Around global and local convexity in digital spaces

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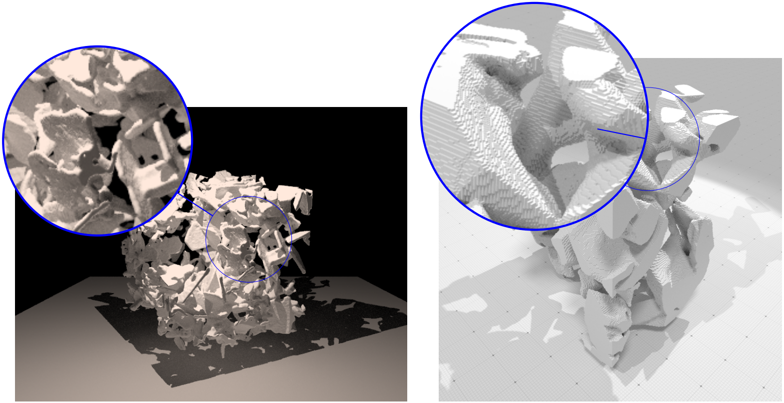
Politecnico di Milano

Around global and local convexity in digital spaces

Geometry of shapes in 3D imaging

MRI, CT, PET, confocal microscopy



snow micro-tomography

Geometry of shapes in 3D imaging

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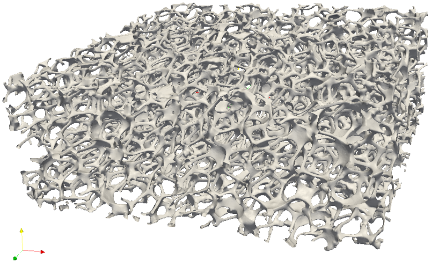

aluminium foam (CT)

Geometry of shapes in 3D imaging

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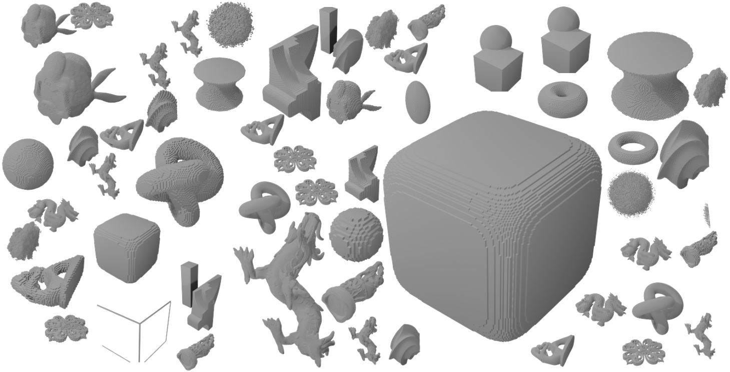


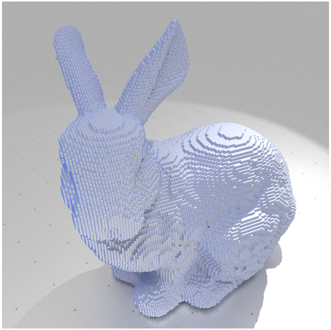
geometric modeling, shape indexing, machine learning

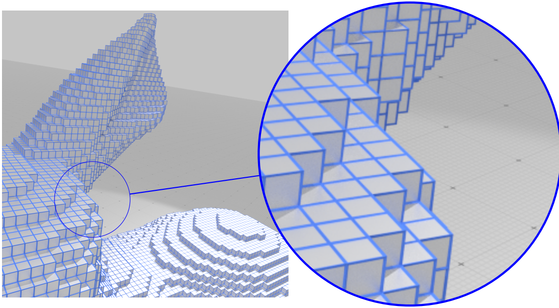
Digital surfaces

Digital geometry

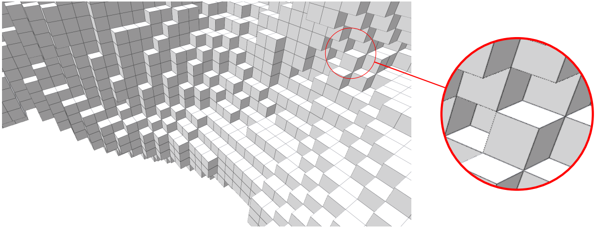
Shape Z subset of $\mathbb{Z}^d = \begin{cases} \text{set of lattice points} \\ \text{set of } \textit{voxels}\text{, i.e. (hyper)cubes.} \end{cases}$

Digital surface ∂Z = boundary of Z, set of unit (hyper)squares.





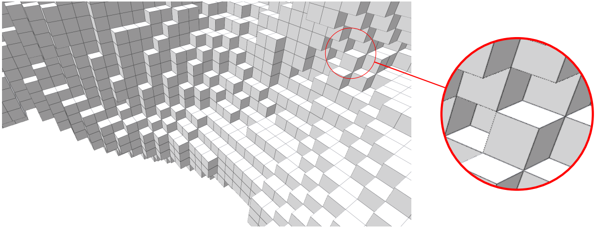
Digital surfaces



Properties of digital surfaces

topology	closed, oriented, but non manifold in general
geometry	approximate positions in $O(h)$, uniform density, few normals ($2d$), lattice points (arithmetic) if h is the gridstep.

