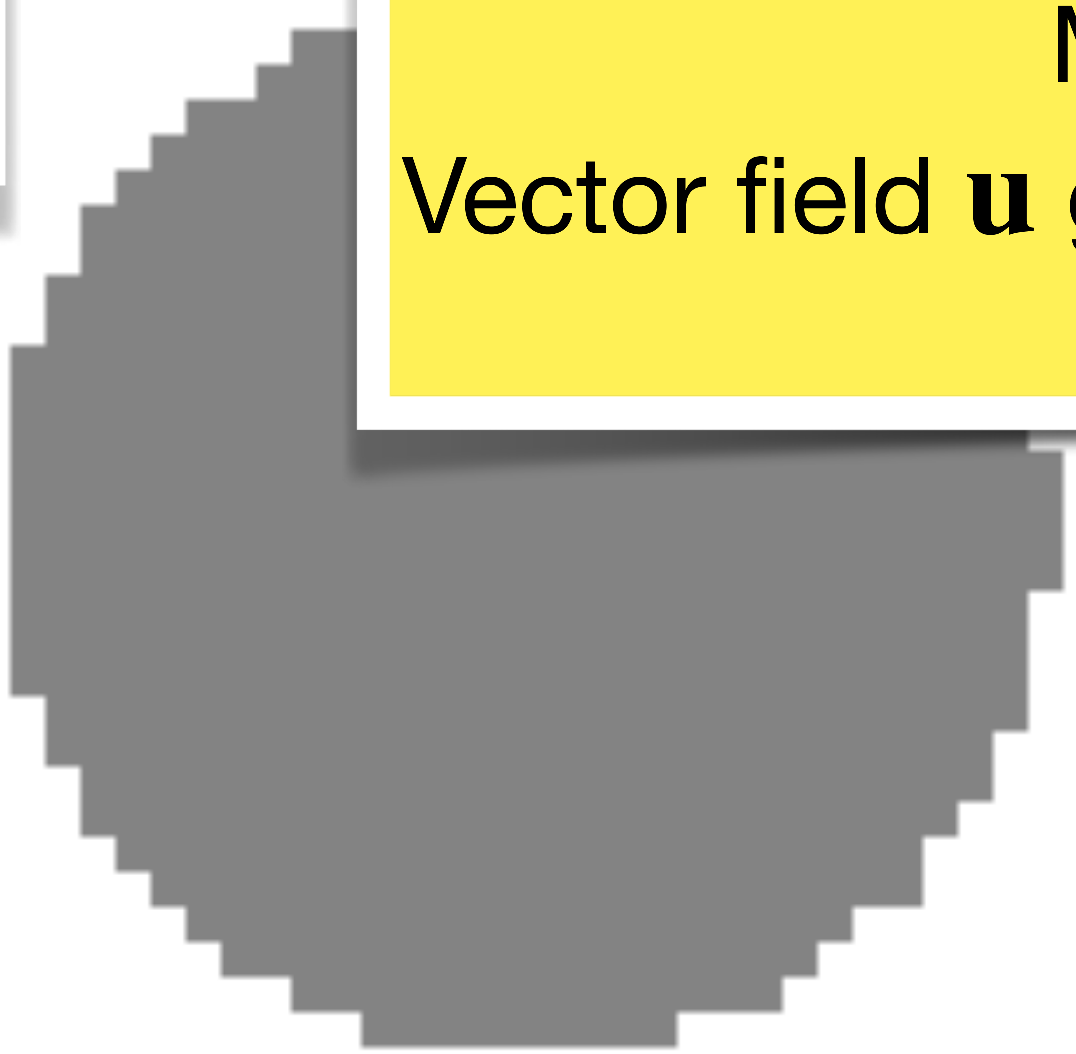
The case of digital contours

Normal Cycle

Curves are not close in $\mathbb{R}^2 \times \mathbb{S}$
Measures are not close !



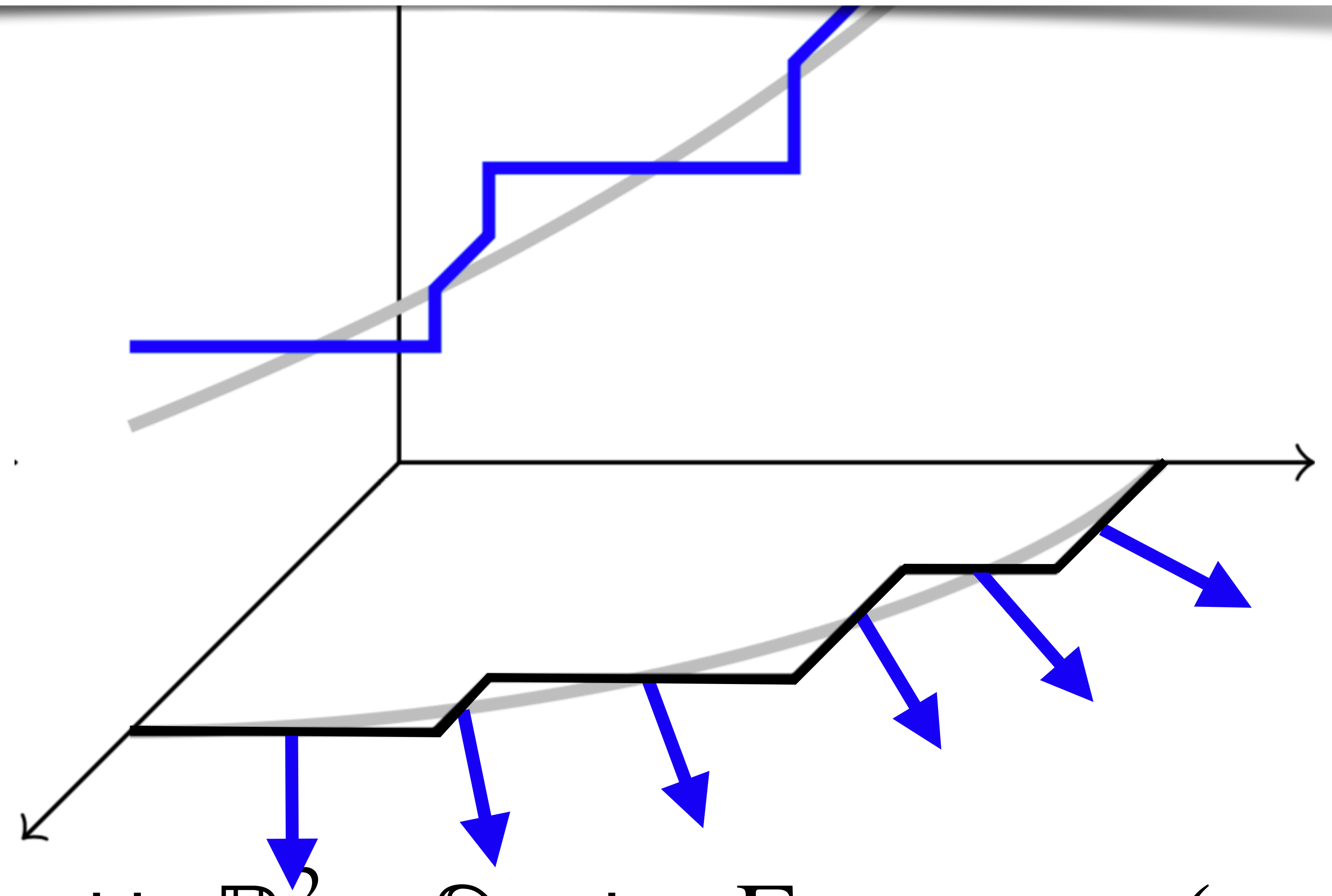
Lifted in $\mathbb{R}^2 \times \mathbb{S}$ using $\Gamma := \mathbf{x} \mapsto (\mathbf{x}, \mathbf{n}(\mathbf{x}))$



Corrected normal current

[Lachaud, Romon, Thibert19]

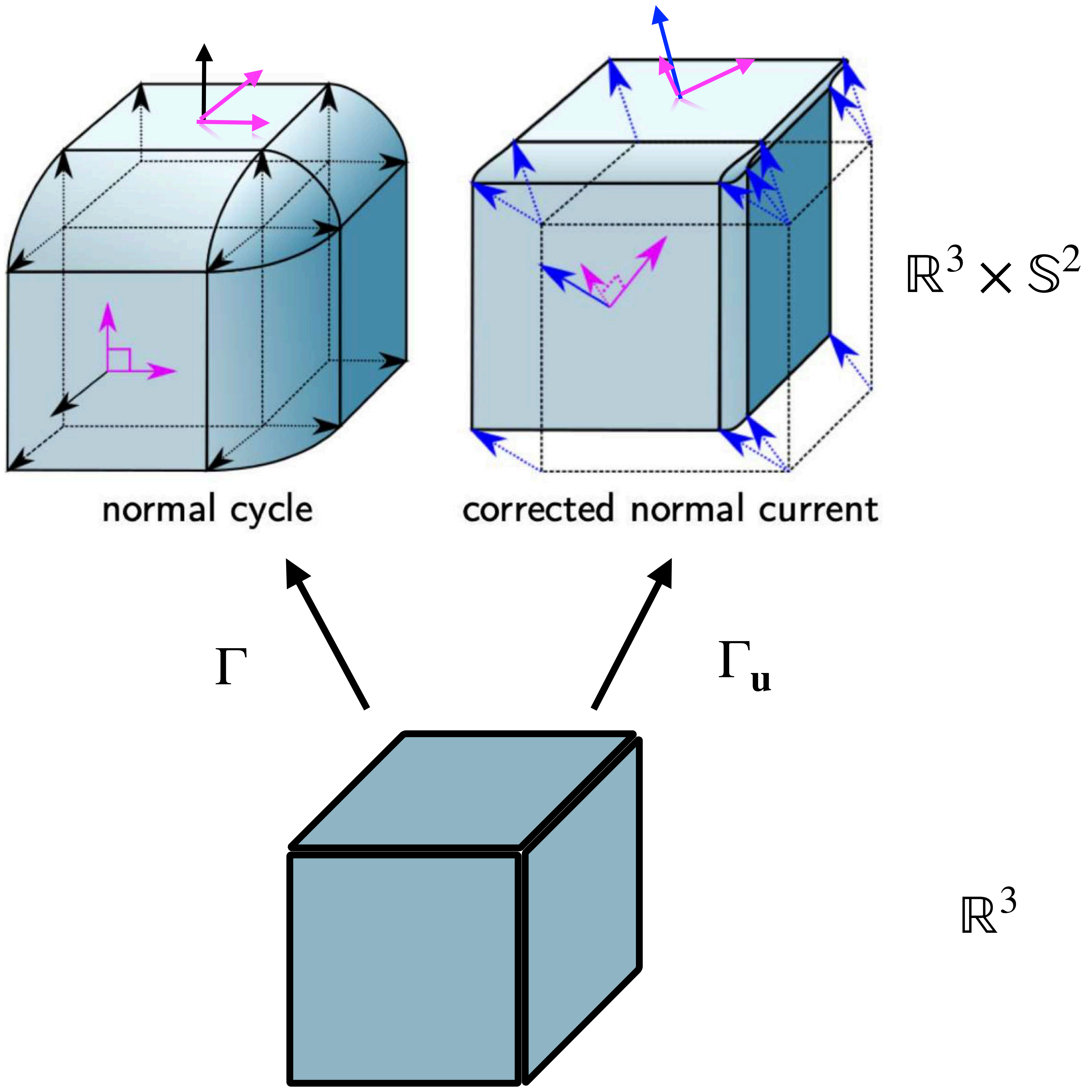
Curves are closer in $\mathbb{R}^2 \times \mathbb{S}$
Measures are thus closer !
Vector field $\mathbf{u}$ given by a convergent normal estimator
(Integral invariant, VCM)



Lifted in $\mathbb{R}^2 \times \mathbb{S}$ using $\Gamma_{\mathbf{u}} := \mathbf{x} \mapsto (\mathbf{x}, \mathbf{u}(\mathbf{x}))$

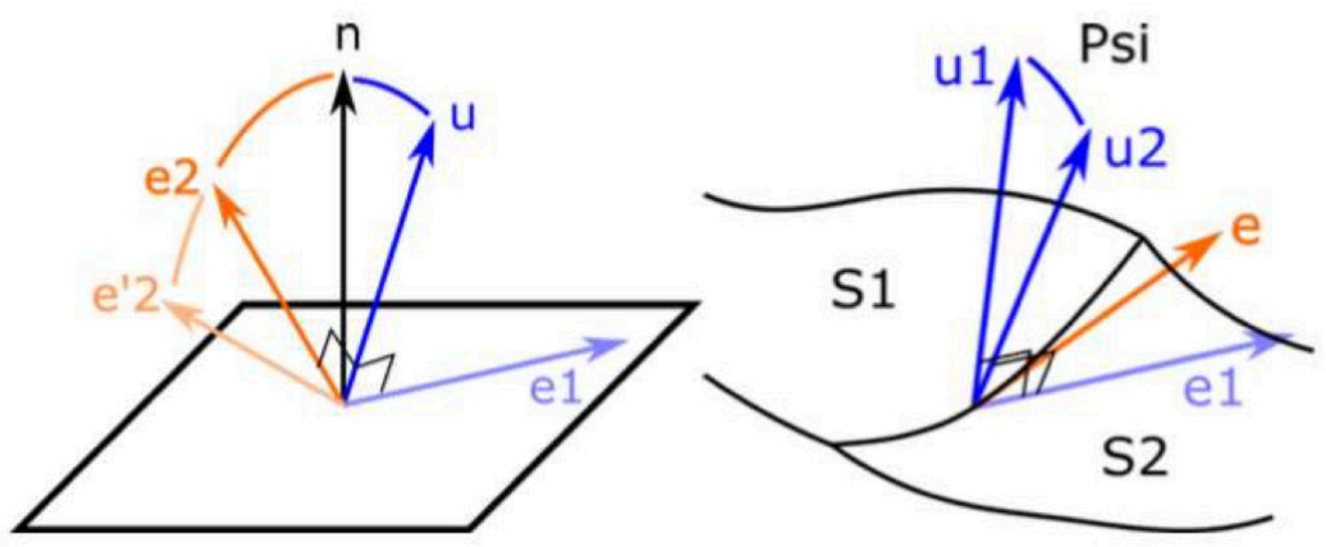
1. Corrected curvature measures

- Given a 2d non-smooth manifold M and $\mathbf{u} : M \to \mathbb{S}^2$ a corrected vector field
- Builds a normal cycle $N(M, \mathbf{u})$ corrected by $\mathbf{u}$
- Corrected curvature measures:
  - Area measure:
    $$\mu^A(B) := \langle N(M,\mathbf{u})_{|B}, \omega^A \rangle = \int_{\Gamma_{\mathbf{u}}(B \cap M)} \omega^A$$
  - Mean curvature measure:
    $$\mu^H(B) := \langle N(M,\mathbf{u})_{|B}, \omega^H \rangle = \int_{\Gamma_{\mathbf{u}}(B \cap M)} \omega^H$$
  - Gaussian curvature measure:
    $$\mu^G(B) := \langle N(M,\mathbf{u})_{|B}, \omega^G \rangle = \int_{\Gamma_{\mathbf{u}}(B \cap M)} \omega^G$$
  - $\omega^A, \omega^H, \omega^G$ are the invariant forms of $\mathbb{R}^3 \times \mathbb{S}^2$



1. Corrected curvature measures (generic case)

Formula:



$\mathbf{e}_1 := \mathbf{u}_1 \times \mathbf{u}_2 / \|\mathbf{u}_1 \times \mathbf{u}_2\|$

