Asymptotic linear digital geometry and applications

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L A M A
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Outline

1. Motivation
2. Around digital straight lines and segments
3. Convexity and asymptotics
4. Applications

Outline

Digital geometry = a geometry in $\mathbb{Z}^n$

- Digital shapes arise naturally in several contexts

arithmetic

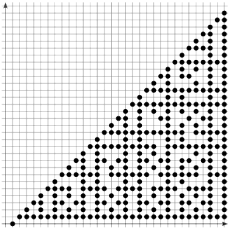
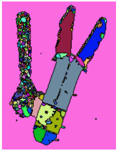
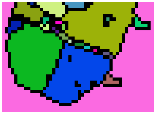


image analysis





- but also : approximation, word combinatorics, tilings, cellular automata, computational geometry, biomedical imaging, . . .
- digital shape analysis requires a sound digital geometry

Why not doing everything in the continuous domain ?

- signal processing and PDE perform well on images : filtering, restoration, known noise removal
- but less on regions and shapes : lack of structure, geometry

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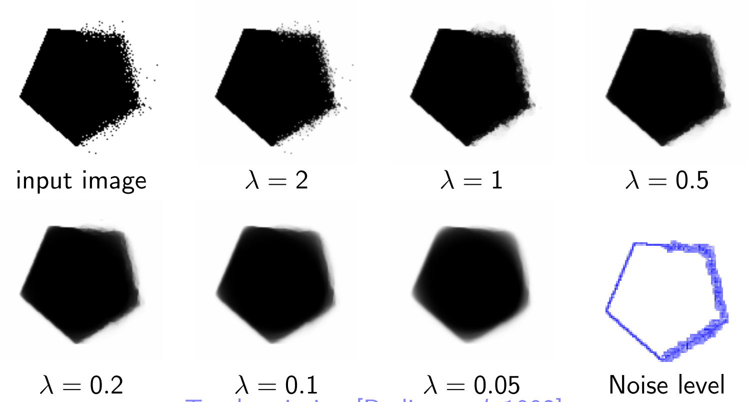

Total variation [Rudin *et al.* 1992]

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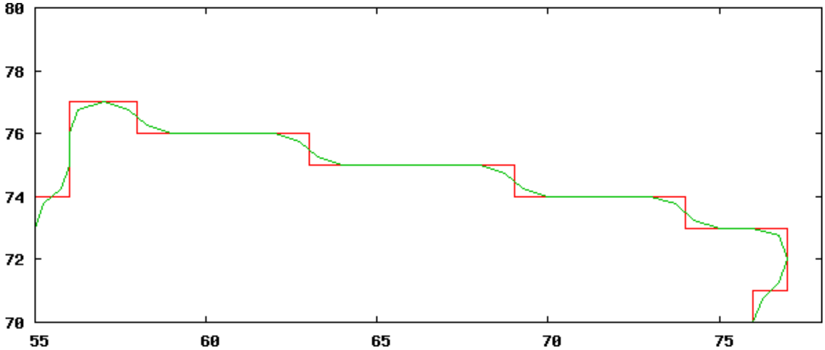

Gaussian smoothing

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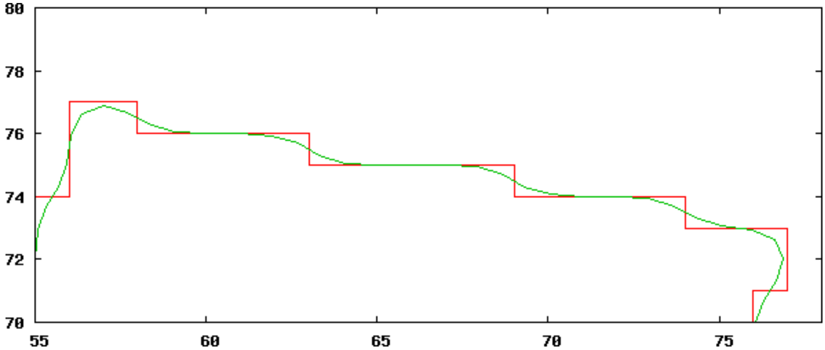

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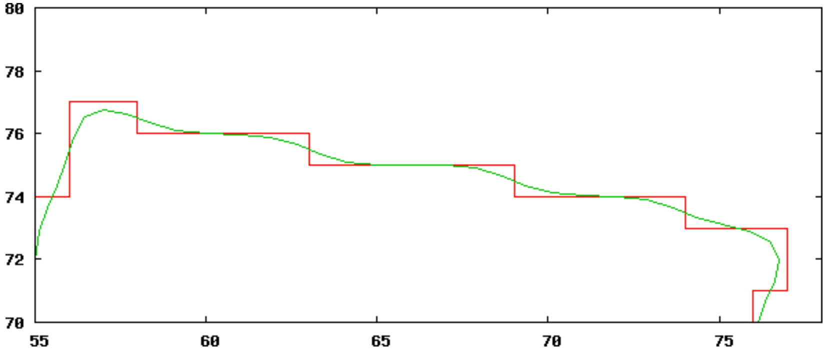

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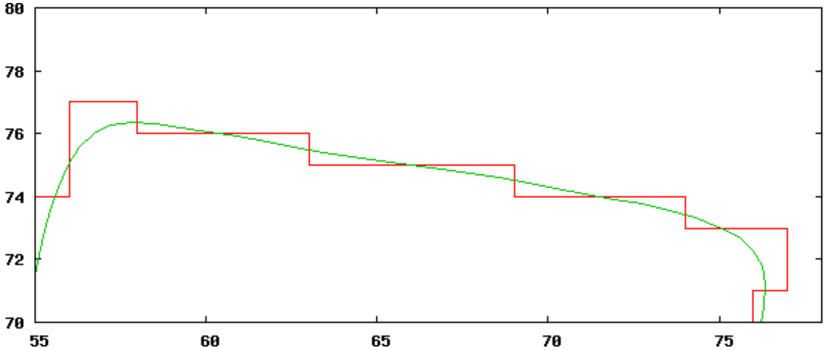

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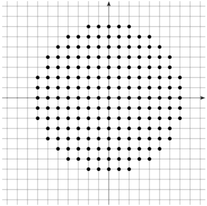
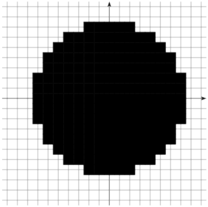
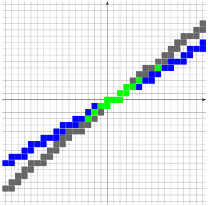
Arithmetic needs also to be taken into account.

Which geometry for $\mathbb{Z}^2$?

- Redefine topology, connectivity, convexity, straightness, etc.

Convexity ?

Intersection ?

Which geometry for $\mathbb{Z}^2$?

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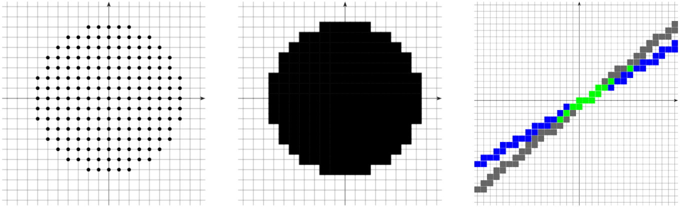
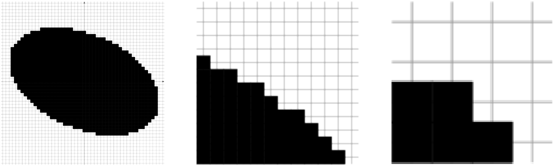


Convexity ? Intersection ?

- Differential approach of geometric quantities ?



Infinitesimal ?

The link with the continuous world or Euclidean geometry

- Digitization : shape $X \subset \mathbb{R}^2$, digitized as $\mathrm{Dig}_h(X) = X \cap (h\mathbb{Z})^2$

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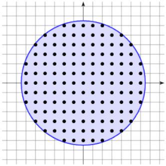

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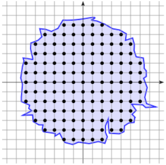
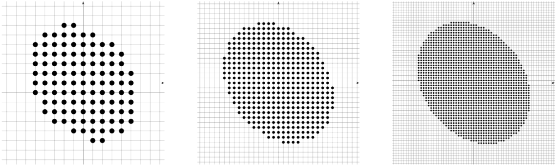


The link with the continuous world or Euclidean geometry

- Digitization : shape $X \subset \mathbb{R}^2$, digitized as $\mathrm{Dig}_h(X) = X \cap (h\mathbb{Z})^2$
- Asymptotic or Multigrid convergence When $h \to 0$ [Serra 82]



$\mathrm{Dig}_h(X)$ $\mathrm{Dig}_{h/2}(X)$ $\mathrm{Dig}_{h/4}(X)$ $\ldots$

Geometric estimator $\hat{\epsilon}$ multigrid convergent for $\mathcal{F}$ to a geom. quantity ϵ
$\forall X \in \mathcal{F}, |\hat{\epsilon}(\mathrm{Dig}_h(X)) - \epsilon(X)| \leq \tau(h)$, with $\lim_{h \to 0} \tau(h) = 0$.