Digital surfaces



Properties of digital surfaces

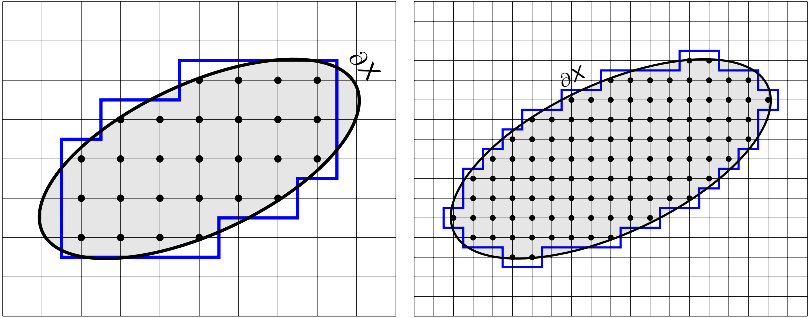
topology: closed, oriented, but non manifold in general

geometry: approximate positions in $O(h)$, uniform density, few normals ($2d$), lattice points (arithmetic)
if h is the gridstep.

What about differential geometry ?

Normals are not differentiable, no clear notion of curvatures

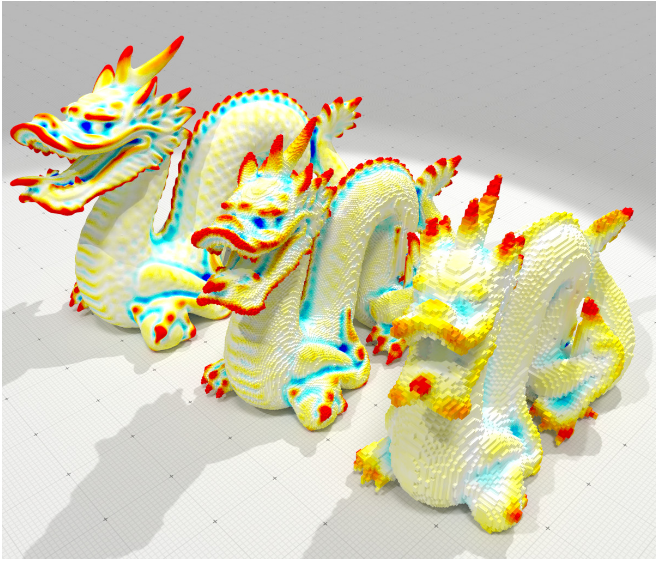
Geometric inference from digitizations



How well can we infer the geometry of X or its boundary ∂X?

- only $Z := X \cap h\mathbb{Z}^d$ is known ($\bullet$)
- or the digital surface ∂Z (⎵⊓)
- define discrete estimator of volume / area / position / normal / curvatures
- convergence of these estimates as $h \to 0$?

Example of convergence of mean curvature estimates



Corrected Normal Current [L., Romon, Thibert 2022]

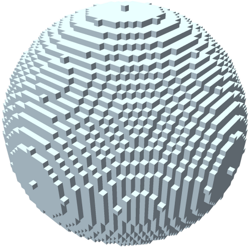
Geometric inference from digitizations, partial bibliography

2D quantity	shapes	method	max error	reference
volume	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	[Huxley 90]
moments	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	[Klette,Žunić 00]
perimeter	cvx polygons	polygonalisation	$\approx 4.5h$	[Kovalevsky,Fuchs 92]
perimeter	cvx polygons	"sausaging"	$\approx 5.8h$	[Klette et al. 98]
perimeter	C^3-convex	grid continuum	$\approx 8h$	[Sloboda,Zatko 96]
kth deriv.	C^{k+1}-functions	bin. convolut.	$O(h^{(\frac{2}{3})^k})$	[Malgouyres et al. 08]
kth deriv.	C^{k+1}-functions	Taylor approx.	$O(h^{\frac{1}{k+1}})$	[Provot,Gérard 11]
tangents	C^3-convex	max. segments	$O(h^{\frac{1}{3}})$	[L,de Vielleville,Vialard 07]

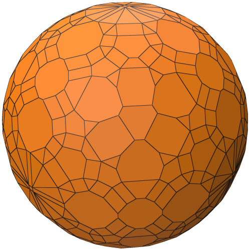
3D quantity	shapes	method	max error	reference
normals	$C^{1,1}$-smooth	integral invariant	$O(h^{\frac{2}{3}})$	[L,Coeurjolly,Levallois 17]
normals	$C^{1,1}$-smooth	Vor. cov. meas.	$O(h^{\frac{1}{8}})$	[Cuel,L.,Thibert 14]
curvatures	C^3-smooth	integral invariants	$O(h^{\frac{1}{3}})$	[Coeurjolly,L.,Levallois 13]
curvatures	C^2-smooth	cor. normal. current	$O(h^{\frac{1}{3}})$	[L.Romon,Thibert 22]

dD quantity	shapes	method	max error	reference
volume	convex	counting	$O(h)$	[Gauss,Dirichlet]
volume	C^3-convex	counting	$O(h^k)$, $k < 2$	[Müller 99],[Guo 10]
moments	$C^{1,1}$-smooth	counting	$O(h)$	[L.,Coeurjolly,Levallois 17]
area	$C^{1,1}$-smooth	$\int$ normals	$O(h^{\frac{2}{3}})$	[L.,Thibert 16]
mean curv.	C^2-smooth	varifold	$O(h^{\frac{1}{3}})$ in 3d	[Buet et al. 18]

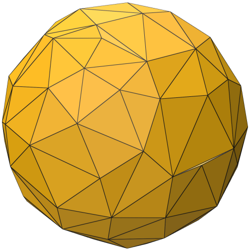
What about using convex hull on convex shapes ?



ball $B_{25} \cap \mathbb{Z}^3$



its convex hull



an 1-approximation

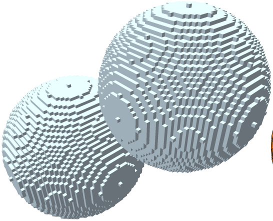
Convex hull

The convex hull $\mathrm{Cvxh}\,(X)$ of $X \subset \mathbb{R}^d$ is the intersection of all the convex set containing X.