$\Psi := \angle \mathbf{u}_1, \mathbf{u}_2$

$\mathbf{e} :=$ tangent to edge

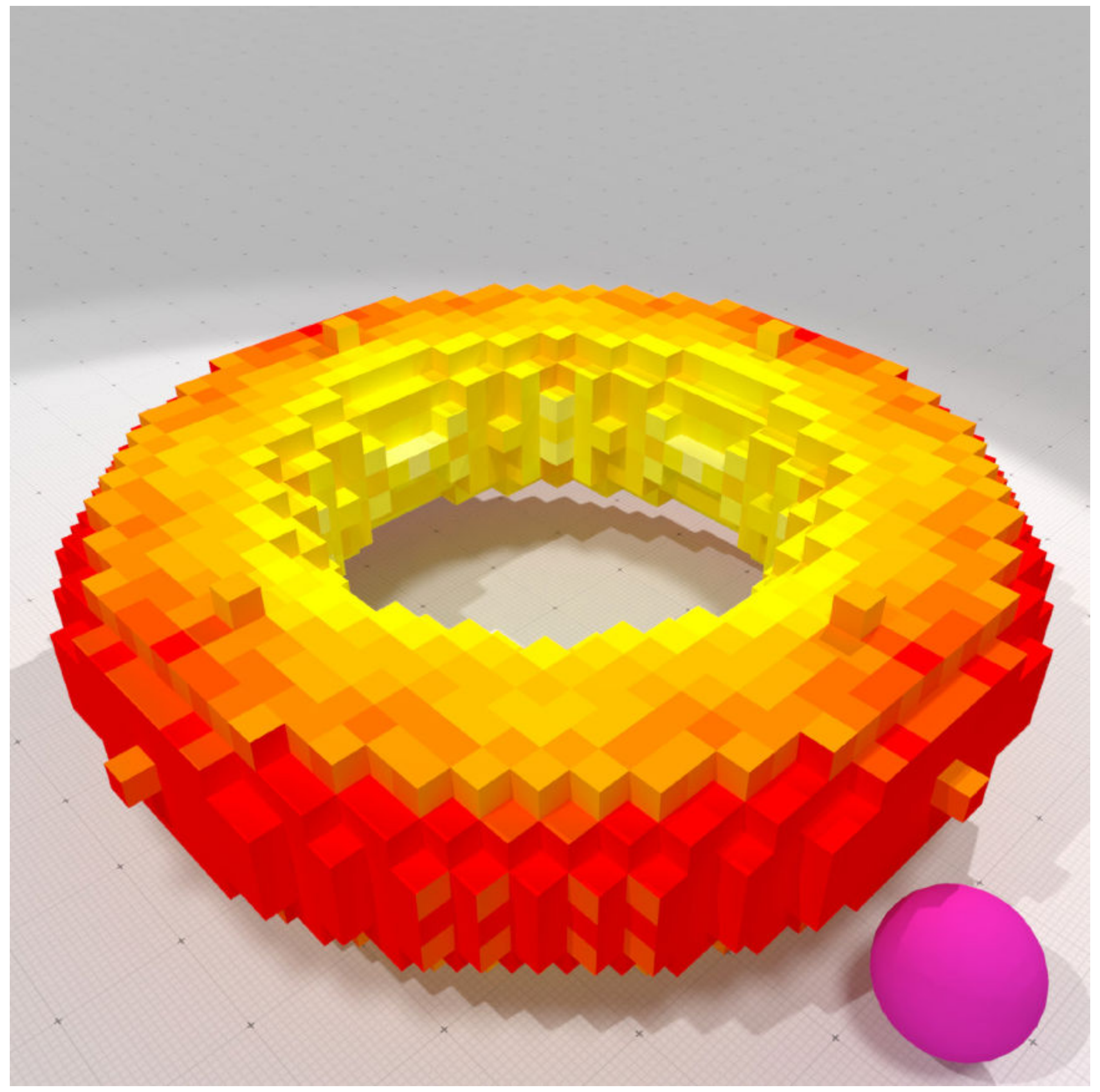
Generic case: S piecewise $C^{1,1}$, $\mathbf{u}$ differentiable per face

$$\mu_0^{S,\mathbf{u}}(B) = \int_{B\cap S} \langle \mathbf{u} \mid \mathbf{n} \rangle \, \mathrm{d}\mathcal{H}^2$$

B is a ball

$$\mu_1^{S,\mathbf{u}}(B) = \int_{B\cap S} \Big(\langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_2 \rangle + \langle \mathbf{u} \mid \mathbf{n} \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_1 \rangle \Big) \, \mathrm{d}\mathcal{H}^2 + \sum_{i \neq j} \int_{B\cap S_{i,j}} \Psi \, \langle \mathbf{e} \mid \mathbf{e}_1 \rangle \, \mathrm{d}\mathcal{H}^1$$

$$\mu_2^{S,\mathbf{u}}(B) = \int_{B\cap S} \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_1 \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_2 \rangle - \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_1 \mid \mathbf{e}_2 \rangle \, \langle \mathrm{d}\mathbf{u} \cdot \mathbf{e}_2' \mid \mathbf{e}_1 \rangle \, \mathrm{d}\mathcal{H}^2 - \sum_{i \neq j} \int_{B\cap S_{i,j}} \tan \frac{\Psi}{2} \langle \mathbf{u}_j + \mathbf{u}_i \mid \mathrm{d}\mathbf{e}_1 \cdot \mathbf{e} \rangle \mathrm{d}\mathcal{H}^1 + \sum_{p \in B \cap \mathrm{Vtx}(S)} \mathrm{AArea}(NC(p, \mathbf{u})).$$

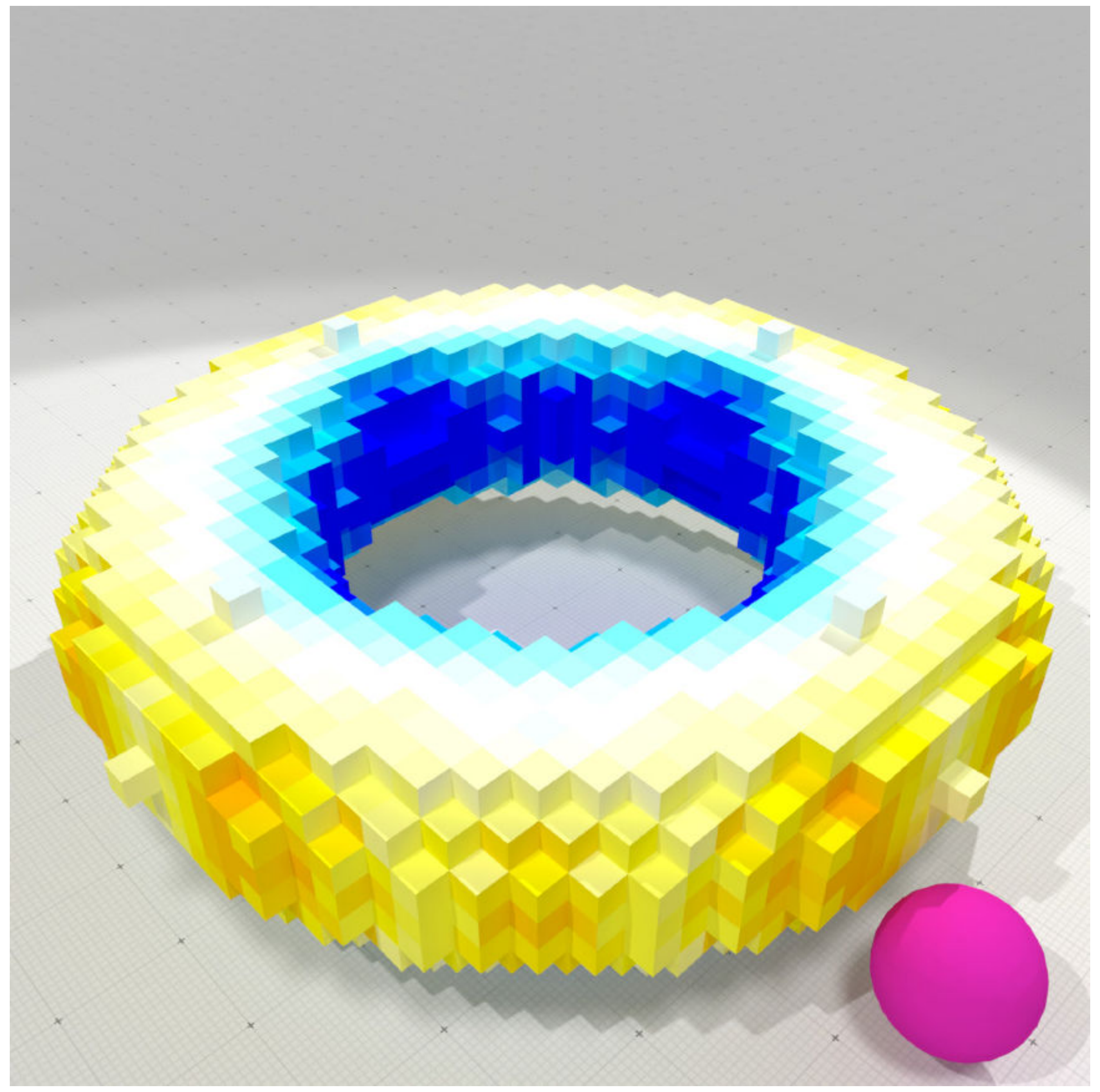


Mean curvature

$$\frac{\mu^H(B)}{\mu^A(B)}$$

Gaussian curvature

$$\frac{\mu^G(B)}{\mu^A(B)}$$



1. Stability of corrected curvature measures

Let S a compact surface of $\mathbb{R}^3$, C^2 smooth, without boundary

Let M a compact mesh without boundary, with $\mathbf{u}$ smooth per face

$$\varepsilon := d_H(S, M) < \mathrm{reach(S)}/2 \quad \text{“position error”}$$

$$\eta := \sup_{\mathbf{x}\in M} \|\mathbf{u}(\mathbf{x}) - \mathbf{n}(\pi_S(\mathbf{x}))\| \quad \text{“normal error”}$$

Then

$$\left| \mu^k_{M,\mathbf{u}}(B) - \mu^k_S(\pi_S(B)) \right| \leq K(\varepsilon + \eta) \quad \text{(for all measures k)}$$

where B is union of triangles of M, and K depends on $\mathrm{Area}(B)$, $\mathrm{Length}(\partial B)$, Lipschitz constant of $\mathbf{u}$, max curvature of S.

2. Interpolated corrected curvature measures

Simple formula on a triangle

Given vertex positions $(\mathbf{x}_l)$ and normals $(\mathbf{u}_l)$, triangle $\tau = (i, j, k)$

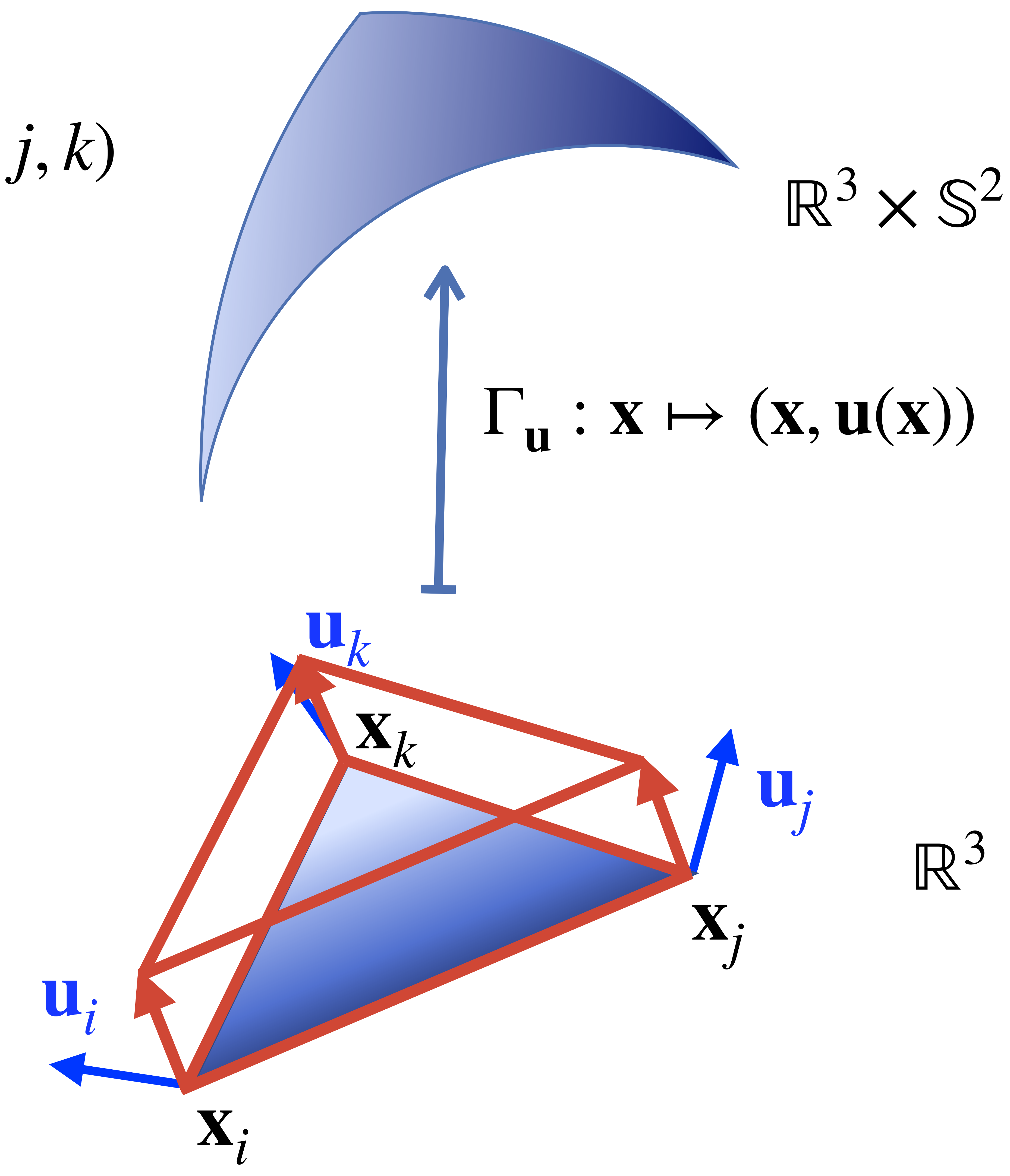
Area measure $\mu^A(\tau) = \int_\tau \Gamma_\mathbf{u}^* \omega^A$

$$= \int_0^1 \int_0^{1-t} \det\left(\mathbf{u}, \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial t}\right) \mathrm{d}s\mathrm{d}t$$

$$= \frac{1}{2} \langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i) \rangle$$

with $\bar{\mathbf{u}} := (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k)/3$

(Using linear interpolation of positions and normals)



2. Interpolated corrected curvature measures

Simple formula on a triangle

Mean curvature measure

$$\mu_{\mathbf{u}}^{H}(\tau) = \int_{\tau} \Gamma_{\mathbf{u}}^{*}\omega^{H}$$
$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_k - \mathbf{u}_j) \times \mathbf{x}_i + (\mathbf{u}_i - \mathbf{u}_k) \times \mathbf{x}_j + (\mathbf{u}_j - \mathbf{u}_i) \times \mathbf{x}_k \rangle$$

Lipschitz-Killing differential forms (Mean and Gaussian forms)

Gaussian curvature measure

$$\mu_{\mathbf{u}}^{G}(\tau) = \int_{\tau} \Gamma_{\mathbf{u}}^{*}\omega^{G}$$
$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_j - \mathbf{u}_i) \times (\mathbf{u}_k - \mathbf{u}_i) \rangle$$

Pointwise curvatures

Measures are extended to arbitrary balls in a canonic way

$$\mu^{(k)}(B) := \sum_{\tau} \mu^{(k)}(\tau) \frac{\mathrm{Area}(\tau \cap B)}{\mathrm{Area}(\tau)}$$