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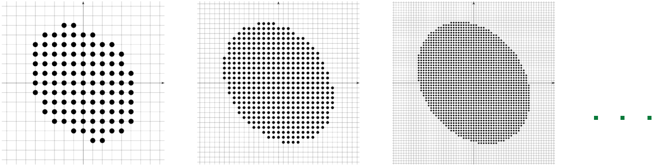

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- [Gauss, Dirichlet] Area of a convex set X by counting. $\tau(h) = O(h)$.
- Moments, perimeter [Klette,Žunić00] [Kovalevsky, Fuchs92] [Sloboda,Zatko96] [Klette *et al.* 98]

Toolbox « Linear digital geometry »

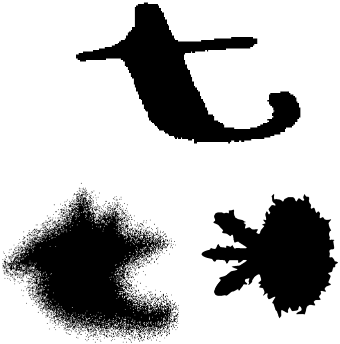
Digital straight lines, maximal segments and their asymptotic properties

Geometric estimators

- convexity/concavity
- tangents
- length
- curvature
- dominant points

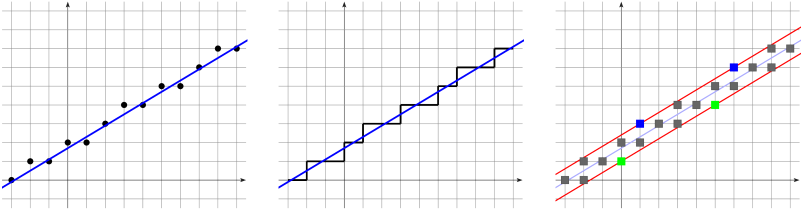
Image analysis in real life

- robust estimators
- automatic detection of noise level



Outline

Standard digital straight line

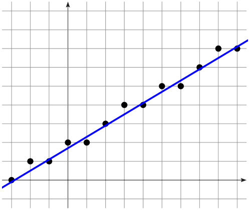


[Bernouilli, 1772]
$y = \left\lfloor \frac{3}{5}x - \frac{17}{10} \right\rceil$

OBQ digitization

Standard line
$-5 \leq 3x - 5y < 3$

Standard digital straight line



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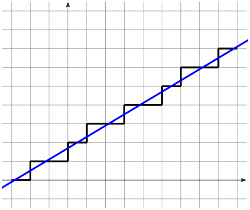

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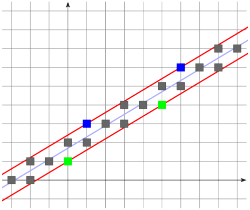


Standard line

$-5 \leq 3x - 5y < 3$

(Arithmetic) standard line [Reveillès 91], [Kovalevsky 90]

$\{(x, y) \in \mathbb{Z}^2, \mu \leq ax - by < \mu + |a| + |b|\}$

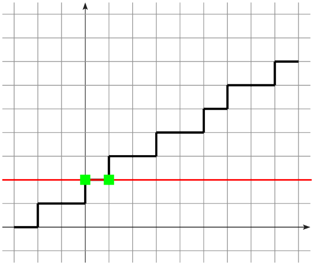
- slope $\frac{a}{b}$, shift to origin μ
- simple 4-connected path in $\mathbb{Z}^2$

Arithmetic definition lead to online recognition

Online recognition algorithm [Debled, Reveillès 95]

Let $S = (P_1, \ldots, P_n)$ be a Digital Straight Segment (DSS)

- Is $S' = (P_1, \ldots, P_n, P)$ also a DSS ?
- If yes, compute its minimal characteristics in O(1)



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ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
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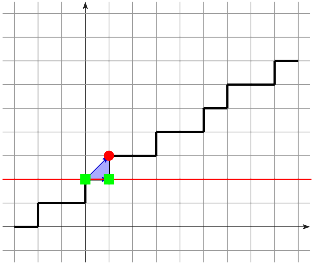

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ajout de (1, 3)

$$ax - by = -3$$

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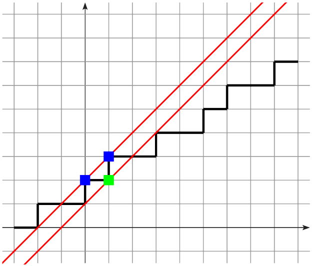

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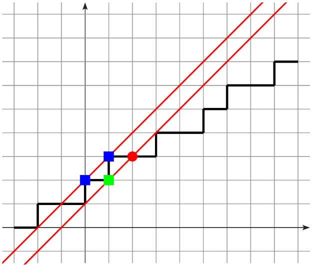

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ajout de (2, 3)

$$ax - by = -1$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
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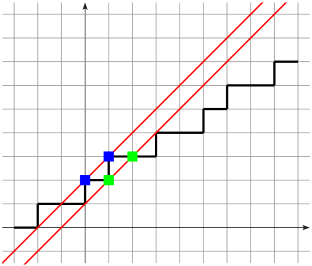

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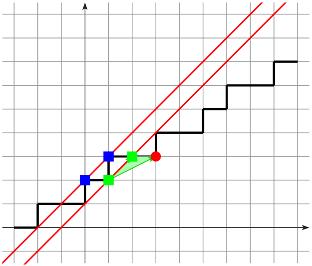

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ajout de (3, 3)

$$ax - by = 0$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
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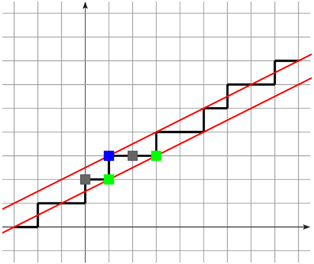

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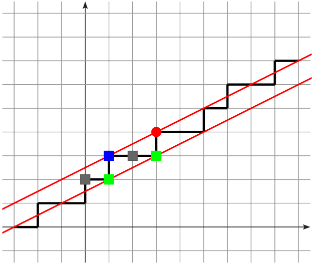

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ajout de (3, 4)

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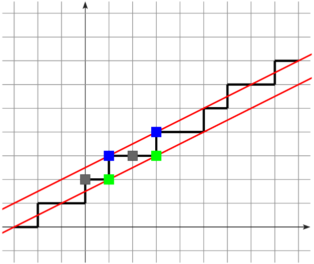

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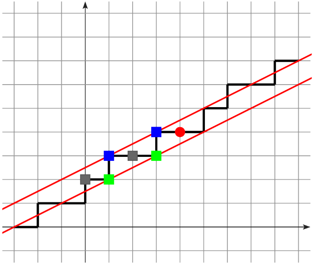

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ajout de (4, 4)

$$ax - by = -4$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
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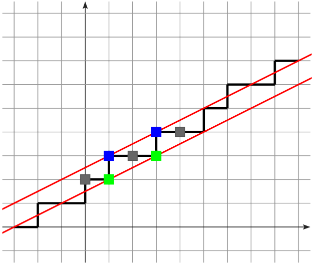

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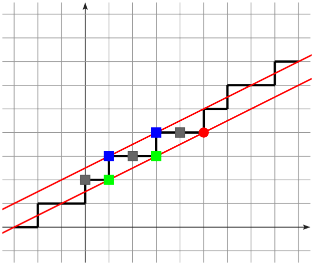

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ajout de (5, 4)

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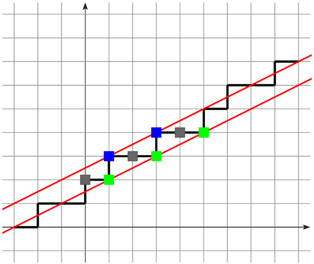