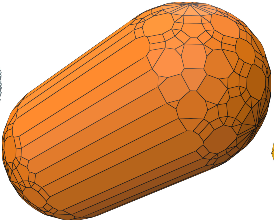
Convex hull for geometric inference

If X is convex and $Z := X \cap \mathbb{Z}^d$, then $\mathrm{Cvxh}\,(X)$ seems to be a good inference of X !

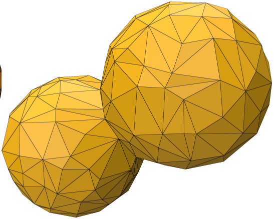
What about non convex shapes ?



2 balls $B_{25} \cap \mathbb{Z}^3$



its convex hull



an 1-approximation

Convex hull for geometric inference

Convex hull is less pertinent, while the 1-approximation does not exploit fully the geometry in **"convex zones"**.

Objectives

1. How good is convex hull for geometric inference ?

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2. Can we define convexity in a local manner along a digital surface ?

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1. How good is convex hull for geometric inference ?
2. Can we define convexity in a local manner along a digital surface ?
3. Can we compute efficiently local convexity / concavity ?

Around global and local convexity in digital spaces

Context: digital geometry and convexity

Geometry of Gauss digitized convex shapes

joint work with

D. Coeurjolly, CNRS

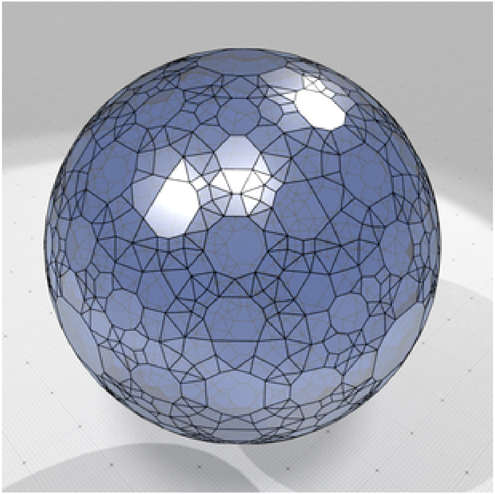
T. Roussillon, INSA Lyon

Locally convex or concave digital shapes

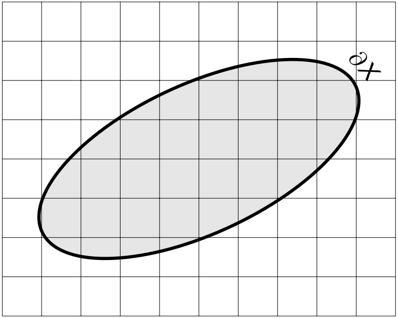
Fast extremal points identification with plane probing

Conclusion

How good is convex hull for geometric inference of digitizations of convex set ?



Notations



- grid $h\mathbb{Z}^d$
- with gridstep $h > 0$
- compact convex shape $X \subset \mathbb{R}^d$, with boundary ∂X

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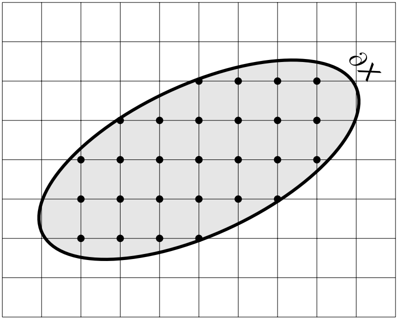

- compact convex shape $X \subset \mathbb{R}^d$, with boundary ∂X
- *Gauss digitization* $X_h := X \cap h\mathbb{Z}^d$

X

X_h •

Notations

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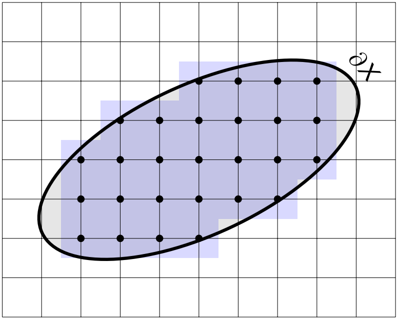

- compact convex shape $X \subset \mathbb{R}^d$, with boundary ∂X — X
- *Gauss digitization* $X_h := X \cap h\mathbb{Z}^d$ — X_h •
- *voxel representation* $\bar{X}_h := X_h \oplus [-\frac{h}{2}, \frac{h}{2}]^d$ — $\bar{X}_h$

Notations

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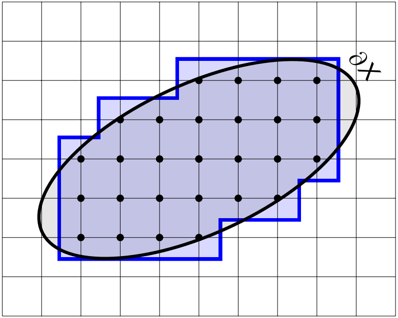

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- *digitized boundary* $\partial_h X := \partial \bar{X}_h$ — $\partial_h X$

Notations

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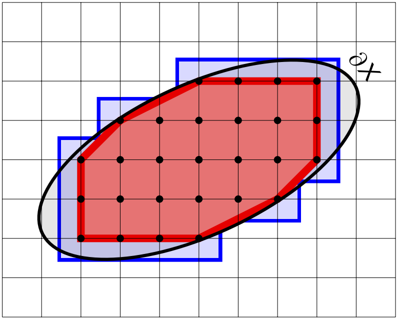