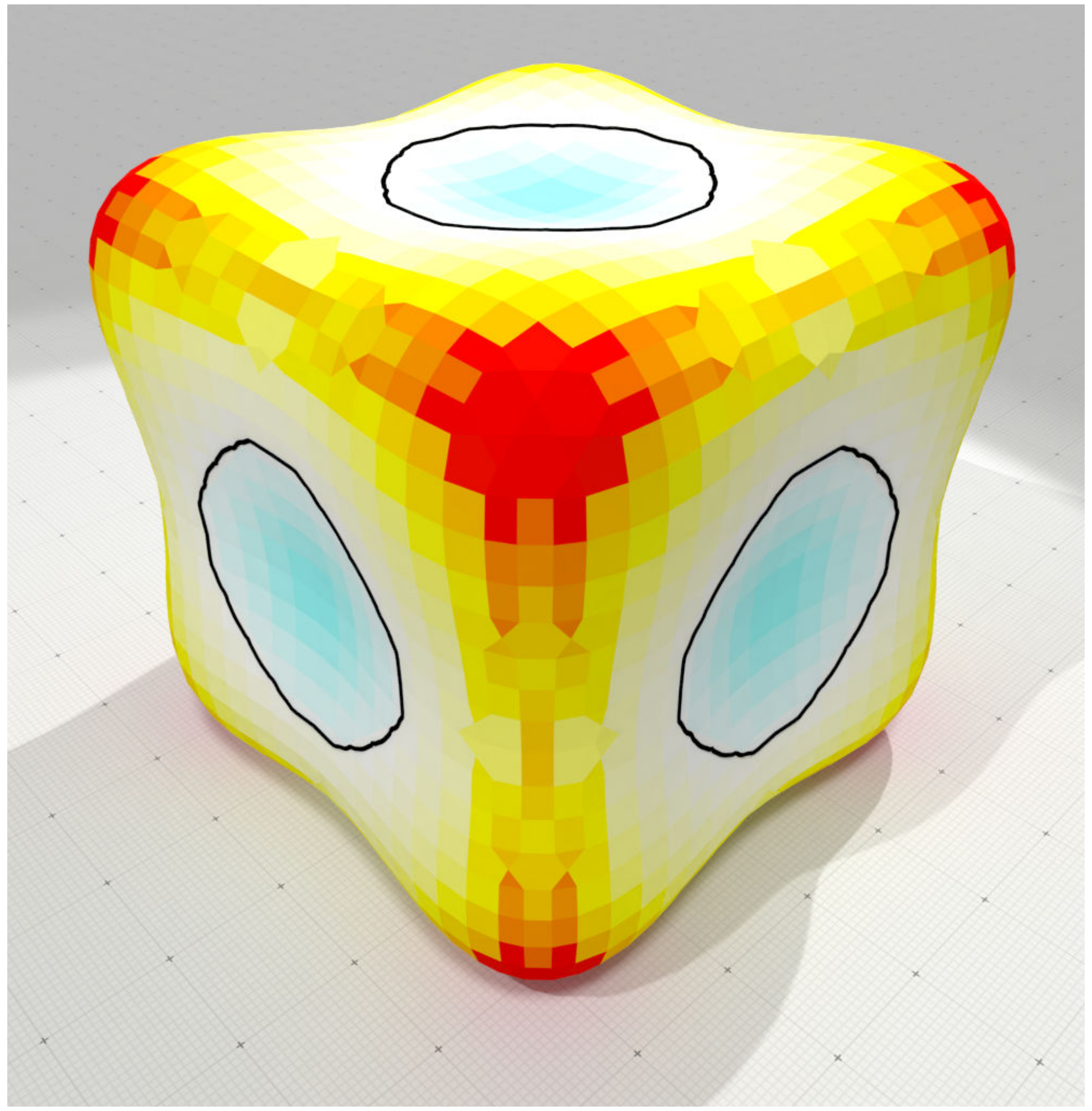
for any ball $B(\mathbf{y}, r)$

Mean curvature

$$H^{CNC}(\mathbf{y}, r) := \frac{\mu^{H}(B \cap S)}{\mu^{A}(B \cap S)}$$

Gaussian curvature

$$G^{CNC}(\mathbf{y}, r) := \frac{\mu^{G}(B \cap S)}{\mu^{A}(B \cap S)}$$



Mean curvature = 0
Gaussian curvature = 0

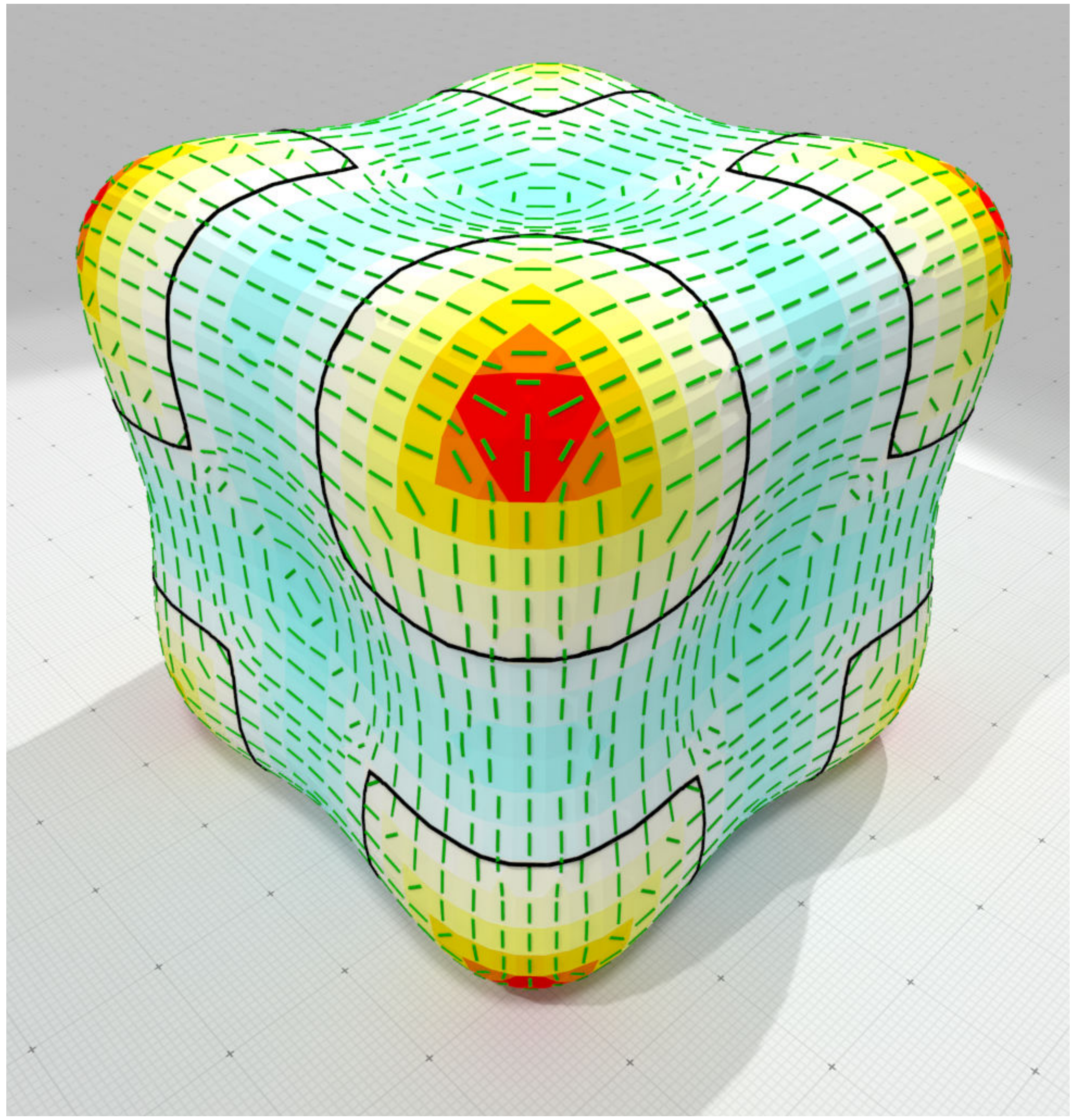
Anisotropic measure, principal curvatures and directions

For vectors $\mathbf{X}$ and $\mathbf{Y}$, the **anisotropic form** is

$$\omega^{(\mathbf{X},\mathbf{Y})}_{(\mathbf{x},\mathbf{u})}(\boldsymbol{\xi},\boldsymbol{\nu}) = \det(\mathbf{u},\mathbf{X},\boldsymbol{\xi}_p)\langle \mathbf{Y} \mid \boldsymbol{\nu}_n\rangle - \det(\mathbf{u},\mathbf{X},\boldsymbol{\nu}_p)\langle \mathbf{Y} \mid \boldsymbol{\xi}_n\rangle$$

Anisotropic measure is

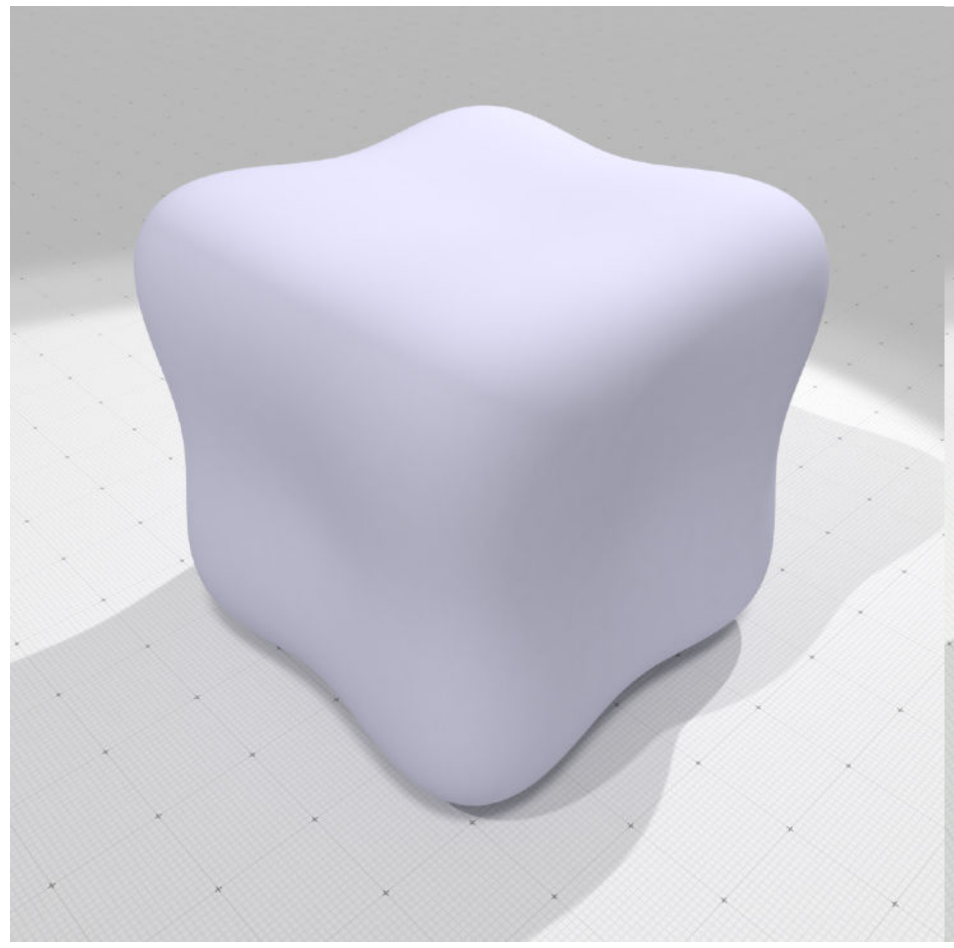
$$\mu^{(\mathbf{X},\mathbf{Y})}(\tau) = \frac{1}{2}\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_k - \mathbf{u}_i\rangle \mathbf{X} \times (\mathbf{x}_j - \mathbf{x}_i)\rangle - \frac{1}{2}\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_j - \mathbf{u}_i\rangle \mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i)\rangle$$



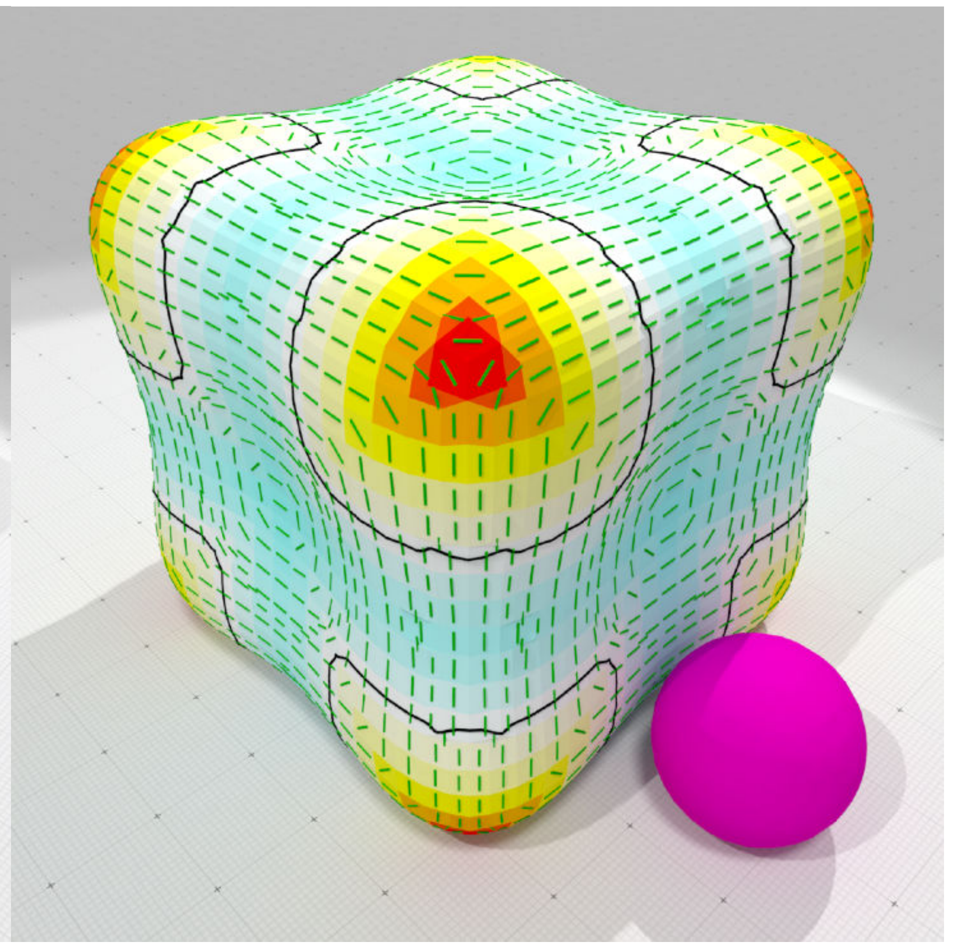
1st principal curvature and direction

Converge to 2nd Fund. Form for tangents $\mathbf{X},\mathbf{Y}$

Stability of 1st principal curvatures error measured as $e := \|\kappa_1 - \hat{\kappa}_1\|_\infty$