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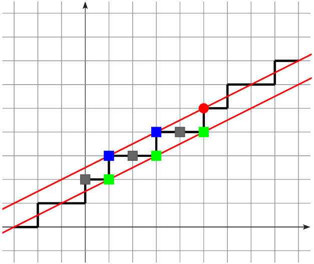
- Is $S' = (P_1, \dots, P_n, P)$ also a DSS ?
- If yes, compute its minimal characteristics in O(1)



$$S = DSS(a = 1, b = 2, \mu = -5)$$
$$-5 \leq ax - by \leq -3$$
$$\frac{a}{b} = \frac{1}{2} = [0; 2] = 0 + \frac{1}{2}$$

ajout de (5, 5)

$$ax - by = -5$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
	Ext	Modif	Ok	Modif	Ext

Arithmetic definition lead to online recognition

Online recognition algorithm [Debled, Reveillès 95]

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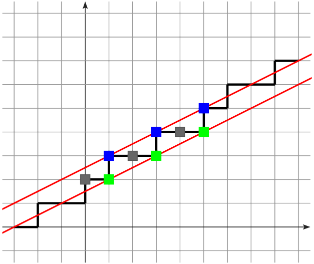

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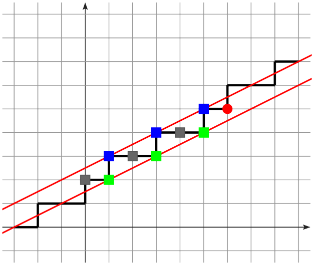

$$S = DSS(a = 1, b = 2, \mu = -5)$$
$$-5 \leq ax - by \leq -3$$
$$\frac{a}{b} = \frac{1}{2} = [0; 2] = 0 + \frac{1}{2}$$

ajout de (6, 5)

$$ax - by = -4$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
	Ext	Modif	Ok	Modif	Ext

Arithmetic definition lead to online recognition

Online recognition algorithm [Debled, Reveillès 95]

Let $S = (P_1, \ldots, P_n)$ be a Digital Straight Segment (DSS)

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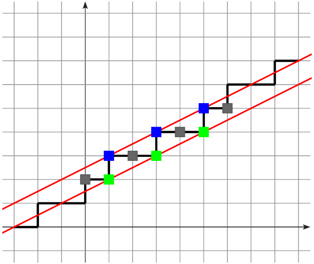

$$S = DSS(a = 1, b = 2, \mu = -5)$$
$$-5 \leq ax - by \leq -3$$
$$\frac{a}{b} = \frac{1}{2} = [0; 2] = 0 + \frac{1}{2}$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
	Ext	Modif	Ok	Modif	Ext

Arithmetic definition lead to online recognition

Online recognition algorithm [Debled, Reveillès 95]

Let $S = (P_1, \ldots, P_n)$ be a Digital Straight Segment (DSS)

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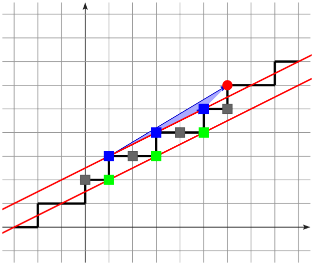

$$S = DSS(a = 1, b = 2, \mu = -5)$$
$$-5 \leq ax - by \leq -3$$
$$\frac{a}{b} = \frac{1}{2} = [0; 2] = 0 + \frac{1}{2}$$

ajout de (6, 6)

$$ax - by = -6$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
	Ext	Modif	Ok	Modif	Ext

Arithmetic definition lead to online recognition

Online recognition algorithm [Debled, Reveillès 95]

Let $S = (P_1, \ldots, P_n)$ be a Digital Straight Segment (DSS)

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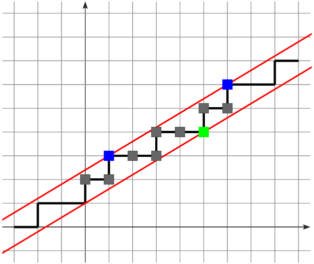
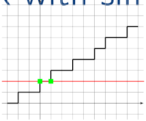


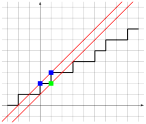
$$S = DSS(a = 3, b = 5, \mu = -12)$$
$$-12 \leq ax - by \leq -5$$
$$\frac{a}{b} = \frac{3}{5} = [0; 1, 1, 2] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

ax-by	$<$	$\mu - 1$	$\mu \leq \cdots \leq \mu + a + b - 1$	$\mu + a + b$	$<$
	Ext	Modif	Ok	Modif	Ext

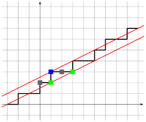
Link with simple continued fractions



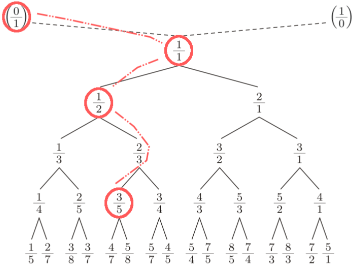
$$\frac{0}{1} = [0] = 0$$



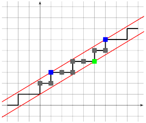
$$\frac{1}{1} = [0;1] = 0 + \frac{1}{1}$$



$$\frac{1}{2} = [0;2] = 0 + \frac{1}{2}$$



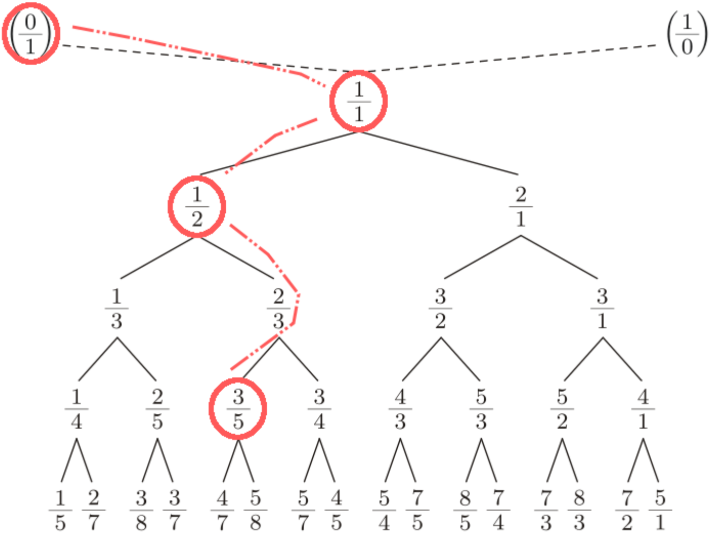
Stern-Brocot tree



$$\frac{3}{5} = [0;1,1,2] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Strong links with arithmetics (continued fractions) and word combinatorics (Christoffel and Sturmian words)

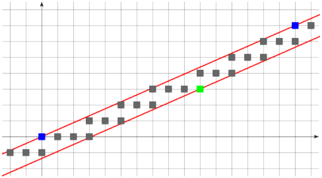
Link with simple continued fractions



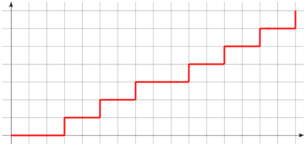
DSS as Patterns

Definition (Pattern)

Freeman chain code between two consecutive upper leaning points of a digital straight line



DSL(7, 16, 0)



00010010010001001001001

= Christoffel words [[Christoffel, 1875]]

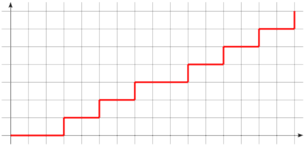
DSS as Patterns

Recursive formula [Berstel, 96] (see also splitting formula [Bruckstein . . .])