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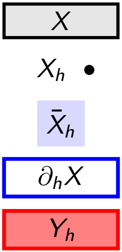
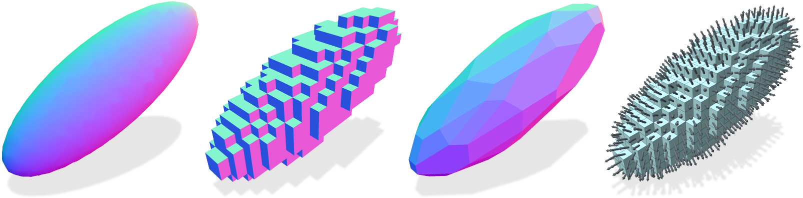
- *digitized boundary* $\partial_h X := \partial \bar{X}_h$
- *digitized convex hull* $Y_h := \mathrm{Cvxh}\,(X_h)$



Are normals to facets of Y_h good normal estimates ?



shape ∂X dig. boundary $\partial_h X$ convex hull Y_h $\partial_h X$ with facet

(color is $\frac{1}{2}(\mathbf{n}+\mathbf{1})$)

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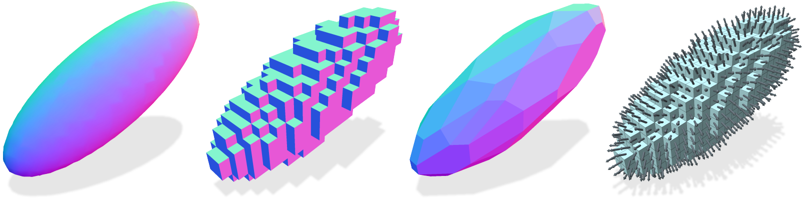

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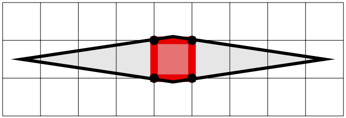
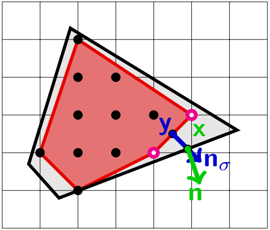
(color is $\frac{1}{2}(\mathbf{n} + \mathbf{1})$)

Two cases to consider

1. the shape X is compact, convex in $\mathbb{R}^d$
2. the shape X has a smooth boundary too

Properties when X is compact, convex (general case)





Lemma

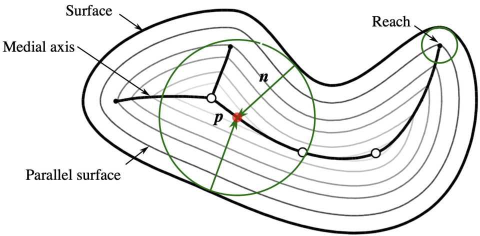
Let $Y \subset X$ be a convex polyhedron. Let $\mathbf{x}$ be an arbitrary point of ∂X, and $\mathbf{n} \in \mathrm{N}_X(\mathbf{x})$. Let $\mathbf{y}$ be the closest point of $\mathbf{x}$ on ∂Y. If $\mathbf{y}$ belongs to the facet σ of Y, with unit normal vector $\mathbf{n}_\sigma$, then the normals are related as:

$$\begin{aligned} \mathbf{n} \cdot \mathbf{n}_\sigma &\geqslant 0, \\ \sin^2 \angle(\mathbf{n}, \mathbf{n}_\sigma) &\leqslant \tfrac{\epsilon^2}{\epsilon^2 + r^2}, \end{aligned} \quad \text{with} \quad \begin{cases} \epsilon := & \|\mathbf{x} - \mathbf{y}\|, \\ r := & \mathrm{d}_{\mathbf{E}}(\mathbf{y}, \partial\sigma). \end{cases}$$

r is the distance from $\mathbf{y}$ to its facet boundary $\partial\sigma$

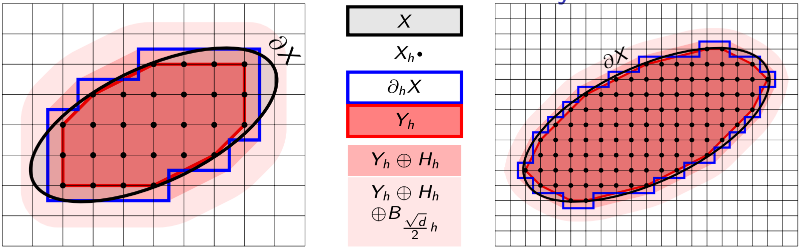
Proof. Uses the support function of Y and X. □

What do we mean by smooth boundary ?



- the medial axis M of ∂X is the locii of points which have more than one closest point on ∂X
- the reach ρ of ∂X is the infimum of the distances of M to ∂X
- if $\rho > 0$ then ∂X is $C^{1,1}$-smooth and a.e. C^2-smooth
- if $\rho > 0$ then it is equal to the inverse of the maximal curvature

Properties when X has smooth boundary



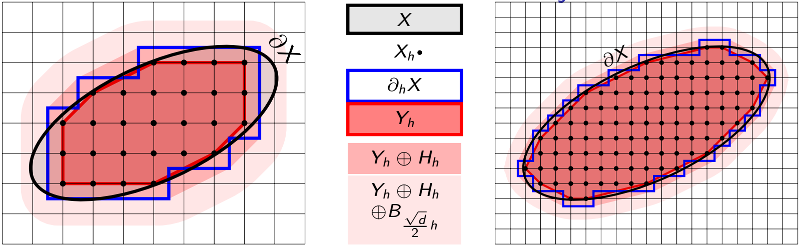
Theorem (Hausdorff closeness)

For all gridsteps h, $0 < h < \frac{2\rho}{\sqrt{d}}$, we have $Y_h \subset X \subset Y_h \oplus H_h \oplus B_{\frac{\sqrt{d}}{2}h}$.