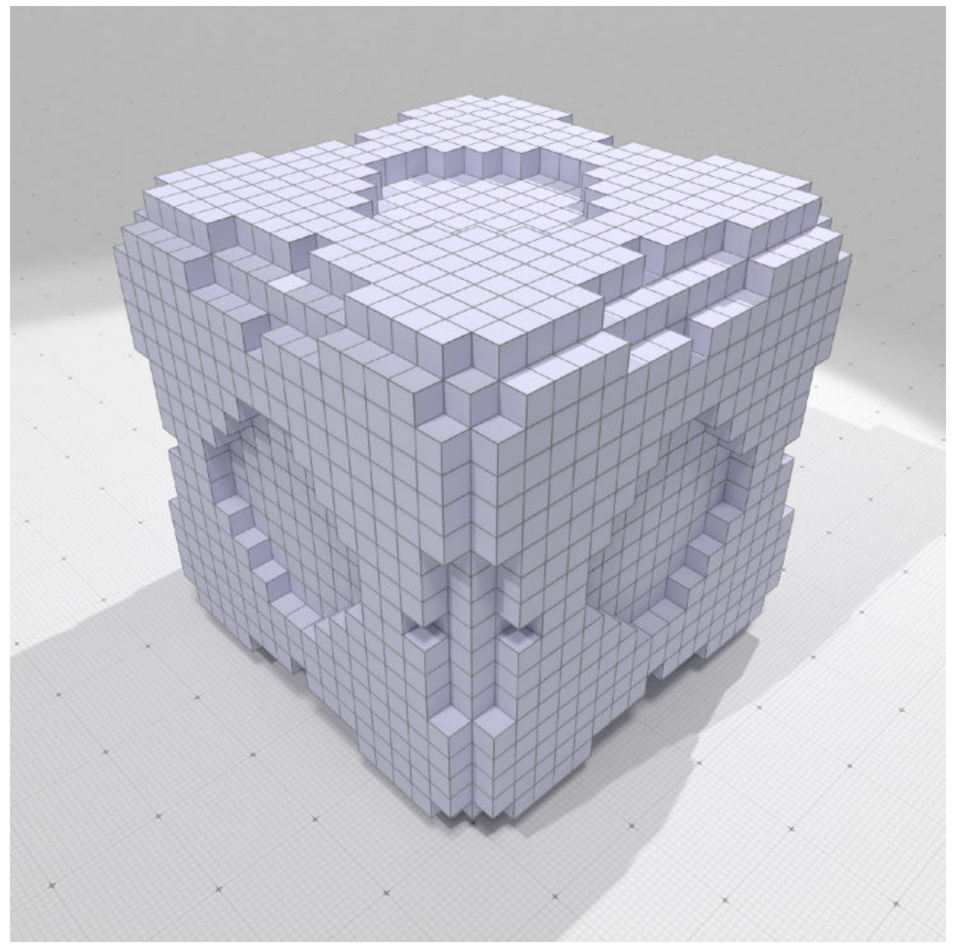


Smooth surface

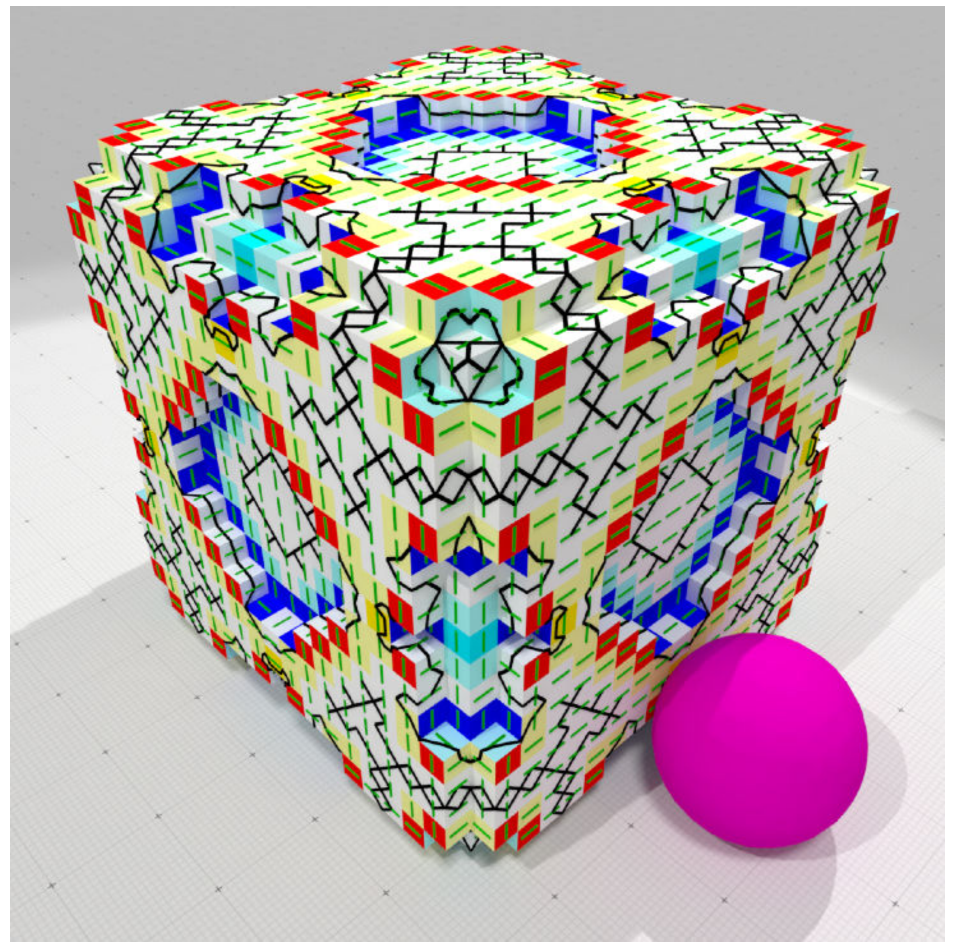


Objective

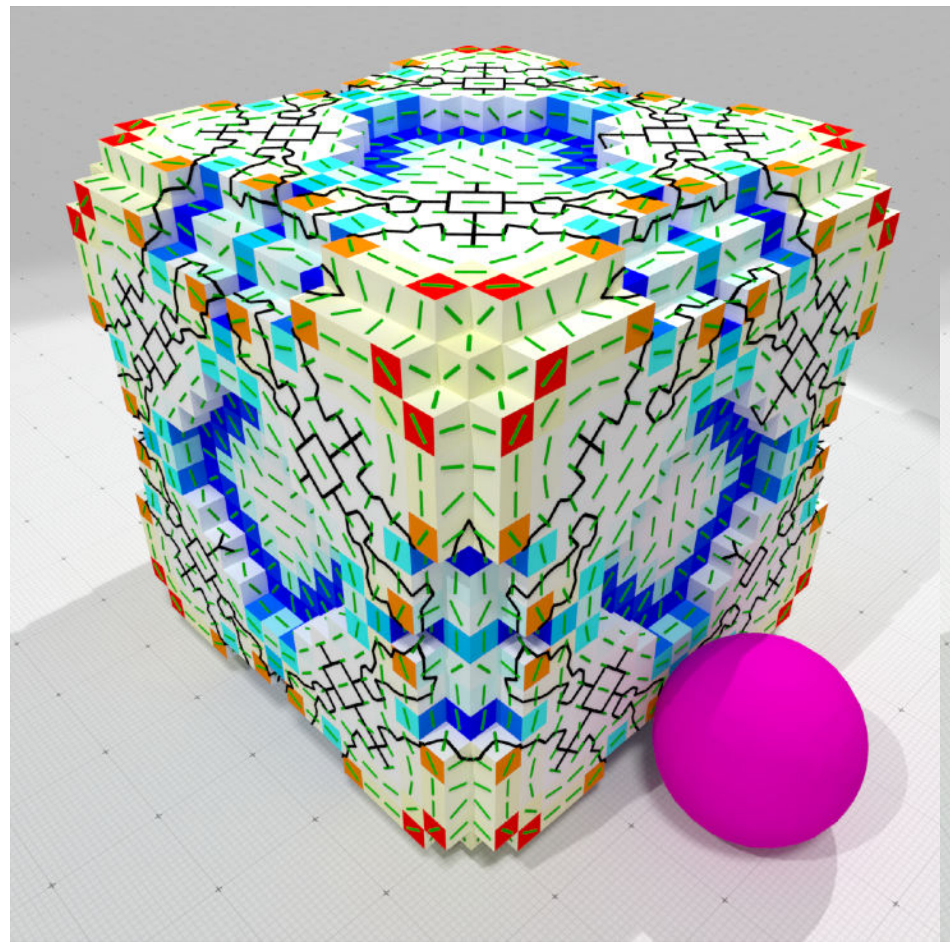
Input data



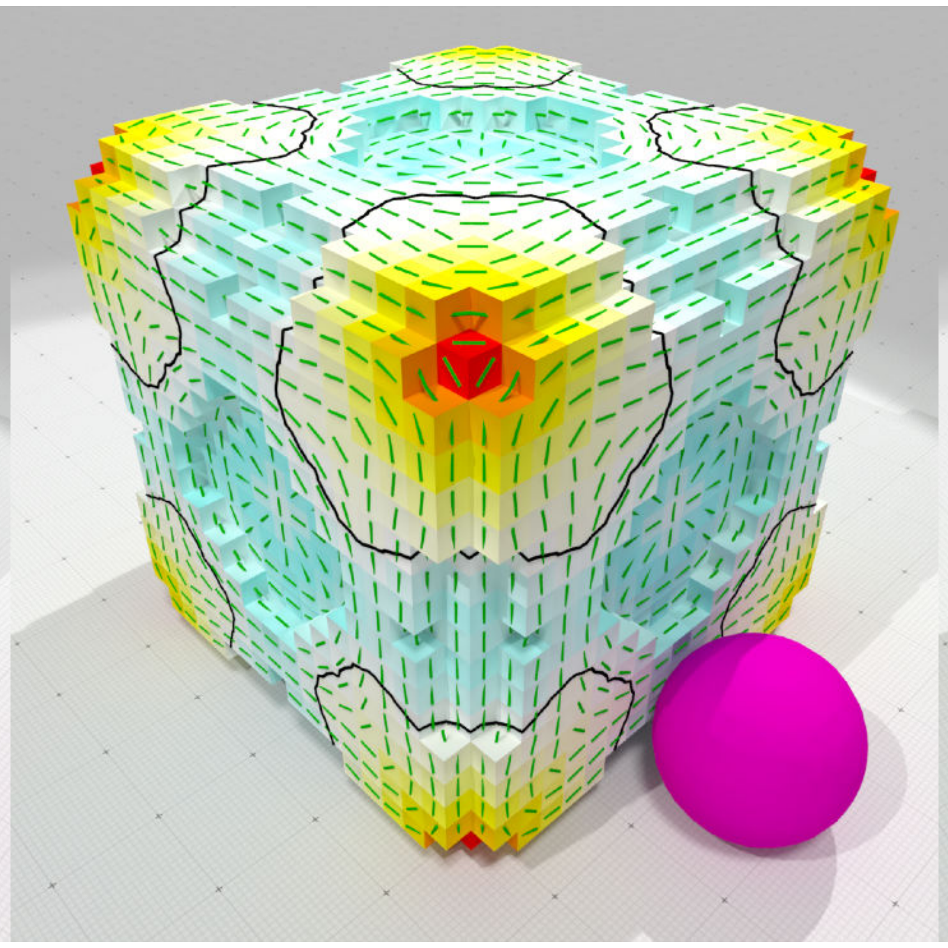
Interpolated Corrected Normal Current



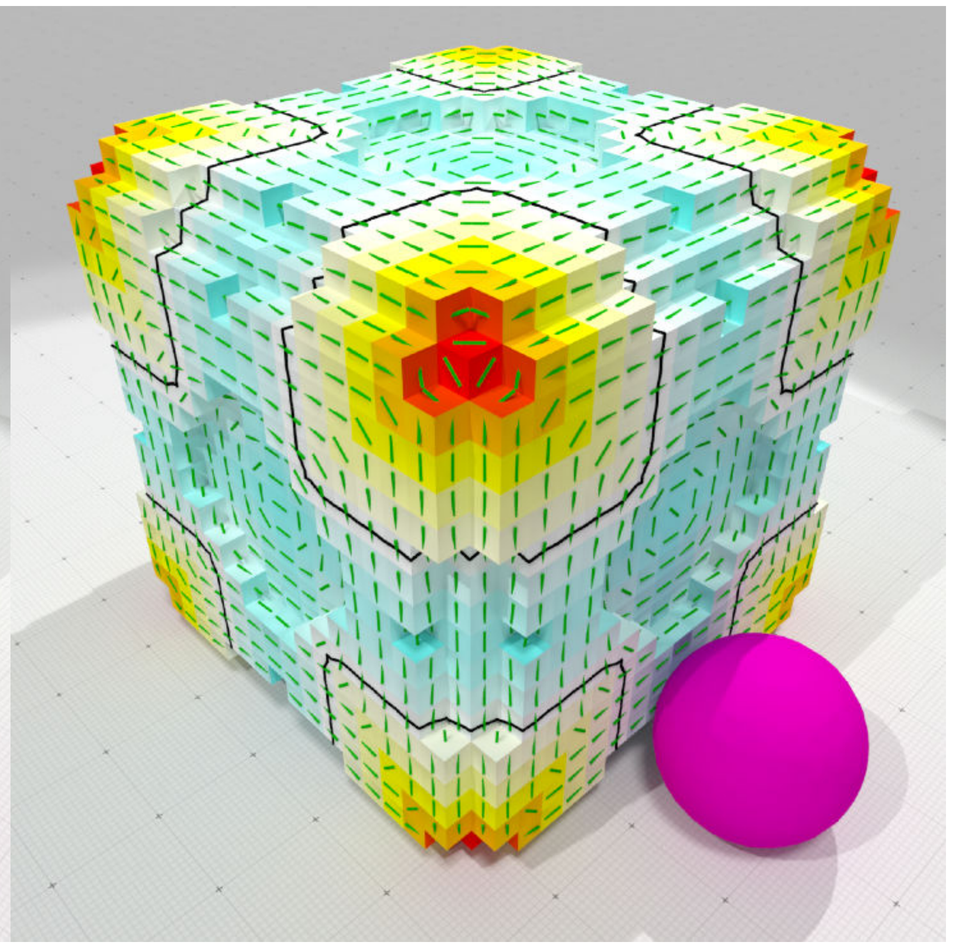
Normal cycle, $e = 1.473$



Naive normals, $e = 0.519$

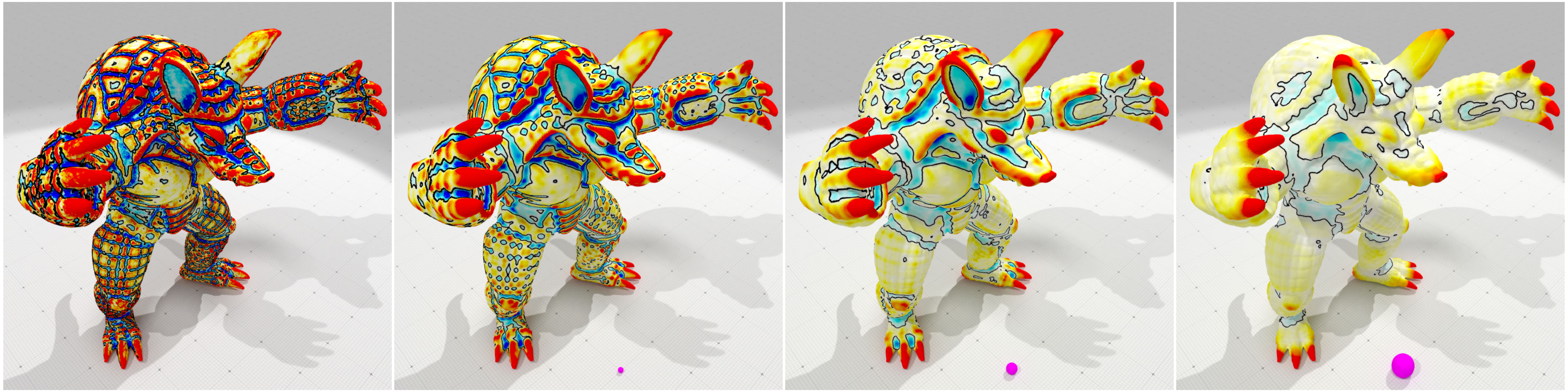


Estimated normals, $e = 0.076$



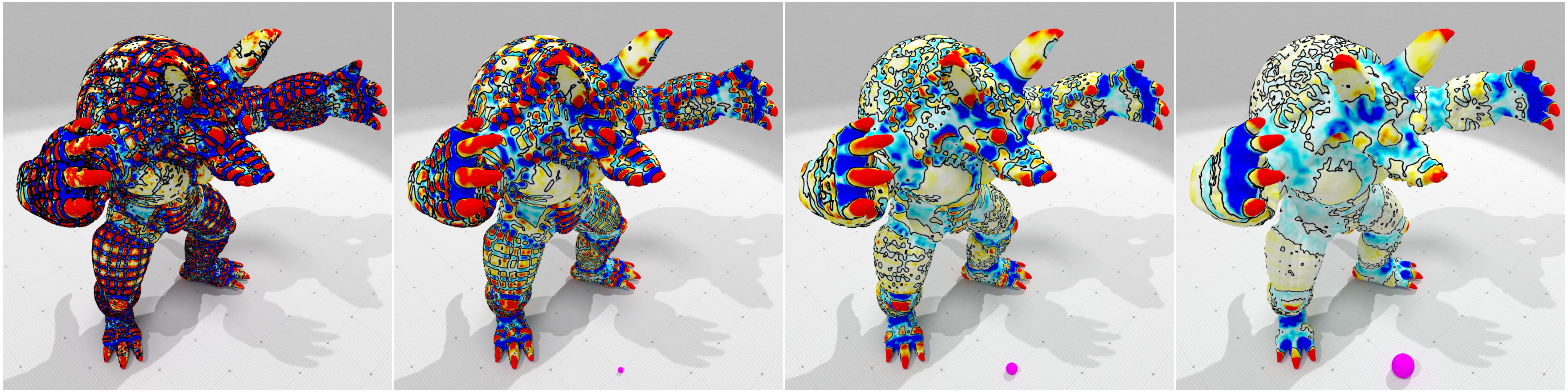
True normals, $e = 0.045$

Scale space continuum for curvatures