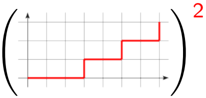
$$\frac{7}{16} = [0, 2, 3, 2]$$

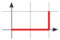
$$E([0, 2, 3, 2]) = E([0, 2, 3])^2 \quad E([0, 2])$$

$$0001001001000100100 1001 = (0001001001)^2 \quad 001$$



=





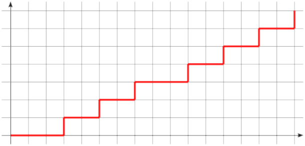
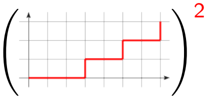
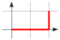
DSS as Patterns

Recursive formula [Berstel, 96] (see also splitting formula [Bruckstein . . .])

$$\frac{7}{16} = [0, 2, 3, 2]$$

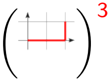
$$E([0,2,3,2]) = E([0,2,3])^2 \quad E([0,2])$$

$$000100100100010010010001001 = (0001001001)^2 \quad 001$$

 =  

$$E([0,2,3]) = E([0]) \quad E([0,2])^3$$

$$0001001001 = 0 \quad (001)^3$$

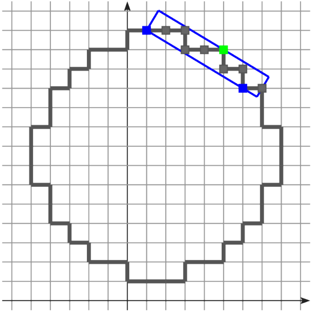
 =  

Tangential cover

Theorem ([Debled, Reveillès 95])

Online recognition of DSS when adding a point to the left or to the right in time $O(1)$.

Maximal segment on contour C : a DSS $S \subset C$ such that $\forall P \in C \setminus S$, $S \cup P$ is not a DSS.



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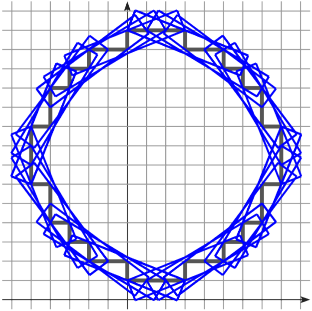


Definition ([Feschet, Tougne, 99])

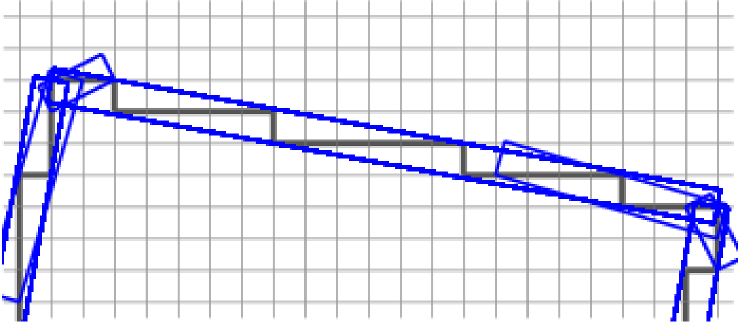
Tangential cover of C : sequence of all maximal segments of C

Theorem ([L., Vialard, de Vieilleville 07])

Updating DSS characteristics when removing a point takes time $O(1)$.

[Dorst, Smeulders, 1991]

Maximal segments capture linear arithmetic geometry



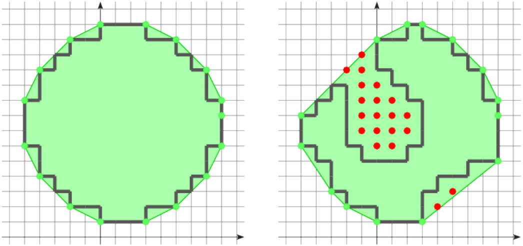
- no parameter
- satisfies convexity
- natural local scale

Outline

Digital convexity

Definition (Convexity of digital shape $O \subset \mathbb{Z}^2$)

O convex iff $\mathrm{Conv}(O) \cap \mathbb{Z}^2 = O$ and O 4-connected.

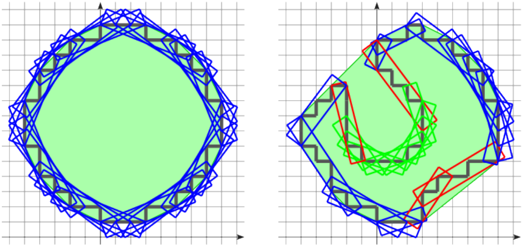


[Kim, Rosenfeld 83], [Minsky, Papert 88] [Hübler, Eckhardt, Klette, Voss, ...], ..., [Brlek, L., Provençal, Reutenauer 09]

Digital convexity and maximal segments

Theorem ([Debled-Rennesson,Reiter-Doerksen 04])

A 4-connected shape $O \subset \mathbb{Z}^2$ is digitally convex iff the directions of its maximal segments are monotonous.



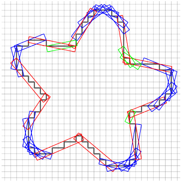
- Splits a digital contour into convex and concave parts, with a straight inflexion zone in-between.
- When $O = \mathrm{Dig}_h(X)$ has an inflexion zone, then X cannot be convex around this point.

Asymptotic behavior of maximal segments

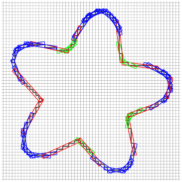
Theorems of multigrid convergence of discrete estimators

Proofs are based on the asymptotic growth of maximal segments along the border of more and more finely digitized shape.

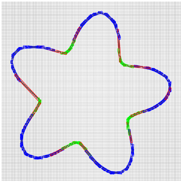
X



$\mathrm{Dig}_1(X)$



$\mathrm{Dig}_{\frac{1}{2}}(X)$



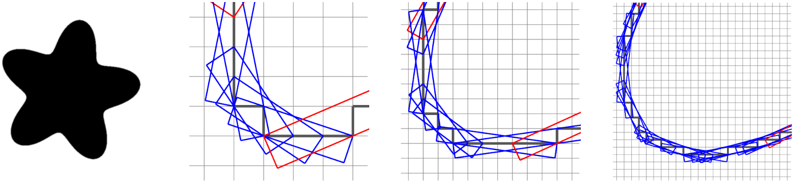
$\mathrm{Dig}_{\frac{1}{4}}(X)$

Asymptotic bounds in number and length of maximal segments ?

Asymptotic behavior of maximal segments

Theorems of multigrid convergence of discrete estimators

Proofs are based on the asymptotic growth of maximal segments along the border of more and more finely digitized shape.



X $\quad$ $\mathrm{Dig}_1(X)$ $\quad$ $\mathrm{Dig}_{\frac{1}{2}}(X)$ $\quad$ $\mathrm{Dig}_{\frac{1}{4}}(X)$

Asymptotic bounds in number and length of maximal segments ?

Asymptotic bounds on maximal segments

Methodology

- Shape is divided into convex and concave parts.

$\Rightarrow$ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

Asymptotic bounds on maximal segments

Methodology

⇒ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

- Theorem [Balog, Bárány 91]].
  $X \in \mathcal{C}^3 -$ convex. Number of edges of its digitizations follows

$$c_1(X)h^{-\frac{2}{3}} \leq n_e(\mathrm{Conv}(\mathrm{Dig}_h(X))) \leq c_2(X)h^{-\frac{2}{3}}$$

Asymptotic bounds on maximal segments

Methodology

⇒ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

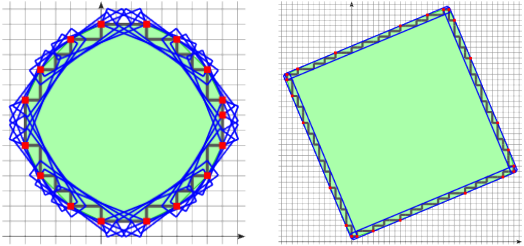
- [Balog, Bárány 91]. $c_1(X)h^{-\frac{2}{3}} \leq n_e(\mathrm{Conv}(\mathrm{Dig}_h(X))) \leq c_2(X)h^{-\frac{2}{3}}$

Asymptotic bounds on maximal segments

Methodology

⇒ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

- [Balog, Bárány 91]. $c_1(X)h^{-\frac{2}{3}} \leq n_e(\mathrm{Conv}(\mathrm{Dig}_h(X))) \leq c_2(X)h^{-\frac{2}{3}}$
- We relate the number n_{MS} of max. seg. to number n_e of edges of convex hull.



$n_{MS} = 24$, $n_e = 16$ $n_{MS} = 4$, $n_e = 24$

Asymptotic bounds on maximal segments

Methodology

⇒ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

- [Balog, Bárány 91]. $c_1(X)h^{-\frac{2}{3}} \leq n_e(\mathrm{Conv}(\mathrm{Dig}_h(X))) \leq c_2(X)h^{-\frac{2}{3}}$
- We relate the number n_{MS} of max. seg. to number n_e of edges of convex hull.