Thus ∂X lies in the strip $Y_h \oplus H_h \oplus B_{\frac{\sqrt{d}}{2}h} \setminus \mathrm{Int}(Y_h)$.

Furthermore $\mathrm{d}_H(X, Y_h) = \mathrm{d}_H(\partial Y_h, \partial X) \leqslant \sqrt{d}h$.

Properties when X has smooth boundary



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Proof.

1. since $X_h \subset X$, $Y_h := \mathrm{Cvxh}\,(X_h) \subset \mathrm{Cvxh}\,(X) = X$

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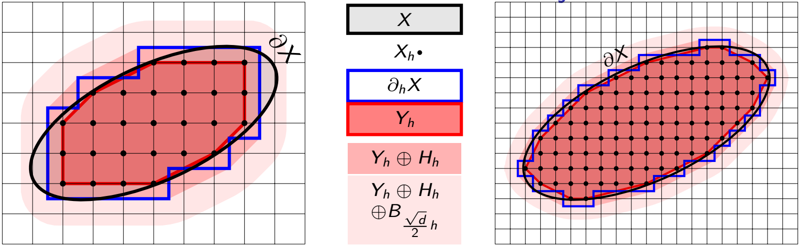

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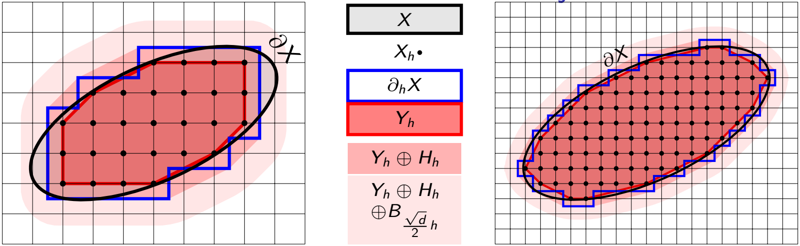
2. $\mathrm{Cvxh}\,(\partial_h X) = \mathrm{Cvxh}\,(\partial(X_h \oplus H_h)) = \mathrm{Cvxh}\,(X_h \oplus H_h) = Y_h \oplus H_h \qquad (1)$

Properties when X has smooth boundary



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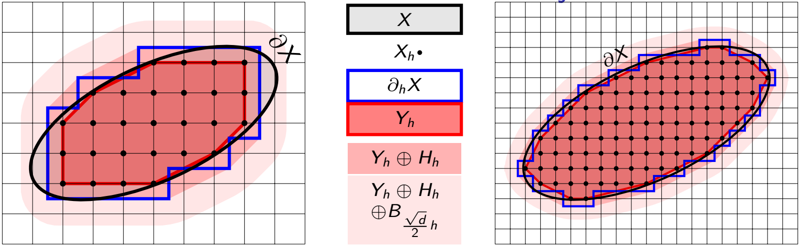

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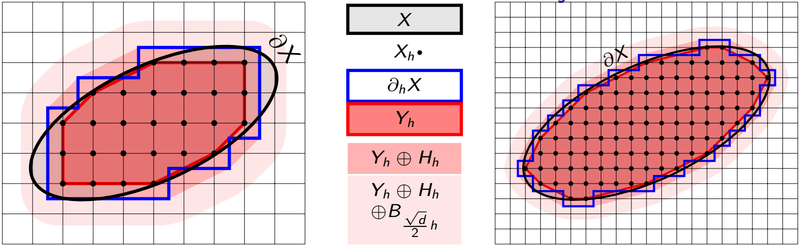
4. $X = \mathrm{Cvxh}\,(\partial X) \subset \mathrm{Cvxh}(\partial_h X \oplus B_{\frac{\sqrt{d}}{2}h}) = Y_h \oplus H_h \oplus B_{\frac{\sqrt{d}}{2}h}$ with (1), (2)

Properties when X has smooth boundary



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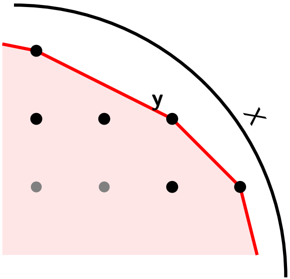
5. support functions and "$\mathrm{d}_H(A, B) = \mathrm{d}_H(\partial A, \partial B)$" [Wills 2007]

Vertices of Y_h are (very) close to X (smooth boundary)



Theorem (mostly [Bárány 90])

Let $\mathbf{y}$ *be a vertex of* Y_h. *Then, for gridsteps* h, $0 < h \leqslant \rho$,

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Vertices of Y_h are (very) close to X (smooth boundary)

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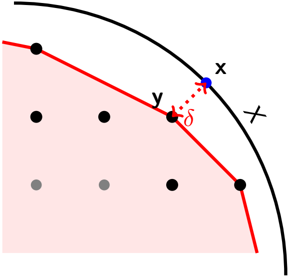

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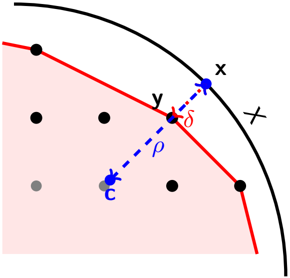

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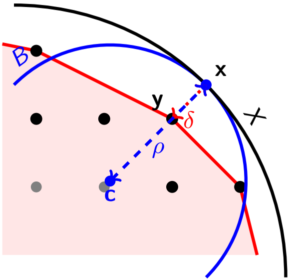