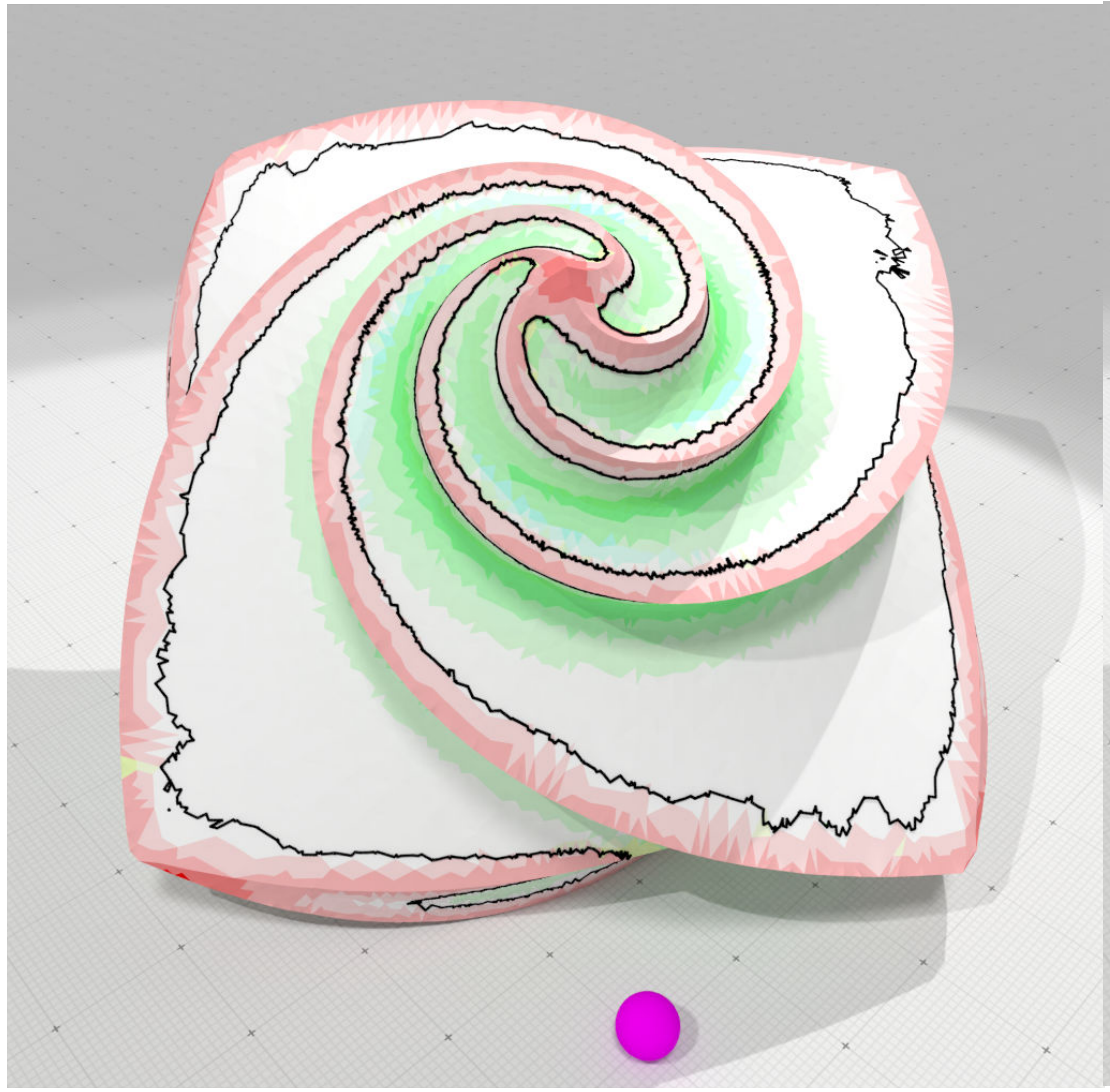
CNC Mean curvature estimation

Scale space continuum for curvatures

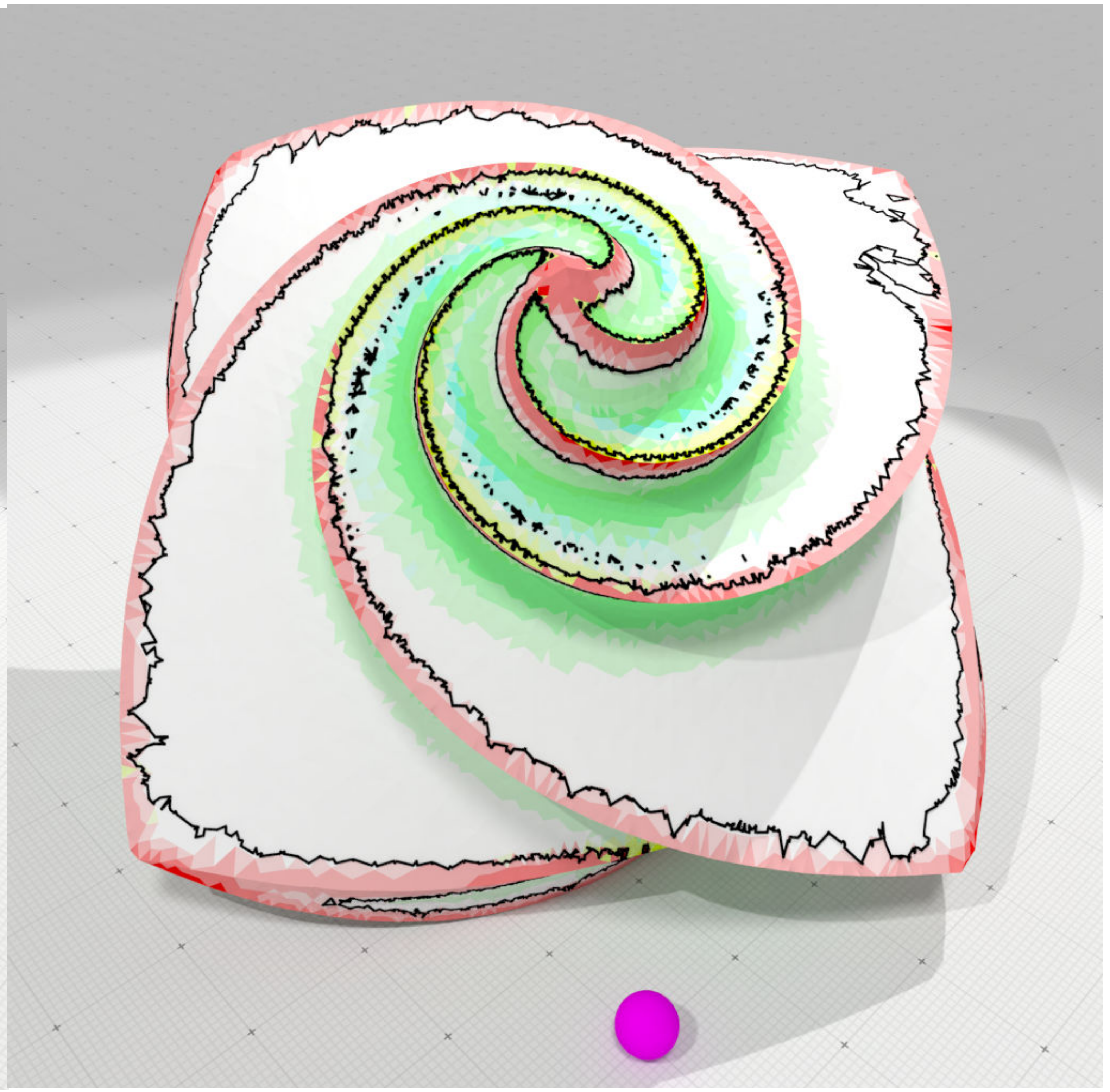


CNC Gaussian curvature estimation

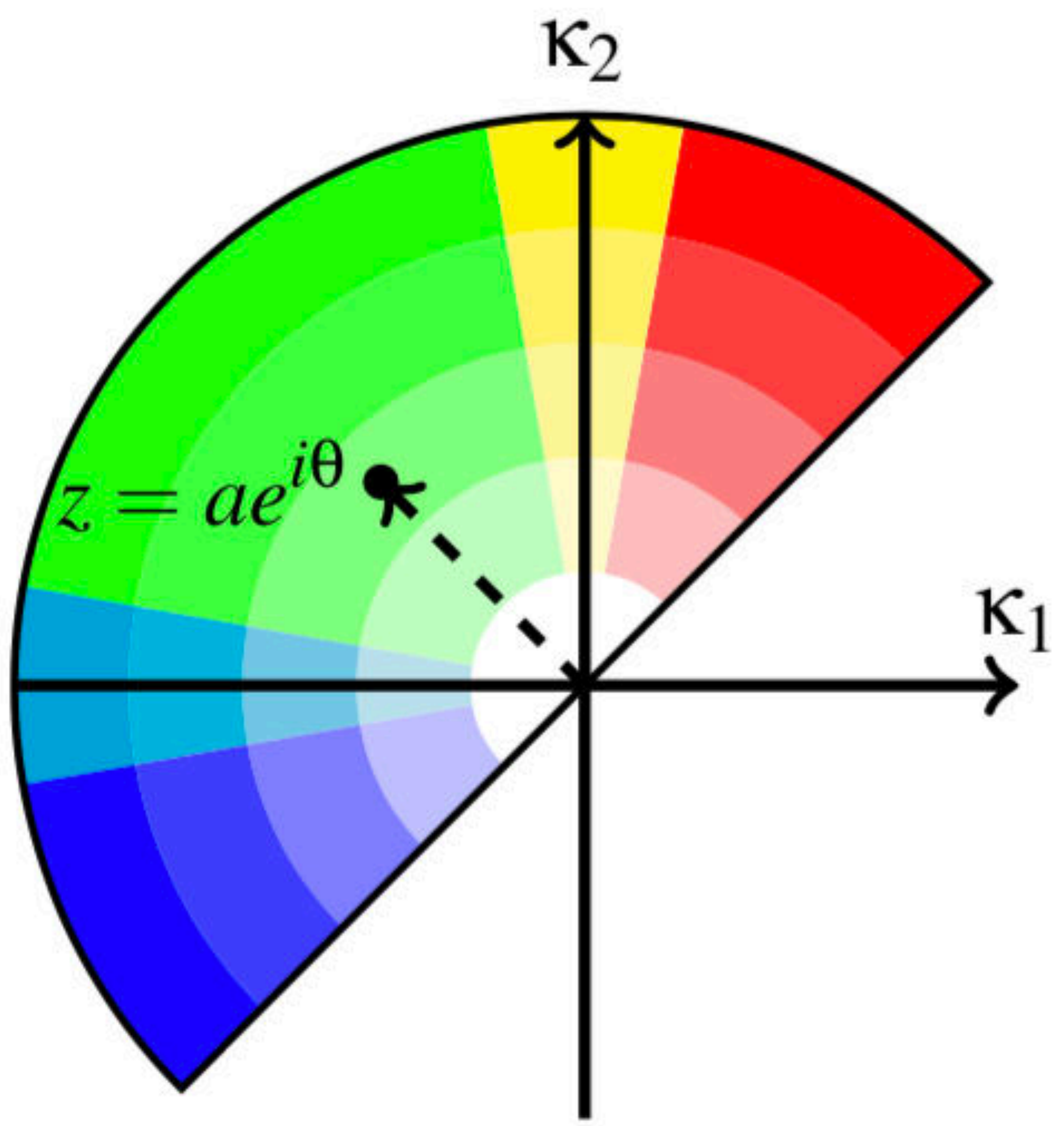
Convex / Concave parts



Interpolated CNC (r=0.2)



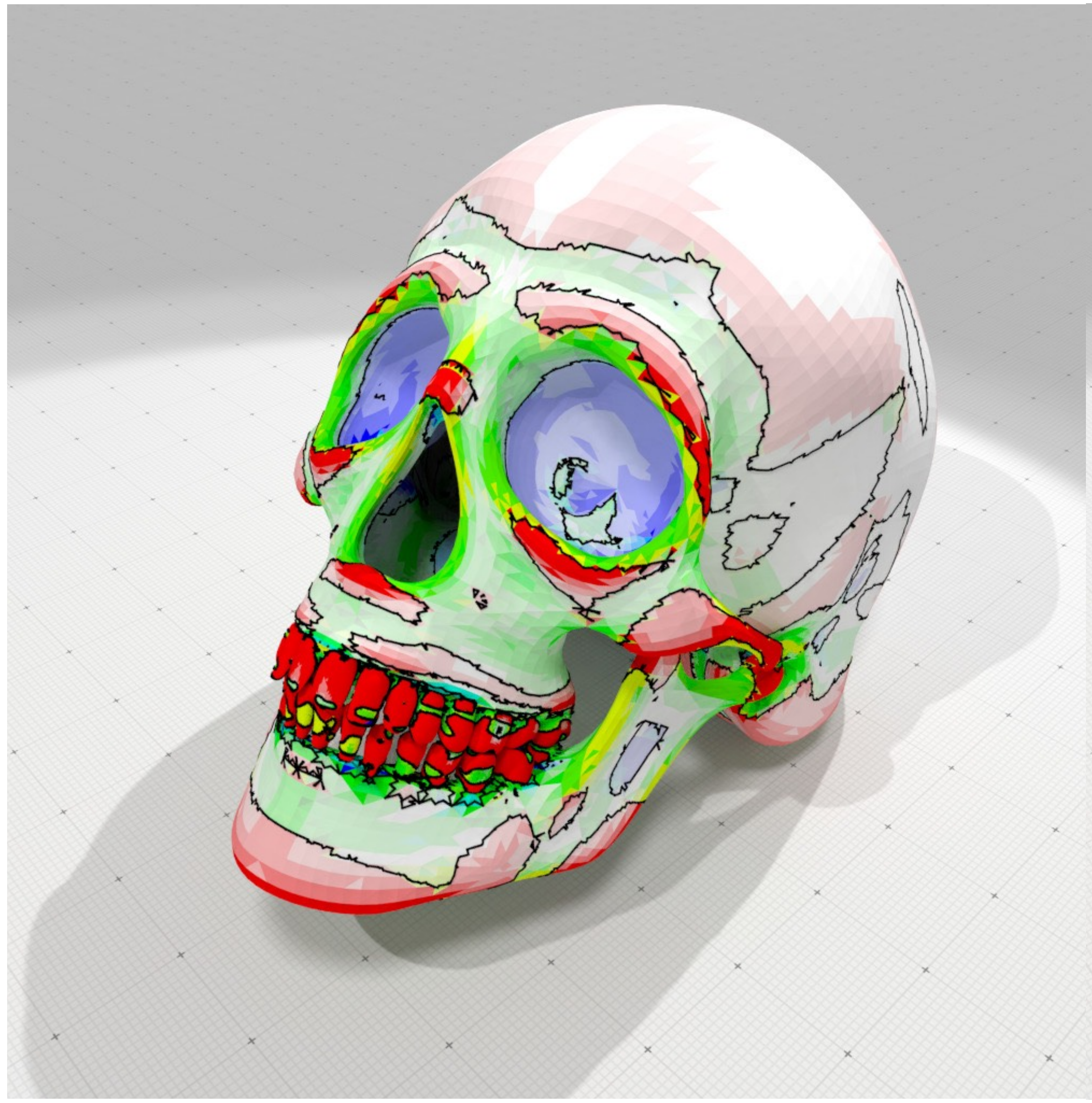
Normal Cycle (r=0.2)



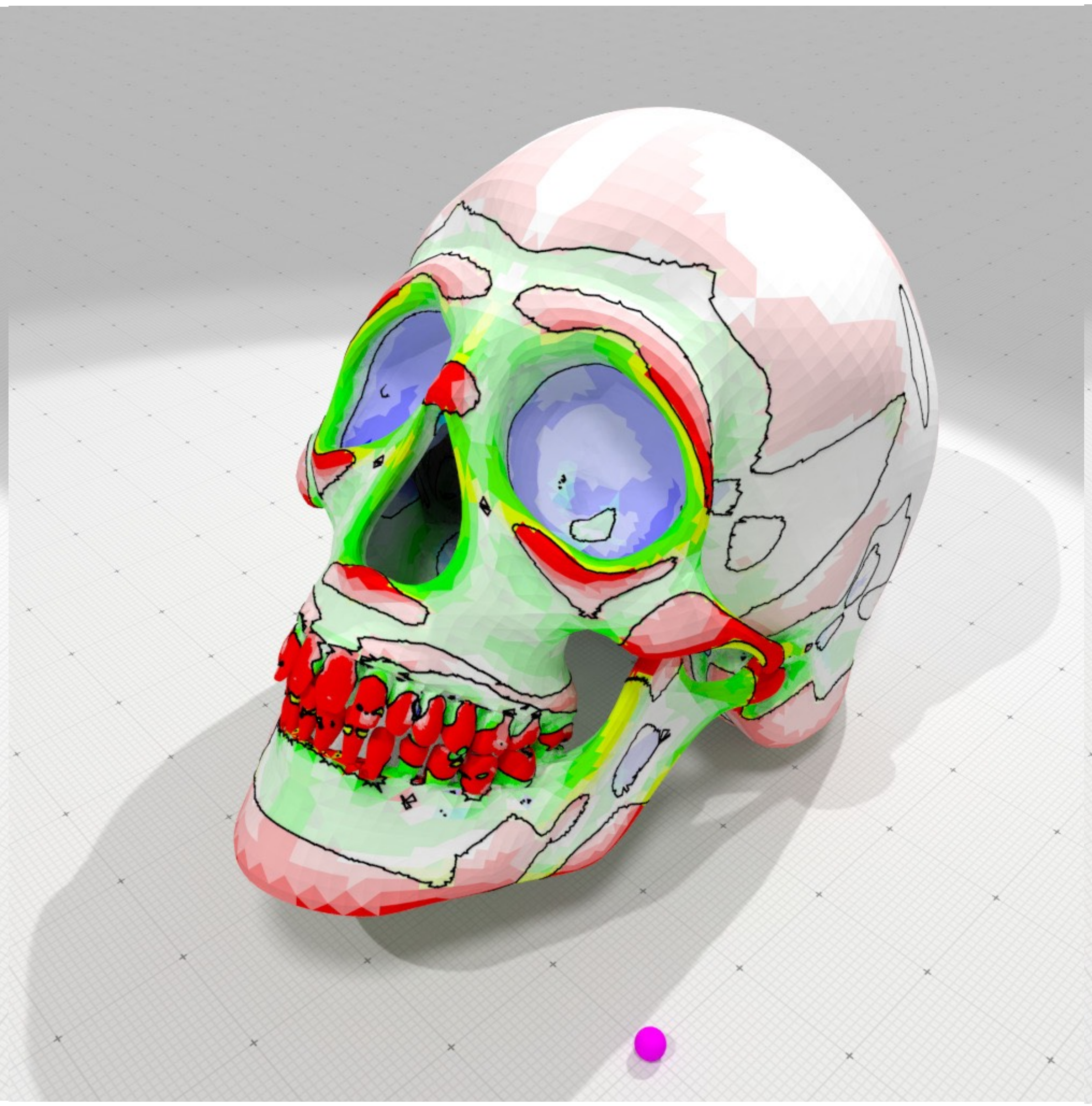
$$M = \max(|\kappa_1|, |\kappa_2|)$$
$$a = \min(M/\kappa_{\max}, 1.0)$$
$$\theta = \mathrm{atan2}(\kappa_2, \kappa_1)$$