Theorem ([de Vieilleville, L., Feschet 07])

$$\frac{n_e(\mathrm{Conv}(\Gamma))}{\Theta(\log \frac{1}{h})} \leq n_{MS}(\partial\Gamma) \leq 3n_e(\mathrm{Conv}(\Gamma)), \qquad \text{avec} \quad \Gamma = \mathrm{Dig}_h(X).$$

Sketch of the proof

- related to continued fraction of DSS slope
- shortest maximal segment which absorbs the greatest number of edges is $[0;2,2,\ldots]$
- inversely, slope complexity upper bounded by $O(\log \frac{1}{h})$

Asymptotic bounds on maximal segments

Methodology

⇒ We only consider finite smooth convex shapes X ($\mathcal{C}^3$ and $\subset [0,1]^2$).

- [Balog, Bárány 91]. $c_1(X)h^{-\frac{2}{3}} \leq n_e(\mathrm{Conv}(\mathrm{Dig}_h(X))) \leq c_2(X)h^{-\frac{2}{3}}$
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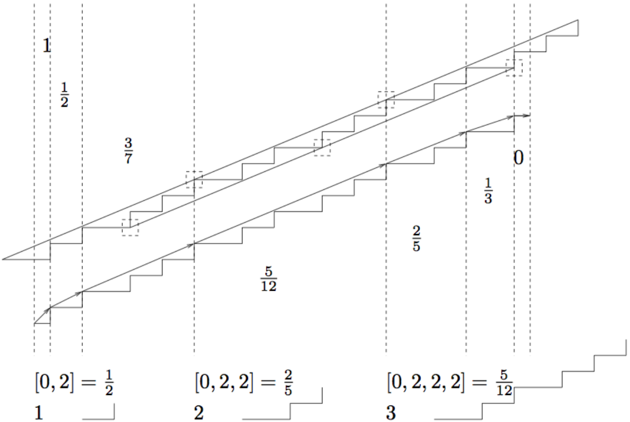
Theorem ([de Vieilleville, L., Feschet 07])

$$\frac{n_e(\mathrm{Conv}(\Gamma))}{\Theta(\log \frac{1}{h})} \leq n_{MS}(\partial\Gamma) \leq 3n_e(\mathrm{Conv}(\Gamma)), \qquad \text{avec} \quad \Gamma = \mathrm{Dig}_h(X).$$

- **average** length $\overline{L_D(MS)}$ of max. seg. (in grid steps)

$$\frac{1}{3}\frac{\mathrm{Per}(\Gamma)}{n_e(\Gamma)} \leq \overline{L_D(MS)} \leq 19\frac{\mathrm{Per}(\Gamma)}{n_e(\Gamma)}\Theta(\log\frac{1}{h})$$

Links between edges of convex hull and maximal segments



$[0; 2, 2, \ldots, 2]$: Shortest maximal segment which absorbs the greatest number of edges.

Links between edges of convex hull and maximal segments

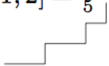
$[0,1] = \frac{1}{1}$

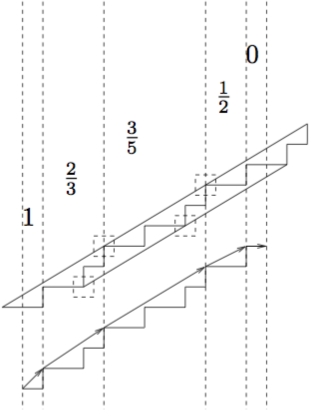
1 

$[0,1,1] = \frac{1}{2}$

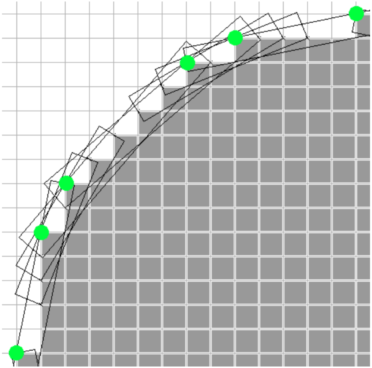
2 

$[0,1,1,2] = \frac{3}{5}$

3 



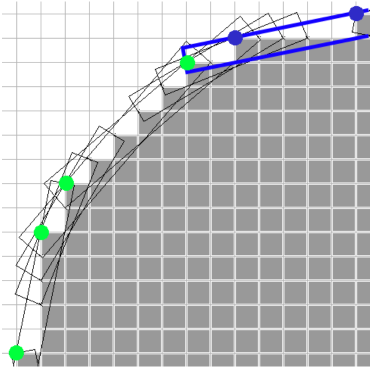
Liens entre arêtes de $\mathrm{Conv}(O)$ et segments maximaux



[de Vieilleville, L., 06]

- convexité $\Rightarrow$2 classes de MS

Liens entre arêtes de $\mathrm{Conv}(O)$ et segments maximaux



[de Vieilleville, L., 06]

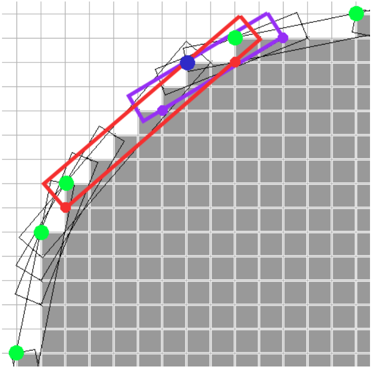
- convexité ⇒2 classes de MS
- Segments maximaux "arete"
  pente z_n = pente arête
  1 MS "arête" par arête

Lemma (basé motifs)

MS contient $\leq 2n+1$ *arêtes*

Ex : pente $z_n = \frac{1}{5} \Rightarrow 3$ arêtes

Liens entre arêtes de $\mathrm{Conv}(O)$ et segments maximaux



[de Vieilleville, L., 06]

- convexité ⇒2 classes de MS
- Segments maximaux "arete"
- Segments maximaux "sommet"

Lemma (basé motifs)

Max. 2 MS "sommet" par sommet
1 prof. pair + 1 prof. impair

gauche $\frac{7}{8} = [0;1,7]$, droite $\frac{3}{5} = [0;1,1,2]$

Lemma (basé motifs)

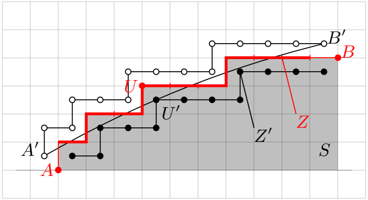
MS contient $\leq 2n$ arêtes

Summary of asymptotic results on max. segments

Along digitizations of C^3-convex shapes (curvature $\kappa > 0$).

	shortest	average	longest
$L_D(MS)$	$\Omega(h^{-\frac{1}{3}})$	$\Theta(h^{-\frac{1}{3}}) \leq \cdot \leq \Theta(h^{-\frac{1}{3}} \log \frac{1}{h})$	$O(h^{-\frac{1}{2}})$
$L(MS)$	$\Omega(h^{\frac{2}{3}})$	$\Theta(h^{\frac{2}{3}}) \leq \cdot \leq \Theta(h^{\frac{2}{3}} \log \frac{1}{h})$	$O(h^{\frac{1}{2}})$

- longest max. segment $= O(h^{-\frac{1}{2}})$ (geometry)
- shortest max. segment $= \Omega(h^{-\frac{1}{3}})$ [L. 06] (separating circles)

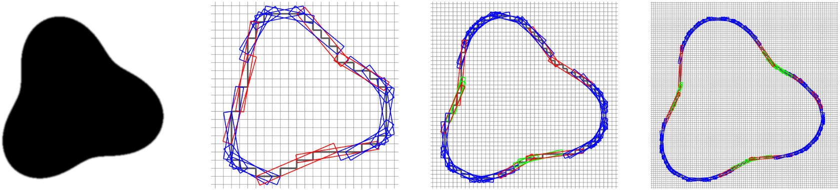


Outline

Multigrid convergence of geometric estimators (I)

Definition (Tangent estimator with maximal segment $\hat{\theta}^{MS}$)

Tangent at point P is the direction of *any* maximal segment covering P.