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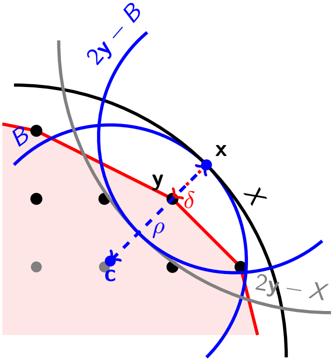

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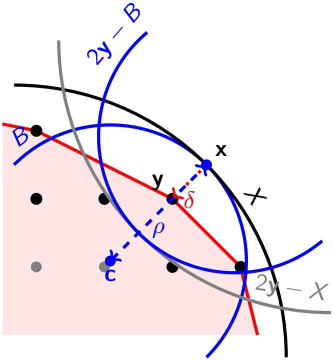

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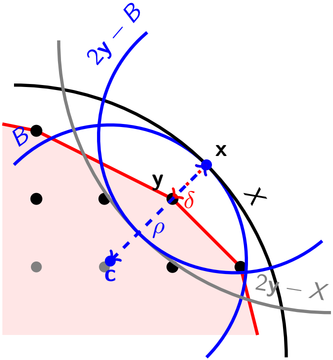

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  $$\mathrm{Vol}^d(S_B) = \mathrm{cst} \cdot \int_0^\delta (\sqrt{2\rho t - t^2})^{d-1} dt = \mathrm{cst} \cdot \rho^{\frac{d-1}{2}} \delta^{\frac{d+1}{2}}$$

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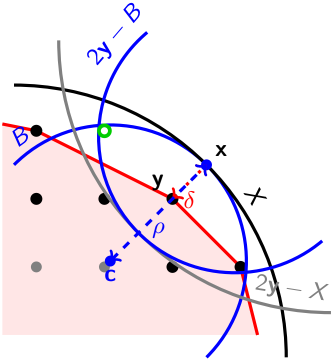

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  $= \mathrm{cst} \cdot \rho^{\frac{d-1}{2}} \delta^{\frac{d+1}{2}}$
- Minkowski theorem (S_X symmetric):
  $\mathrm{Vol}^d(S_B) \leqslant \mathrm{Vol}^d(S_X) \leqslant (2h)^d$

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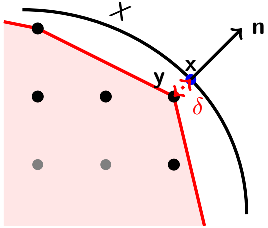
Normals of Y_h are close to normals of X



Theorem

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$$\mathbf{n} \cdot \mathbf{w} \geqslant \frac{1 - \sqrt{d}\frac{h}{\rho}}{1 - \frac{\delta}{\rho}} \geqslant 1 - \sqrt{d}\frac{h}{\rho} > 0 \qquad \textit{i.e.} \qquad \angle(\mathbf{n}, \mathbf{w}) \leqslant O\left(\sqrt{\frac{h}{\rho}}\right).$$

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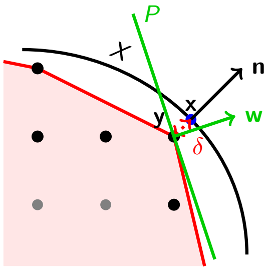

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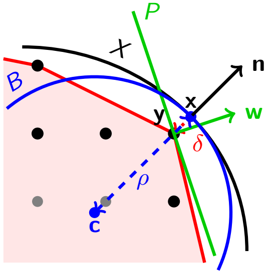

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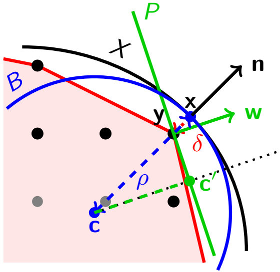

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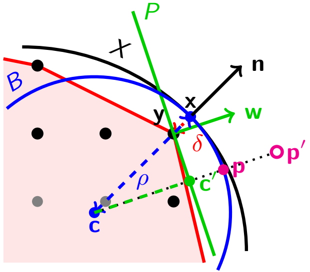

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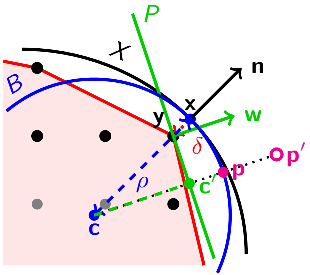

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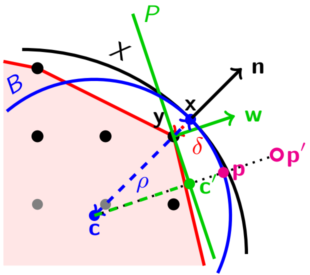

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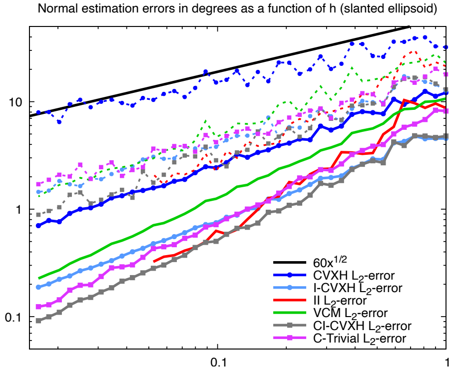
The convergence $\sqrt{h}$ is tight. The constant $\sqrt{2\sqrt{d}/\rho}$ is almost reached (20%).

And in practice, normals of Y_h are good ?