Robustness to noise

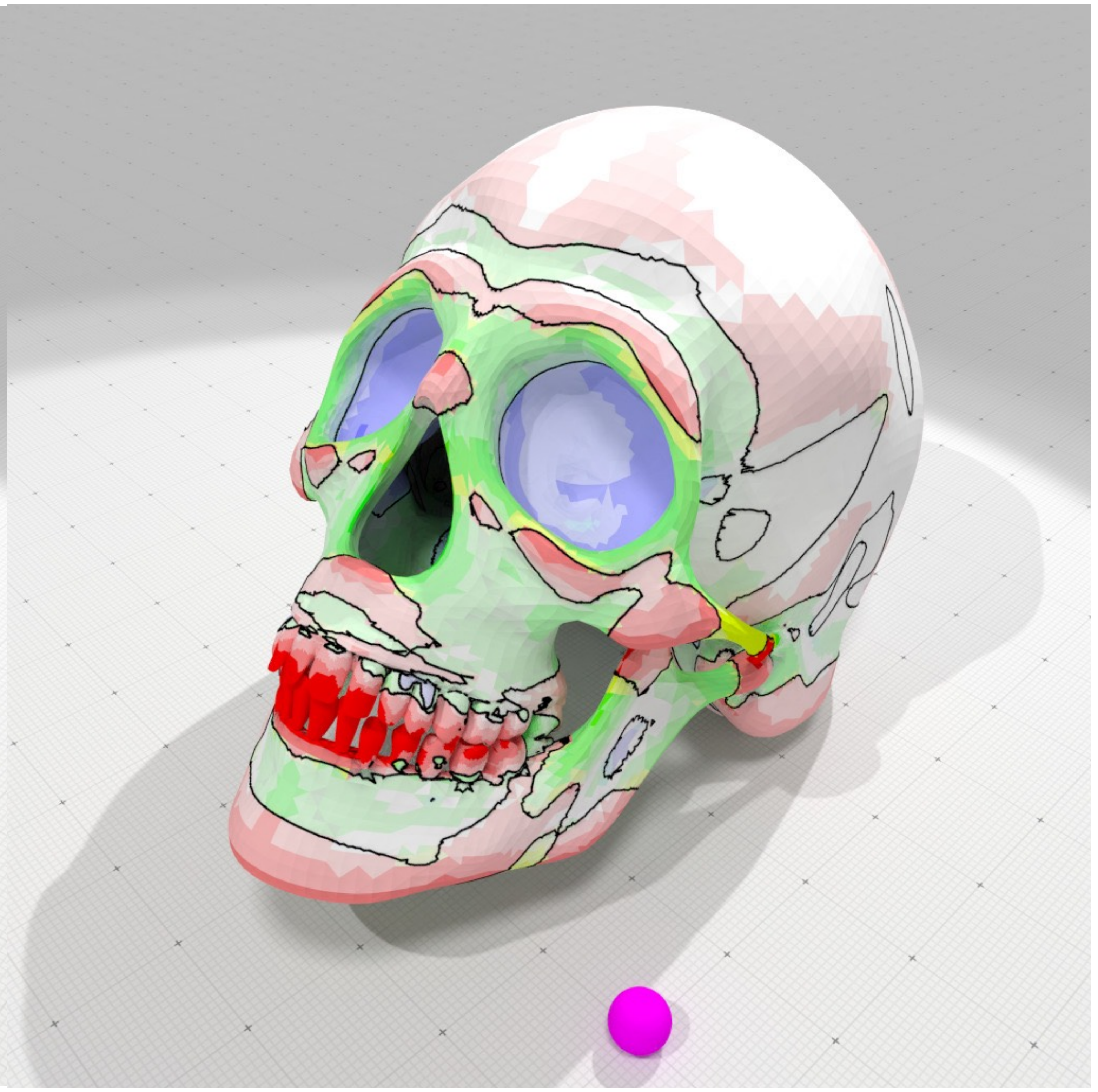
(normal field $\mathbf{u}$ is naive normals $\mathbf{n}$ averaged 4 times)



CNC, $r = 0$, noise = 0%



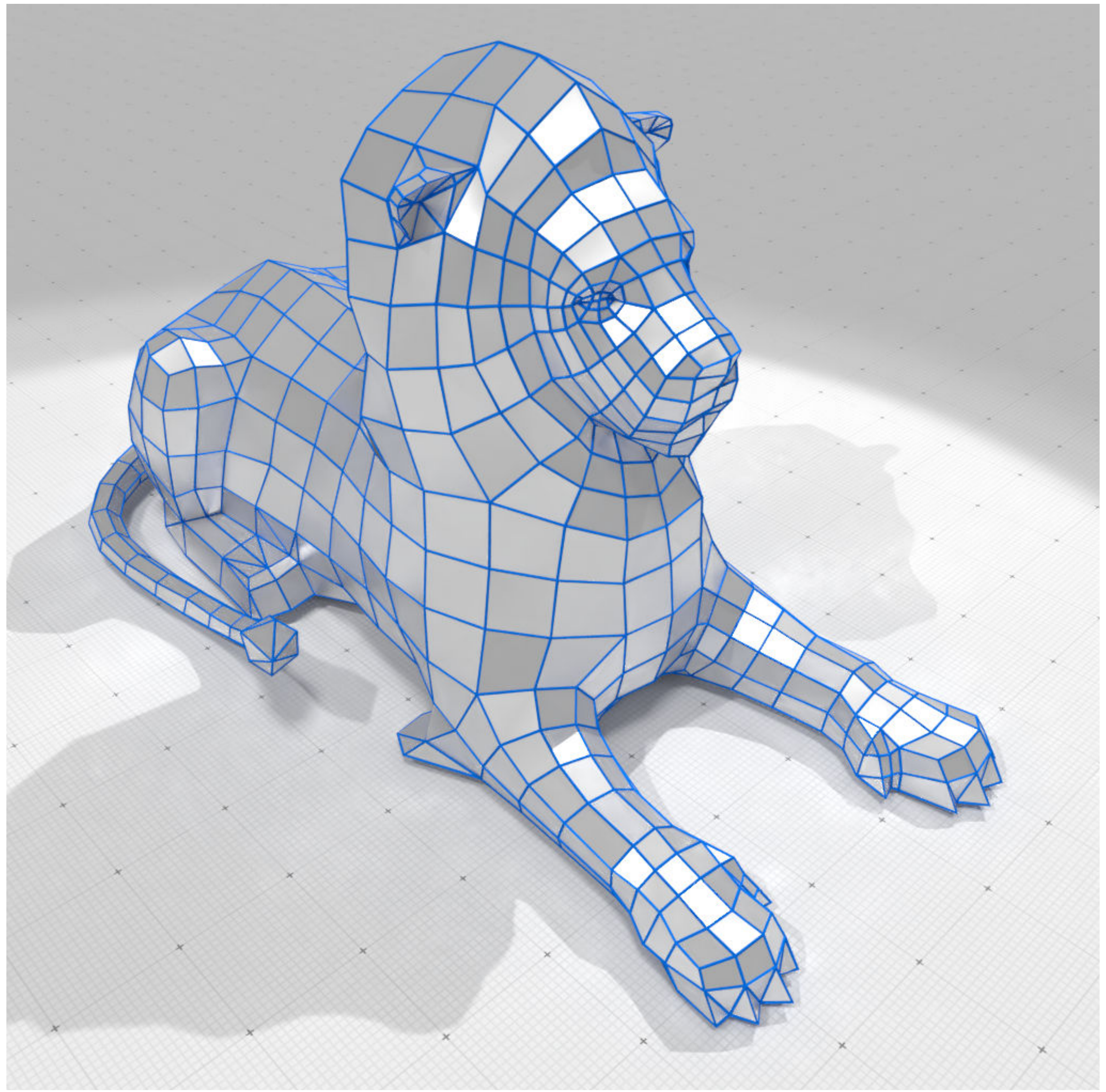
CNC, $r = 0.1$, noise = 0%



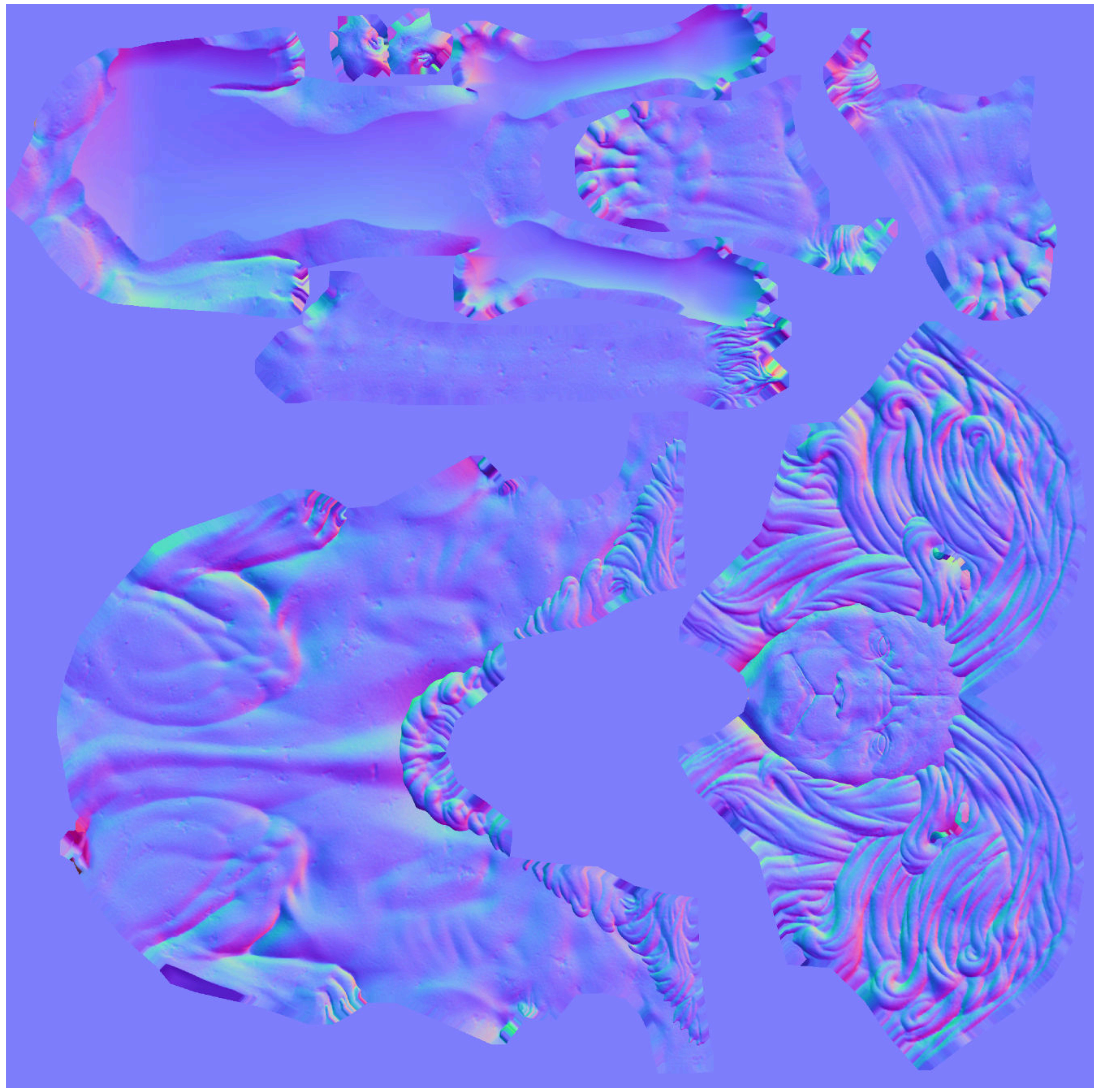
CNC, $r = 0.2$, noise = 0%

Separated positions $\mathbf{x}$ and normals $\mathbf{u}$

$\Rightarrow$ curvatures with normals maps



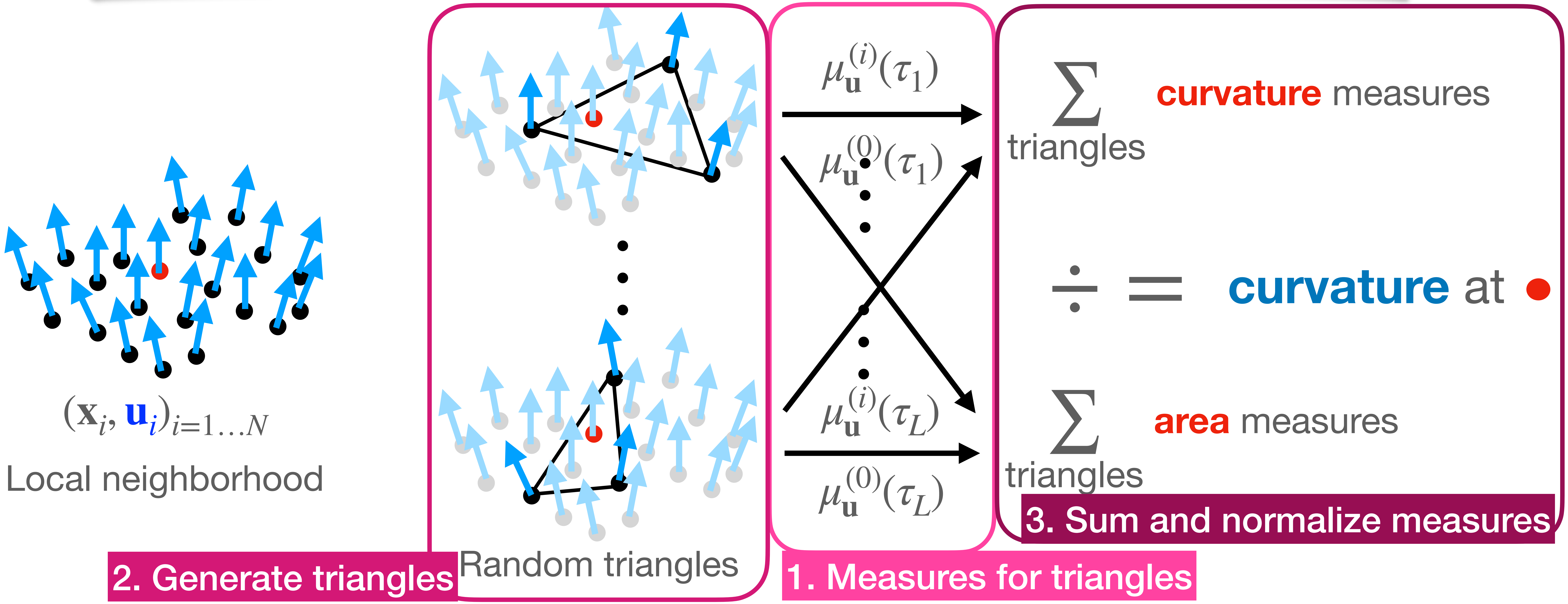
Mesh geometry



Normal map

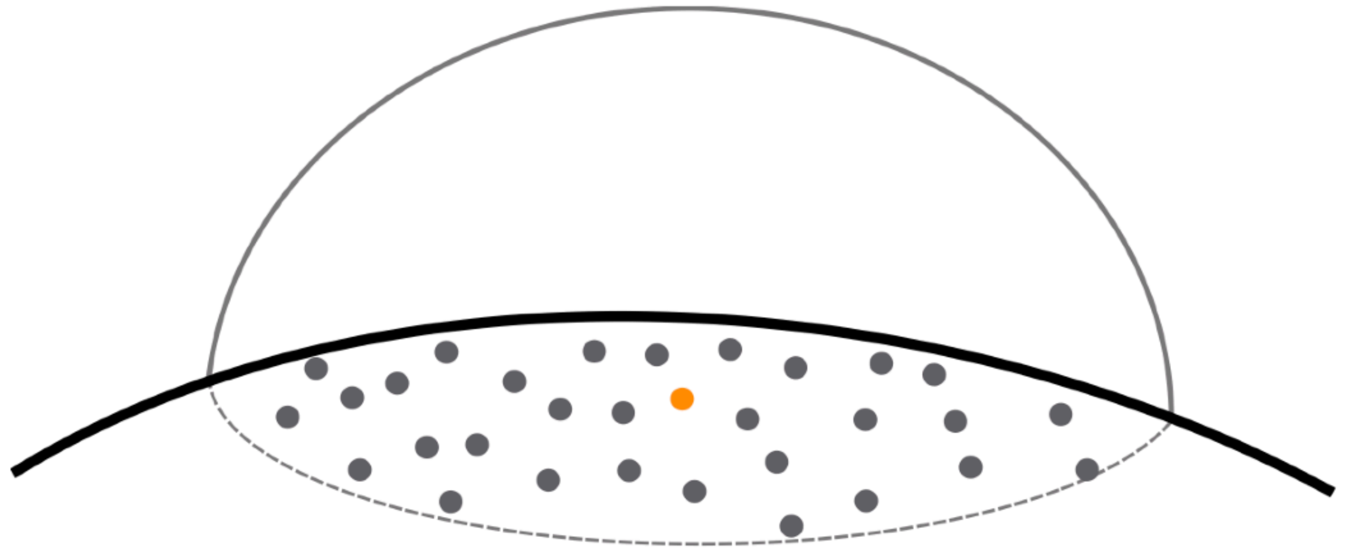
3. Extend a surface theory to oriented point clouds