Theorem (L., Vialard, de Vieilleville 07 + L. 06)

Tangent estimators $\hat{\theta}^{MS}$ are **uniformly** *multigrid convergent in $O(h^{\frac{1}{3}})$ for shapes with $\mathcal{C}^3$-boundary and finite number of inflexion points.*

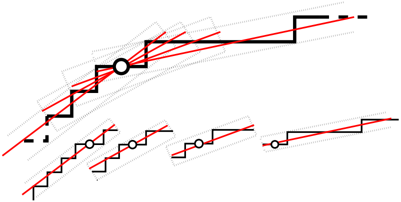
Examples of geometric estimators (I)

Definition (Tangent estimator λ-MST [L. *et al.* 06])

Let $(MS_i)_{i=1\ldots k}$ be the maximal segments covering a point P, and (θ_i) their respective directions.

$$\hat{\theta}(P) = \frac{\sum_{i=1\ldots k} \lambda(e_i(P))\theta_i}{\sum_{i=1\ldots k} \lambda(e_i(P))}. \quad (1)$$

where $e_i(P) \in [0,1]$ is the eccentricity of P in MS_i, and λ is some map in $\mathbb{R}^+$, $\lambda(0) = \lambda(1) = 0$.

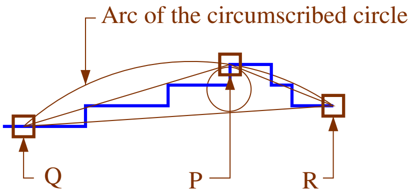


Corollary

The λ-MST is uniformly multigrid-convergent in $O(h^{\frac{1}{3}})$.

Examples of geometric estimators (II)

Definition (Curvature by circumscribed circle [Coeurjolly *et al.* 01])



- simple and fast to implement
- convergent iff maximal segments grow in $O(h^{\frac{1}{2}})$.
- false almost everywhere, not convergent in practice.

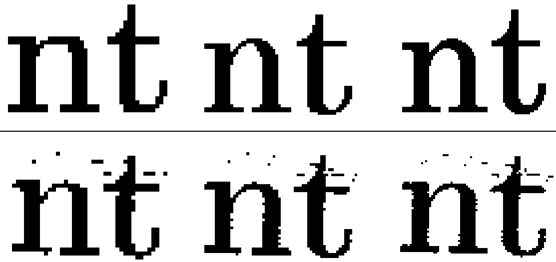
Multigrid convergence of geometric estimators (II)

Quantity	estimator	Uniform convergence	Exp. average convergence
position	$\hat{x}^{\mathrm{conv}}$	$O(h)$	$O(h^{\frac{4}{3}})$
tangent	sym. tan.	no	?
tangent	$\hat{\theta}^{\mathrm{conv}}$	?	$O(h^{\frac{2}{3}})$
tangent	$\hat{\theta}^{\mathrm{MS}}$	$O(h^{\frac{1}{3}})$	$O(h^{\frac{2}{3}})$
curvature	Circum. circle	no	exp. no
curvature	Var. of sym. tan.	no	no

Quantity	estimator	Convergence
length	$\int \hat{\theta}^{\mathrm{MS}}$	$O(h^{\frac{1}{3}})$

What about noisy digital shapes ?

200 dpi	300 dpi	400 dpi

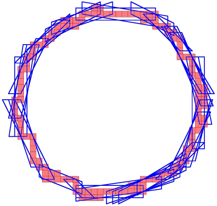


Printed at 600dpi, then scanned at specified resolution (Roman 14pt font)

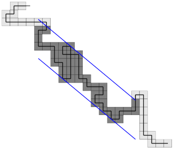
What about noisy digital shapes ?

Definition (Blurred segments [Debled *et al.*06])

Segments of maximal thickness ν (given by the user)



$\nu = 2$

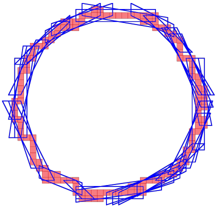


$\nu = 4$

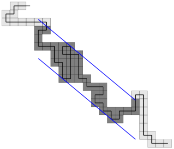
What about noisy digital shapes ?

Definition (Blurred segments [Debled *et al.*06])

Segments of maximal thickness ν (given by the user)

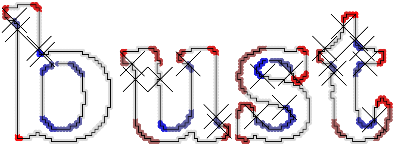


$\nu = 2$



$\nu = 4$

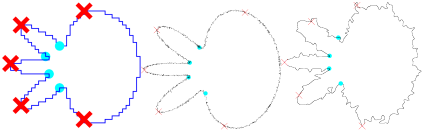
Curvature estimator



$\arg\min_X \int_{\partial X} \kappa^2$, and $\mathrm{Dig}(X) = O$

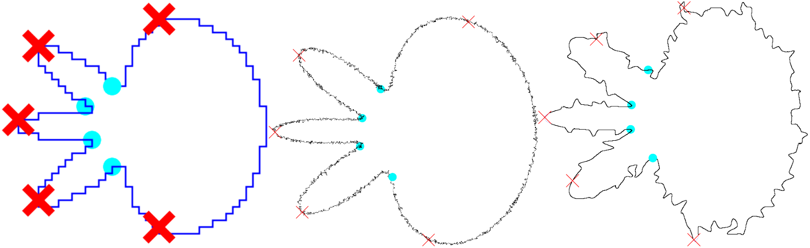
[Kerautret, L. 09]

Dominant points



[Kerautret, L. 08]

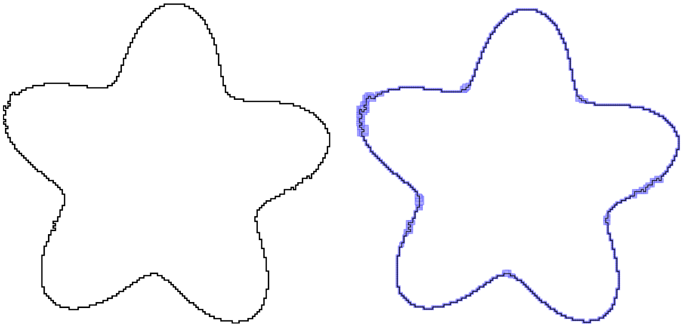
Dominant points



Automatic detection of meaningful scale / noise

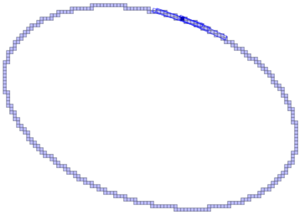
Noise / meaningful scale along digital contour [Kerautret, L. 10]

- asymptotic properties of maximal segments along digitizations of ideal shapes
- properties are estimated locally by multiresolution
- comparisons with ideal case determines if the contour is damaged

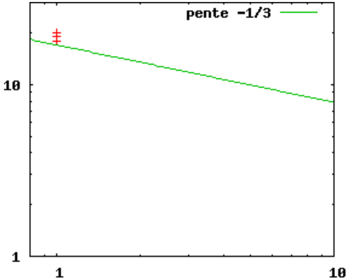


Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		



$h_1 = h,\ L^{h_1} = (18, 20, 19)$

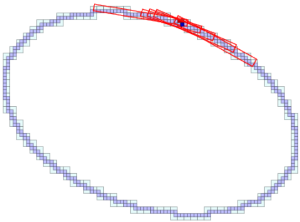


fine ← − − − − → coarse

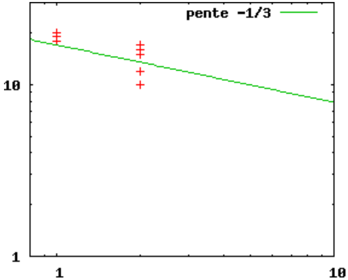
$(h, L(h))$ in logscale

Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		



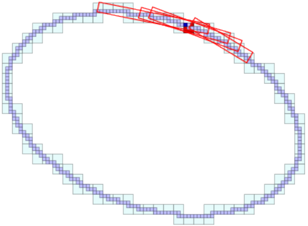
$h_2 = 2h,\ L^{h_2} = (15, 10, 12, 17, \ldots)$



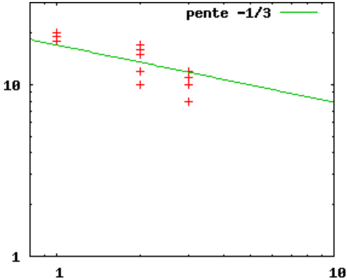
fine ← − − − − → coarse
$(h, L(h))$ in logscale

Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		



$h_3 = 3h,\ L^{h_3} = (12, 10, 11, 8, \ldots)$

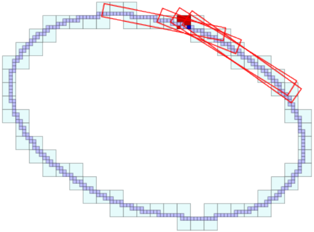


fine ← − − − − → coarse

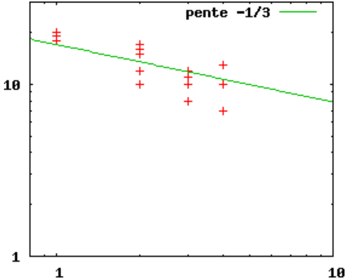
$(h, L(h))$ in logscale

Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		



$h_4 = 4h,\ L^{h_4} = (10, 7, 13, 13, \ldots)$

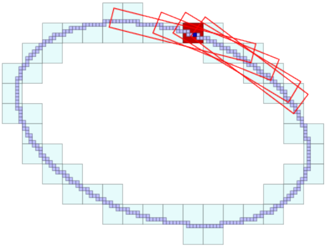


fine ← − − − − → coarse

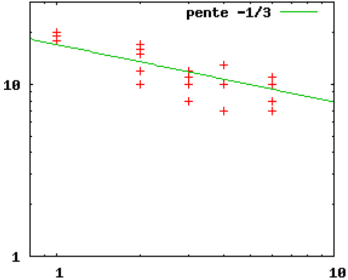
$(h, L(h))$ in logscale

Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		



$h_6 = 6h,\ L^{h_6} = (8, 7, 8, 8, \ldots)$

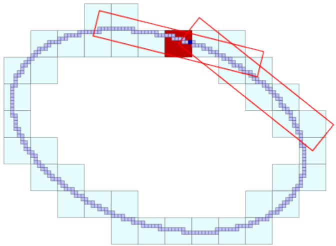


fine ← − − − − → coarse

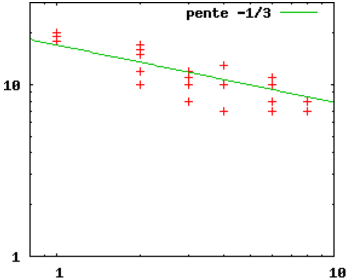
$(h, L(h))$ in logscale

Local shape analysis by multiresolution

local geometry	digital Length $L_D(\frac{1}{h})$	slope in logscale
convex, concave	$\Omega((\frac{1}{h})^{\frac{1}{3}}) \leq \cdot \leq O((\frac{1}{h})^{\frac{1}{2}})$	$-\frac{1}{2} \leq \cdot \leq -\frac{1}{3}$
flat	$\Theta(\frac{1}{h})$	≈ -1
noise otherwise		

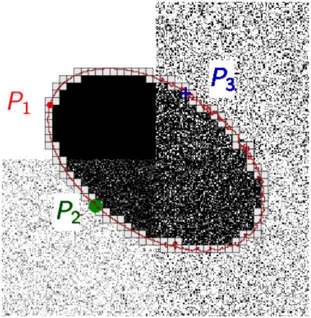


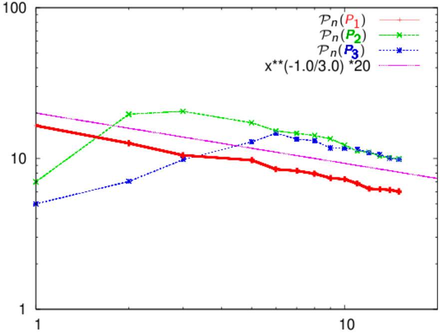
$h_8 = 8h$, $L^{h_8} = (8, 7, \ldots)$



fine ← − − − − → coarse
$(h, L(h))$ in logscale

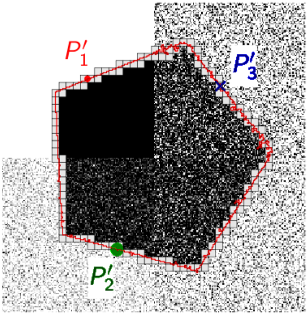
Local profile and noise level

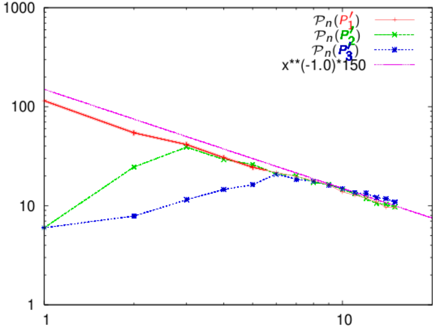




Smooth shape

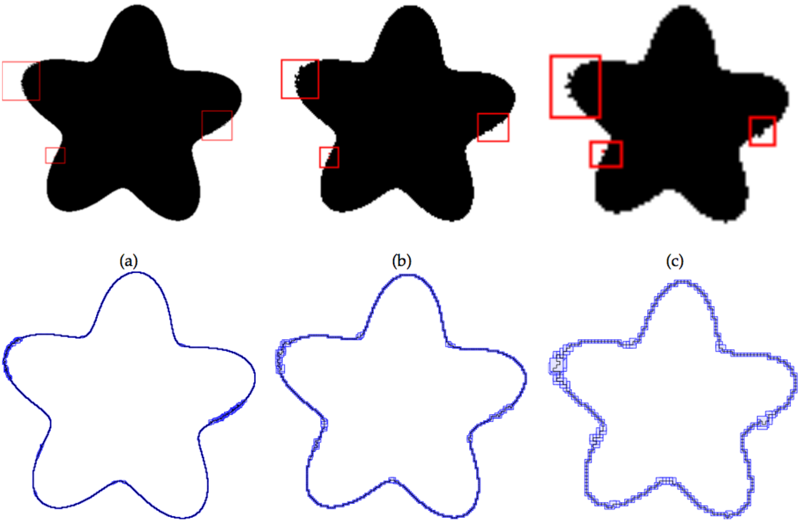
Local profile and noise level





Polygonal shape

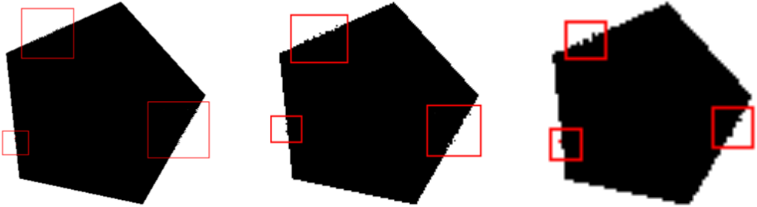
Local profile and noise level

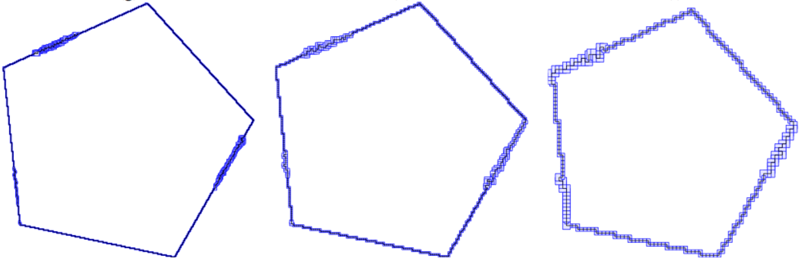


(a) (b) (c)

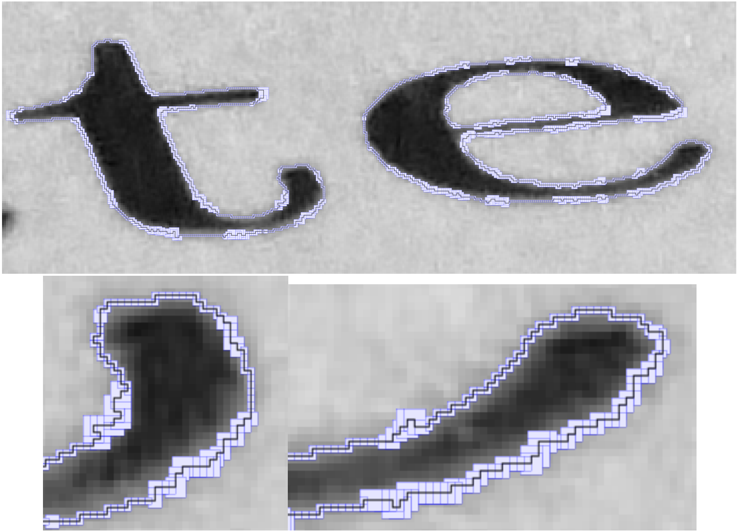
(d) (e) (f)

Local profile and noise level


(g) (h) (i)