II
: Integral invariant, expected $O(h^{\frac{2}{3}})$

VCM
: Voronoi Covariance Measure, expected $O(h^{\frac{1}{8}})$

CVXH
: normals of Y_h, expected $O(h^{\frac{1}{2}})$

I-CVXH
: interpolation of normals of Y_h at vertices, expected $O(h^{\frac{1}{2}})$

CI-CVXH
: interpolation of normals of Y_h at vertices, convolved by smoothing kernel, expected ?

CTrivial
: interpolation of trivial normals of $\partial_h X$, convolved by smoothing kernel, expected ?

Around global and local convexity in digital spaces

Context: digital geometry and convexity

Geometry of Gauss digitized convex shapes

Locally convex or concave digital shapes

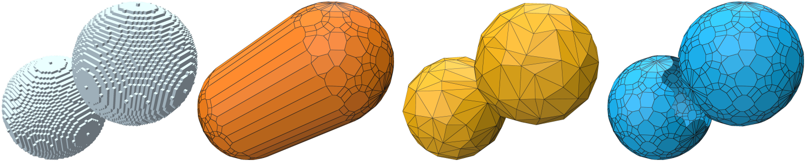
joint work with

T. Roussillon, INSA Lyon

Fast extremal points identification with plane probing

Conclusion

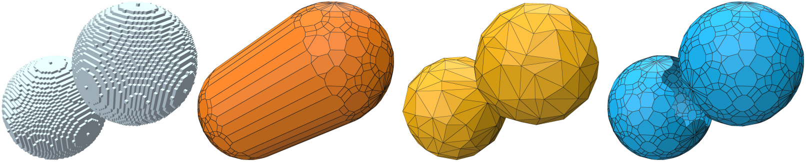
What should be local convexity ?



2 balls $B_{25} \cap \mathbb{Z}^3$ — its convex hull — an 1-approximation — local convexity

- identify vertices that are (locally) extremal in some direction
- identify edges and faces joining them
- edges should form convex angles,
- faces around vertices should form convex cones
- edges and faces should stay close to the digital surface

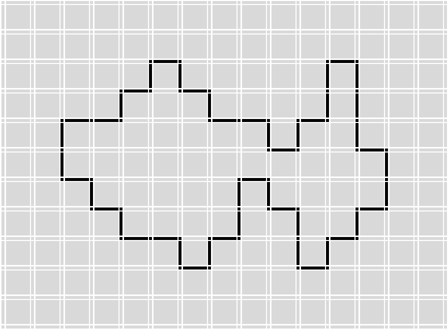
What should be local convexity ?



2 balls $B_{25} \cap \mathbb{Z}^3$ its convex hull an 1-approximation local convexity

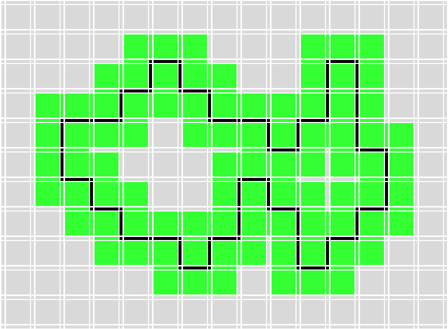
- identify vertices that are (locally) extremal in some direction
- identify edges and faces joining them
- edges should form convex angles,
- faces around vertices should form convex cones
- edges and faces should stay close to the digital surface
- edges and faces should not cross the digital surface

Tangency / full subconvexity



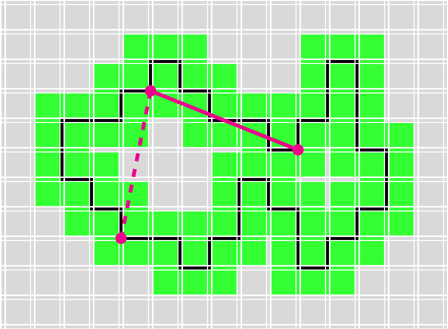
- cubical grid $\mathcal{C}^d$, with $\mathcal{C}^d_0 = \mathbb{Z}^d$
- Input: digital surface S

Tangency / full subconvexity



- cubical grid $\mathcal{C}^d$, with $\mathcal{C}^d_0 = \mathbb{Z}^d$
- Input: digital surface S
- $\mathrm{Star}(Y) := \{c \in \mathcal{C}^d, \bar{c} \cap Y \neq \emptyset\}$
  $\mathrm{Star}(S)$ is a strip around S

Tangency / full subconvexity



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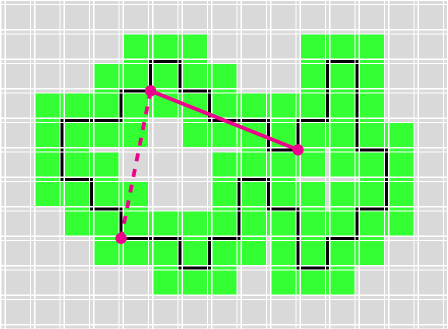
  $\mathrm{Star}\,(S)$ is a strip around S
- $A \subset \mathbb{Z}^d$ is **tangent to** S iff
  $\mathrm{Star}\,(\mathrm{Cvxh}\,(A)) \subset \mathrm{Star}\,(S)$

Example of tangent edges in S (●—●)
and non tangent edges in S (● - -●)