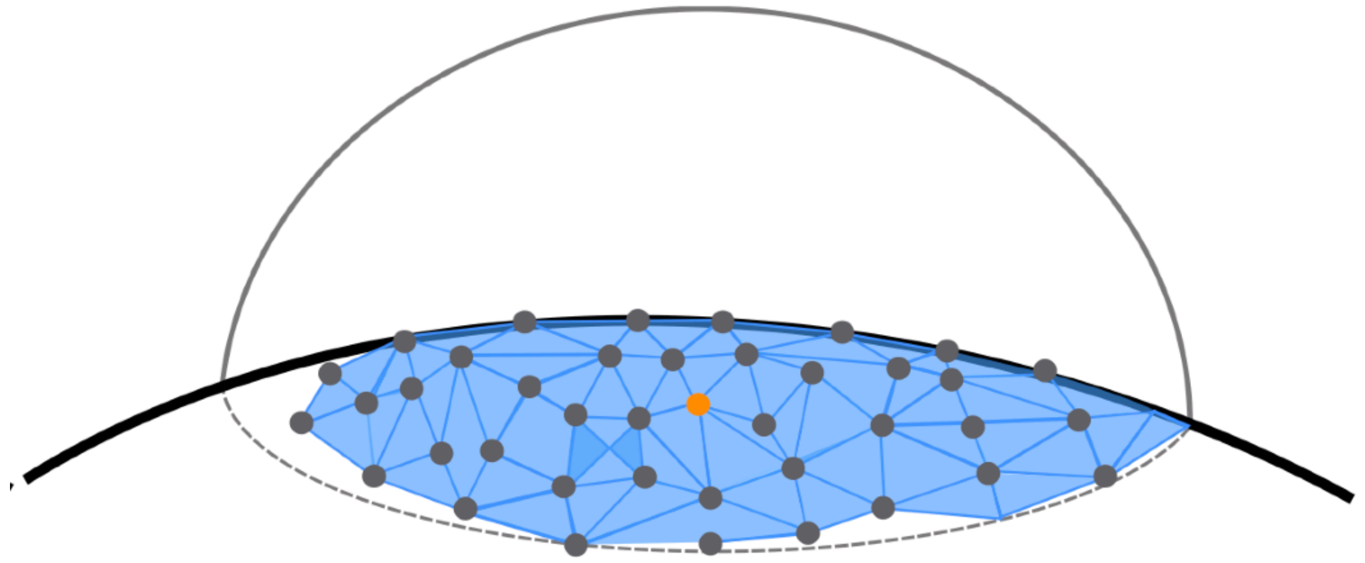
Key idea:
measures do not need consistent mesh topology



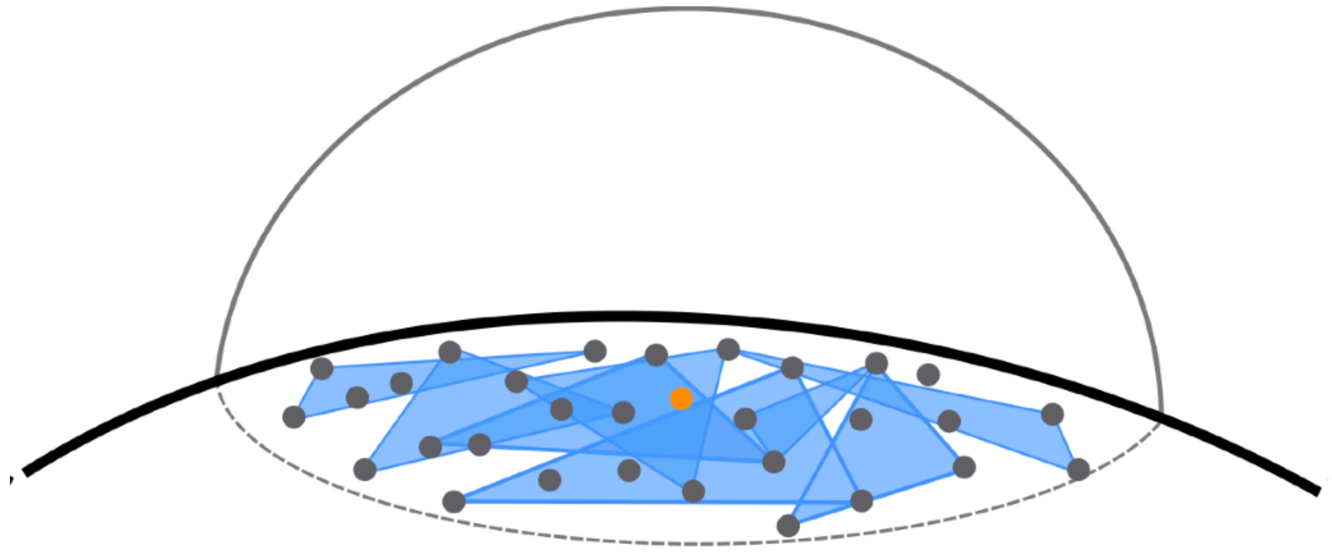
3.2 Per point $\mathbf{x}$, generate locally random triangles

- Neighbours of $\mathbf{x}$: either K nearest or within Ball($\mathbf{x}$, δ)
- Choose a strategy to build L triangles within

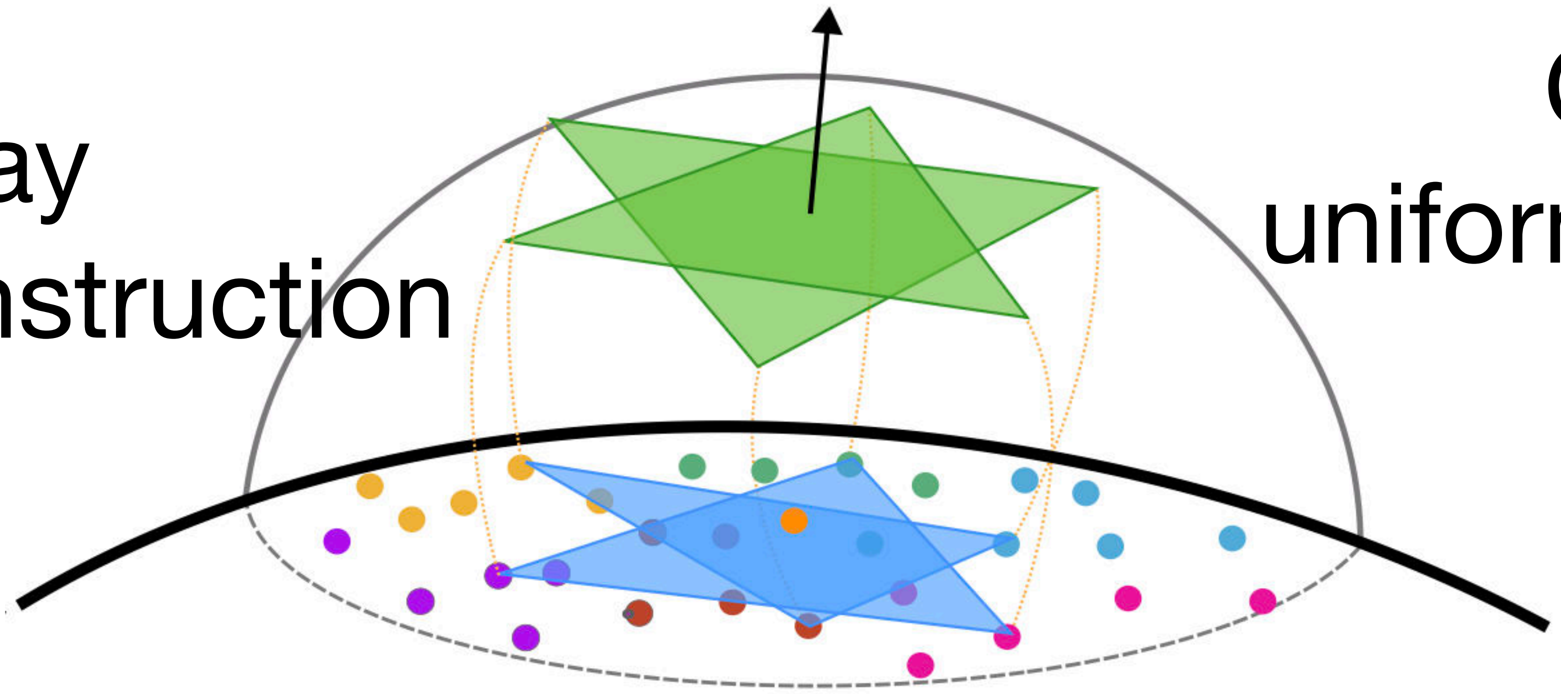




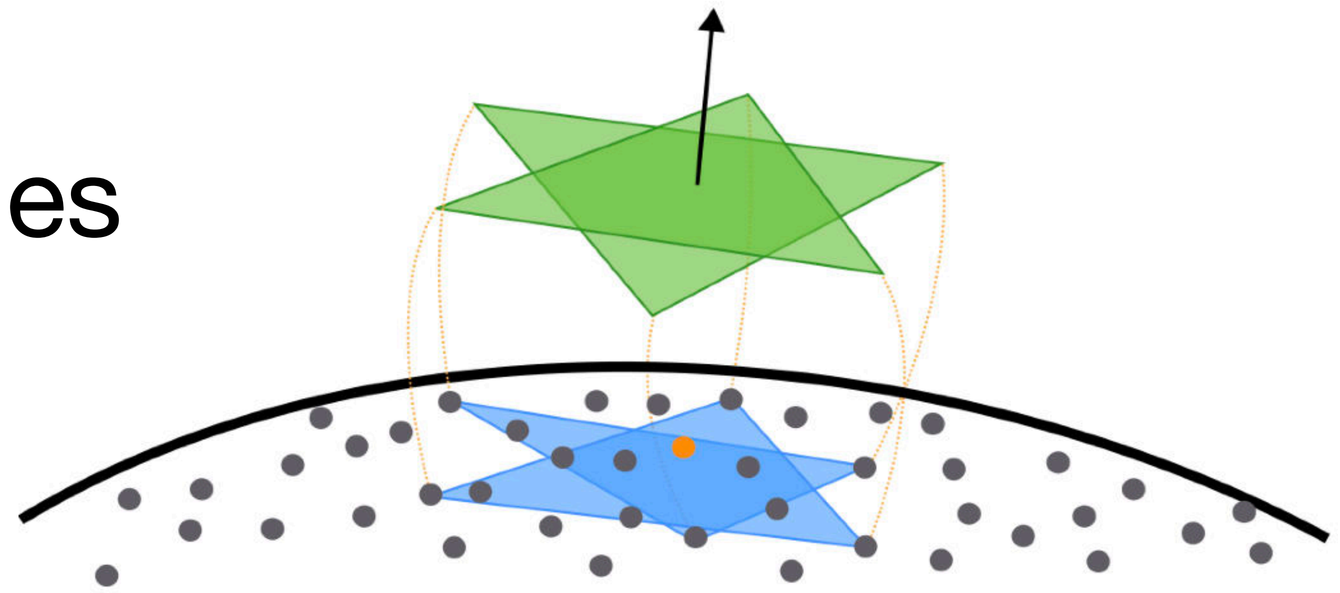
CNC-Delaunay
local Delaunay reconstruction



CNC-Uniform
uniform random triangles

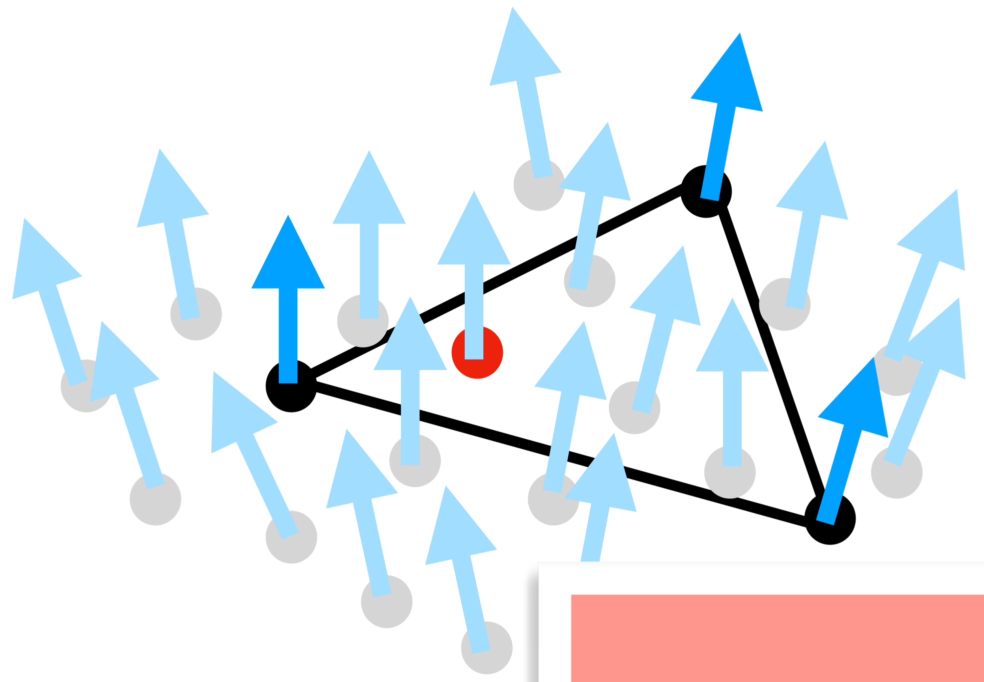


CNC-AvgHexagram
2 triangles with average nearest points



CNC-Hexagram
2 triangles with nearest points

3. Sum up results and normalise curvature measures



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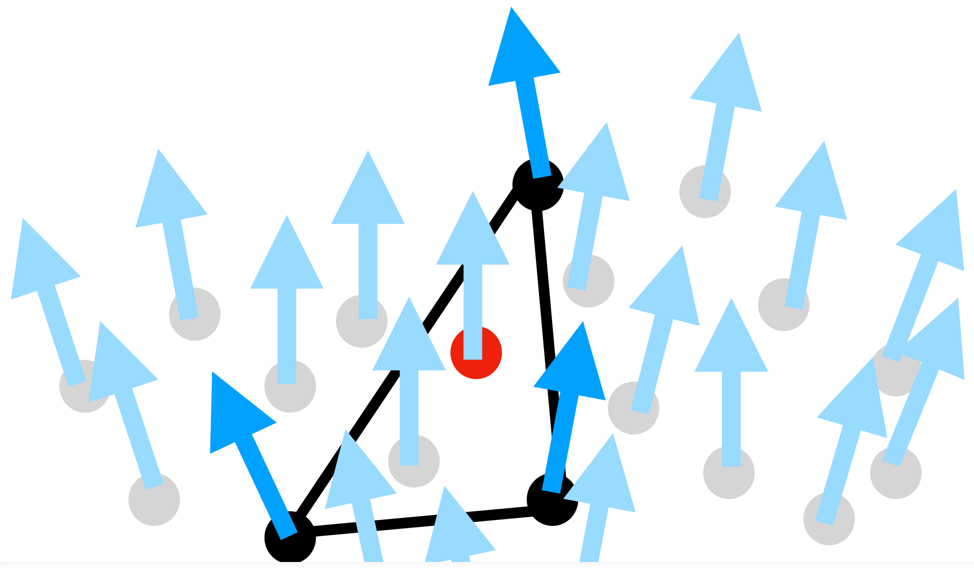


Triangles oriented such that $\mu_{\mathbf{u}}^{A}(\hat{\tau}_l) \geq 0$

$\mu_{\mathbf{u}}^{A}(\hat{\tau}_1) \geq 0, \mu_{\mathbf{u}}$

Lightweight computations:

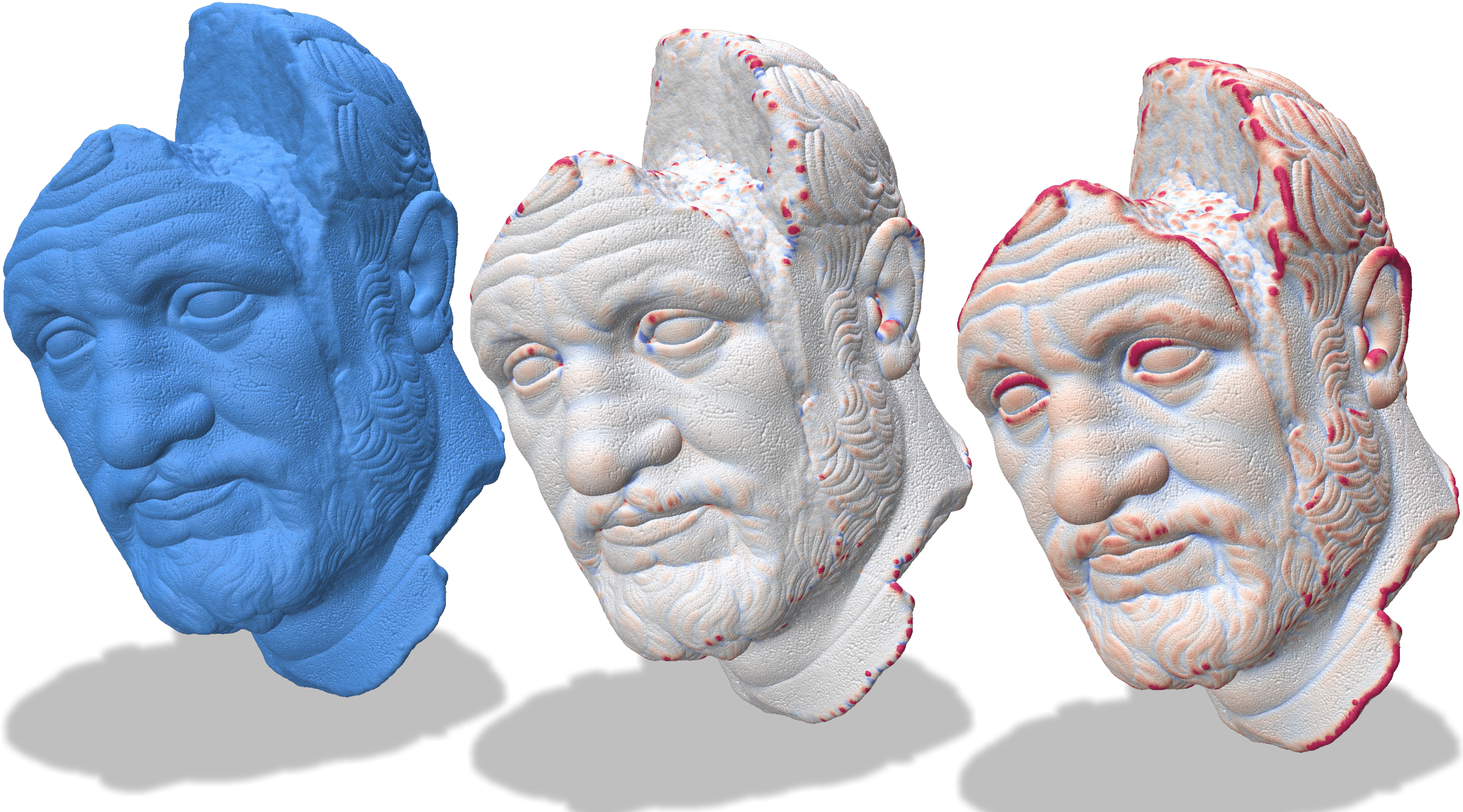
- either K-NN
- or 6-NN for CNC-Hexagram
- sums of $\langle \cdot \mid \cdot \rangle$ and $\cdot \times \cdot$ formulas per triangle

- Sum area measu… Sum … …ussian curvature meas.

$$\hat{A}^A = \sum_{l=1}^{L} \mu_{\mathbf{u}}^{A}(\hat{\tau}_l) \qquad \hat{A}^H = \sum_{l=1}^{L} \mu_{\mathbf{u}}^{H}(\hat{\tau}_l) \qquad \hat{A}^G = \sum_{l=1}^{L} \mu_{\mathbf{u}}^{G}(\hat{\tau}_l)$$

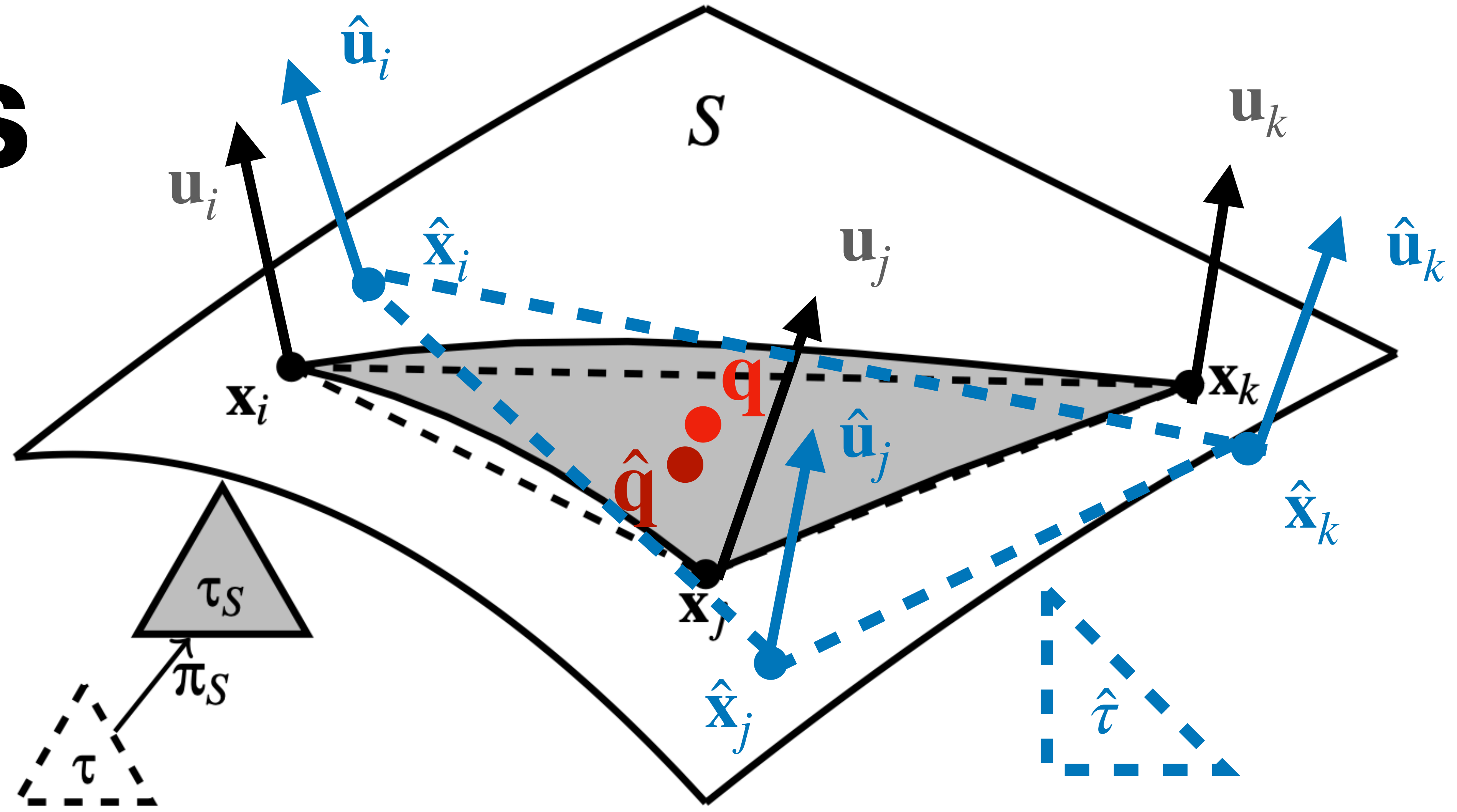
Mean curvature $\hat{H}(\mathbf{x}) = \hat{A}^H / \hat{A}^A$

Gaussian curvature $\hat{G}(\mathbf{x}) = \hat{A}^G / \hat{A}^A$



Works well in practice
What tells the theory ?

Stability of curvature estimates (noisy data)



Estimated curvature

True curvature on S

Theorem If $\hat{A}^{(0)}/L = \Theta(\delta^2)$ then

$$\left|\hat{H}(\hat{\mathbf{q}}) - H(\mathbf{q})\right| \leq \left|O(\delta) + \Theta(\delta^{-2})\left(\bar{Z}_L^{(1)} - \bar{Z}_L^{(0)} H(\mathbf{q})\right)\right| / \left|1 + \Theta(\delta^{-2})\bar{Z}_L^{(0)}\right|$$

Variation of H + …

Sum of mean curvature measure errors

Sum of area measure errors

$\hat{A}^{(0)}/L$ = average area of triangles

$$\bar{Z}_L^{(0)} = \frac{1}{L}(\hat{A}^{(0)} - A^{(0)})$$ area error law

$$\bar{Z}_L^{(1)} = \frac{1}{L}(\hat{A}^{(1)} - A^{(1)})$$ mean curvature error law

Property of error laws

Convergence if $\bar{Z}_L^{(0)}$ and $\bar{Z}_L^{(1)}$ are below $O(\delta^2)$

Property 2 The error laws $\bar{Z}_L^{(0)}$ and $\bar{Z}_L^{(1)}$ have both null expectations. Their variance follows, for C and C' some constants:

$$\mathbb{V}\left[\bar{Z}_L^{(0)}\right] \leq \frac{C}{L}\left((\sigma_\xi^2\delta^2+\sigma_\varepsilon^2)\delta^2+\sigma_\varepsilon^2\sigma_\xi^2\delta^2+\sigma_\varepsilon^4(1+\sigma_\xi^2)\right)$$

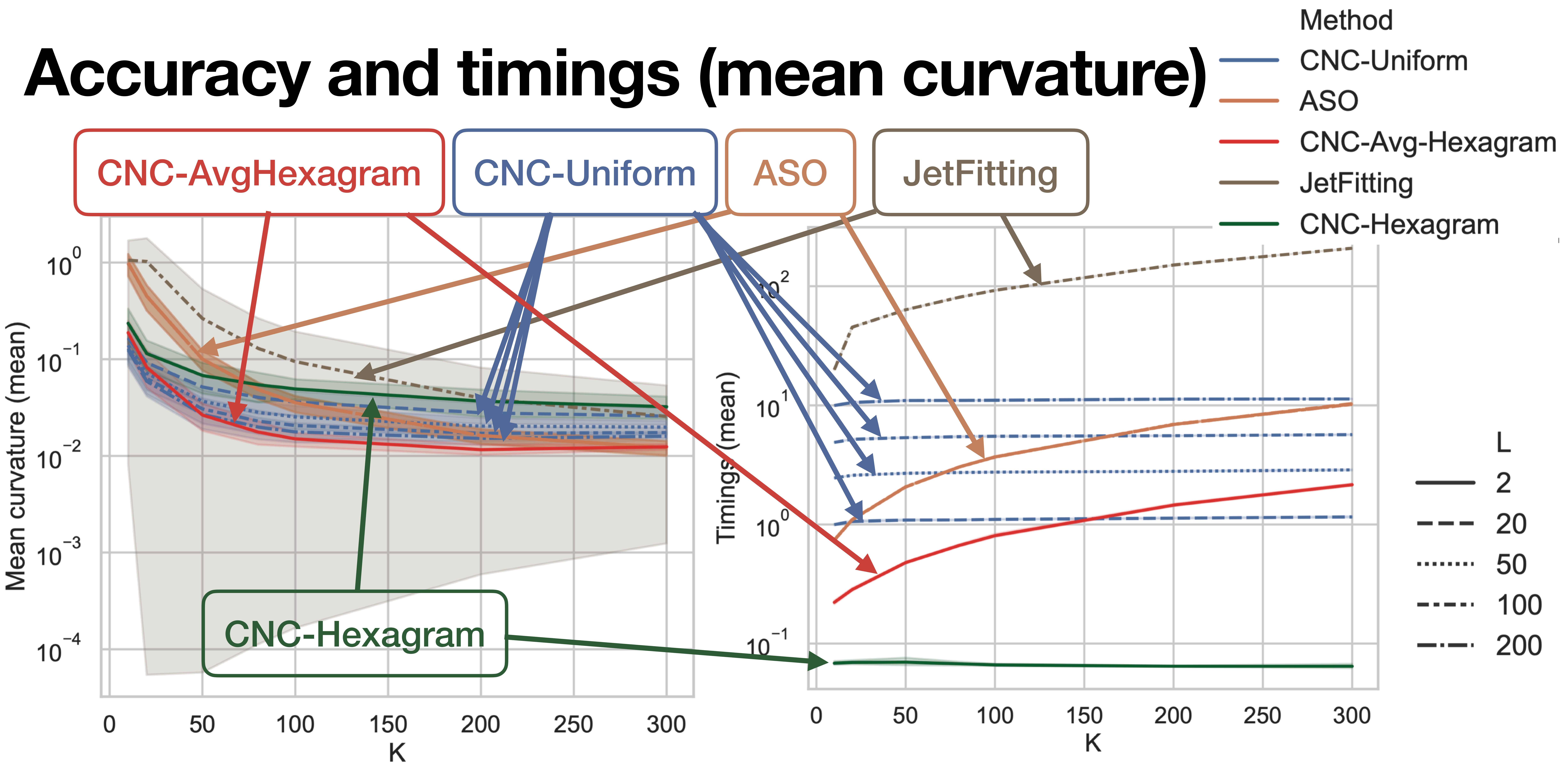
$$\mathbb{V}\left[\bar{Z}_L^{(1)}\right] \leq \frac{C'}{L}\left(\sigma_\xi^2\delta^2+\sigma_\varepsilon^2+\sigma_\varepsilon^2\sigma_\xi^2+\sigma_\xi^4\delta^2+\sigma_\varepsilon^2\sigma_\xi^4\right).$$

Area error is very low (order 4)

Mean curvature error is essentially σ_ϵ^2

Increasing $L =$ #triangles decreases both errors !

Accuracy and timings (mean curvature)



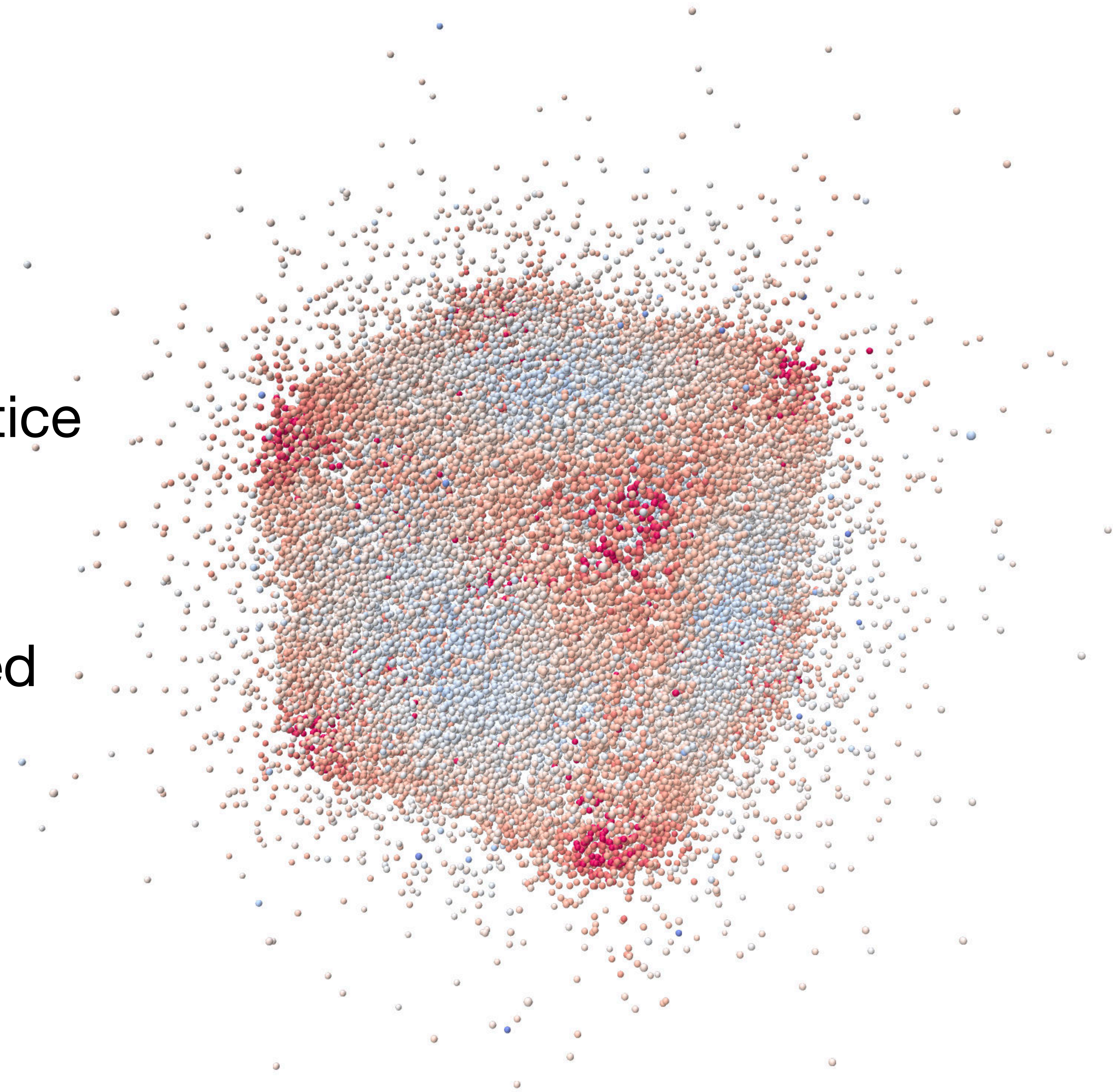
- **Goursat** shape : $N \in \{10000, 25000, 50000, 75000, 100000\}$, $\sigma_\epsilon, \sigma_\xi \in \{0, 0.1, 0.2\}$

Simplest Hexagram gives multiscale geometric information

- 1,8M of points, 4s for all $H, G, \kappa_1, \kappa_2, \mathbf{v}_1, \mathbf{v}_2$ curvatures, 81s for NN

Conclusion & Future works

- A unified framework for geometric analysis of both smooth and discrete data
- Theoretical guarantees and simple formulas in practice
- Extend this approach to unoriented point clouds
- Higher-order differential quantities using an extended Grassmannian
- Relation with Geometric algebra since choosing the right space and differential forms made geometry simple



Close geometry implies close measures