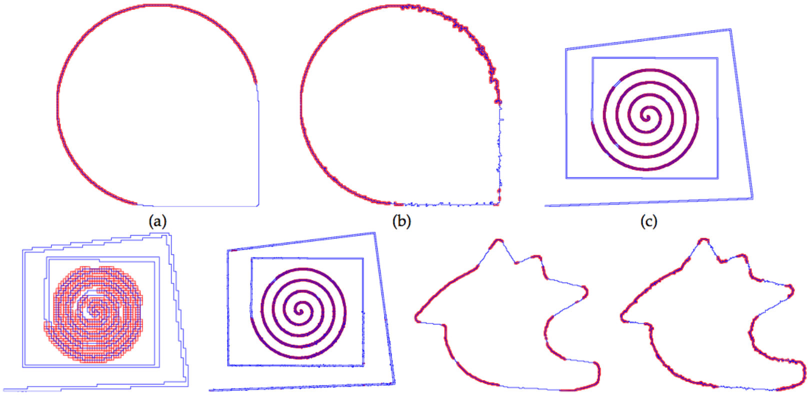

(j) (k) (l)

Local profile and noise level



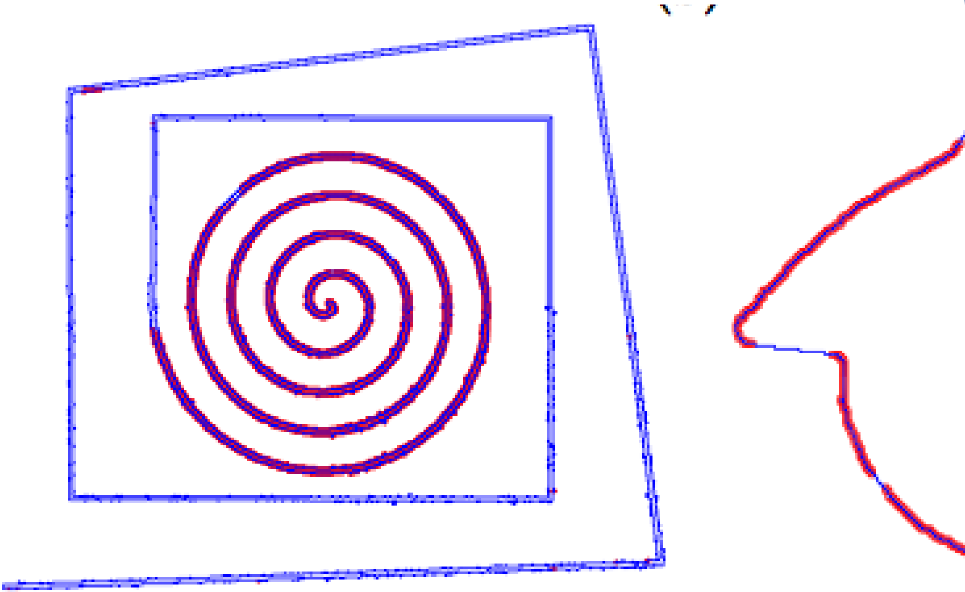
Photography

Local profile and flat/curve discrimination



(a) (b) (c) (d) (e) (f) (g)

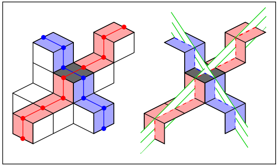
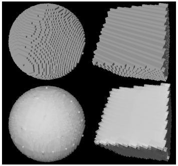
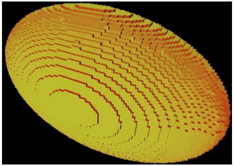
Analysis of linear parts ⇒ highlights curved zones !



(e)

What about 3D, even ND ?

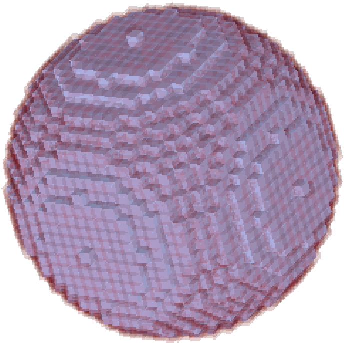
- ND estimators by crossing several 2D geometries [Lenoir 97] [Debled et Tellier 99] [L. et Vialard 03]

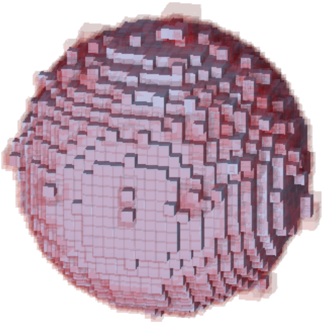




$N-1$ paths per surfel — normal $\hat{\mathbf{n}}$ orth. to $(\hat{\theta}_i)$ — Area $= \sum_\sigma |\hat{\mathbf{n}} \cdot \mathbf{e}_{\perp\sigma}|$

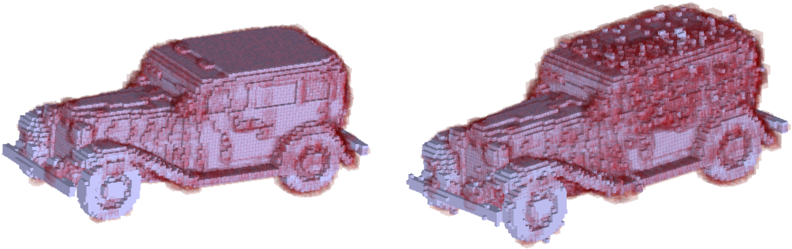
- What about convergence ?

3D noise detection





3D noise detection



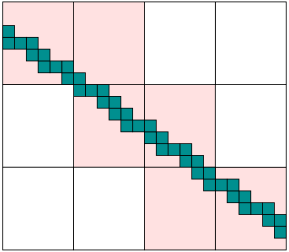
Open problems on discrete geometric estimators

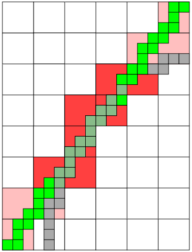
Influence of depth of continued fractions

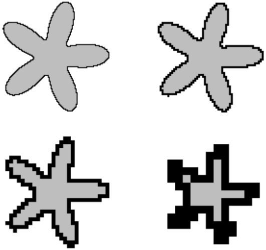
- Position error : theoretically $O(h)$, exp. $O(h^{\frac{4}{3}})$ $\times O(h^{\frac{1}{3}})$
- Tangent error : theoretically $O(h^{\frac{1}{3}})$, exp. $O(h^{\frac{2}{3}})$ $\times O(h^{\frac{1}{3}})$
- related to depth of continued fractions
  - bad estimation around $1/88 = [0; 88]$,
  - good estimation around $34/55 = [0; 1, 1, 1, 1, 1, 1, 1, 2]$
- average analysis of depth and partial quotients of slopes along digitizations. How ?

[de Vieilleville, L., Proc. ISVC 2006]

Open problems on analytical multiresolution (I)



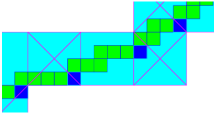




Analytical formulae for digital straight segments

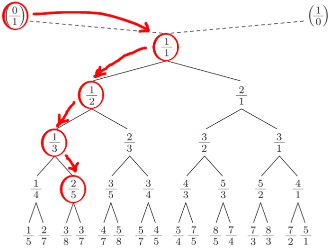
- (h, v)-covering of a digital straight line (a, b, μ).
  Thm[Said, L., Feschet 2009] : It is the standard line (α, β, ν), with $\alpha = \frac{ah}{g}$, $\beta = \frac{bv}{g}$, $g = gcd(ah, bv)$, $\nu = \ldots$
- What about the (h, v)-covering of a segment ?

Open problems on analytical multiresolution (II)

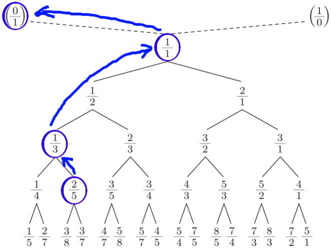


Recognition of a segment within a known digital line.

top-down



bottom-up



Conclusion and future works

- Digital straightness : a very rich toolbox
  - nice arithmetic, geometric and combinatorial properties
  - numerous applications to shape analysis problems
- Other not presented applications : minimum perimeter polygon, 3D extensions
- Natural question : similar theory for maximal digital circular arcs ?

Parts of these works were made in collaboration with :
S. Brlek, F. de Vieilleville, F. Feschet, B. Kerautret, X. Provençal, C. Reutenauer, M. Said, A. Vialard

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Questions ?