Tangency / full subconvexity



- cubical grid $\mathcal{C}^d$, with $\mathcal{C}_0^d = \mathbb{Z}^d$
- Input: digital surface S
- $\mathrm{Star}\,(Y) := \{c \in \mathcal{C}^d, \bar{c} \cap Y \neq \emptyset\}$
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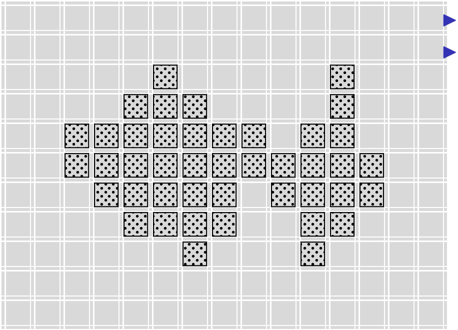
  $\mathrm{Star}\,(\mathrm{Cvxh}\,(A)) \subset \mathrm{Star}\,(S)$
- Tangent edges are close to S
  but may cross the surface

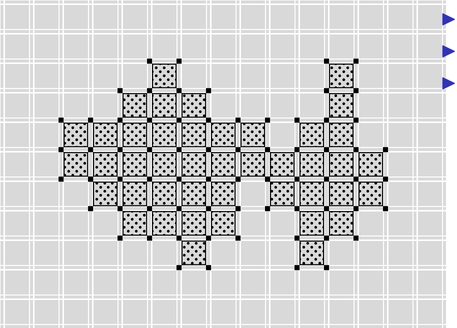
Example of tangent edges in S (●—●)
and non tangent edges in S (● - -●)

The digital surface must be oriented



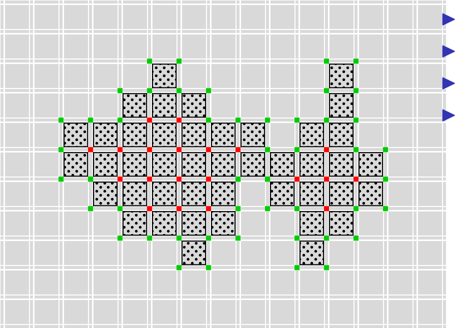
- cubical grid $\mathcal{C}^d$, with $\mathcal{C}_0^d = \mathbb{Z}^d$
- Input: set of d-cells K

The digital surface must be oriented



- cubical grid $\mathcal{C}^d$, with $\mathcal{C}^d_0 = \mathbb{Z}^d$
- Input: set of d-cells K
- X_K the 0-cells of the closure of K

The digital surface must be oriented



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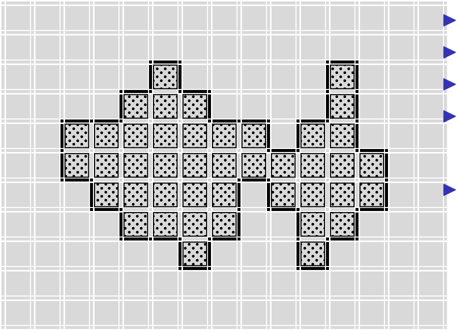

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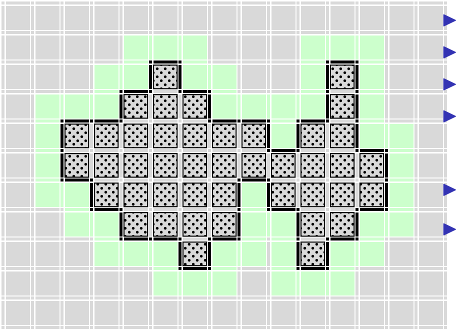
The digital surface must be oriented



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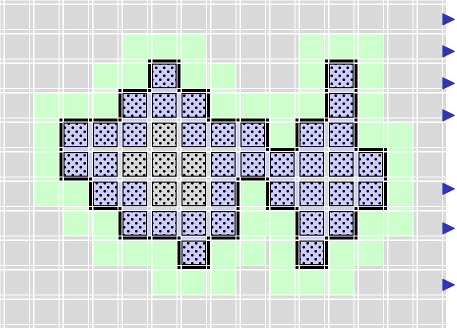
The digital surface must be oriented



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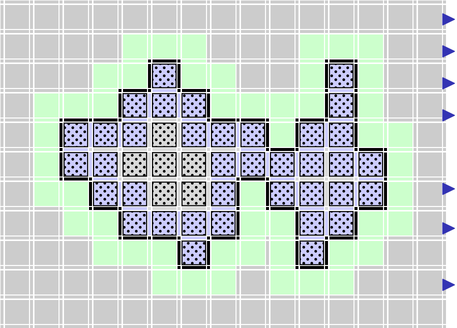
- $\mathrm{In}(K) := \{c \in \mathcal{C}^d, \mathrm{Extr}(c) \cap Z_K \neq \emptyset$ and $\mathrm{Extr}(c) \cap I_K \neq \emptyset\}$

The digital surface must be oriented



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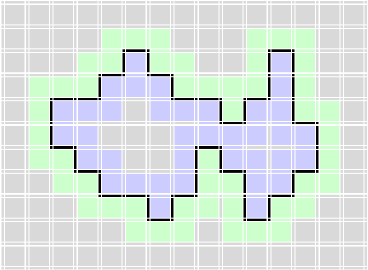
We have $\mathrm{Star}(\mathrm{Bd}(K)) = \mathrm{Bd}(K) \sqcup \mathrm{Out}(K) \sqcup \mathrm{In}(K)$

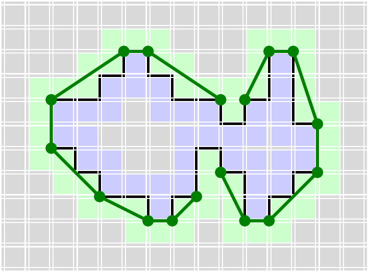
Convex and concave visibility



Cover of $Y \subset \mathbb{R}^d$

$\text{Cover}(Y) := \{c \in \mathcal{C}^d, c \cap Y \neq \emptyset\}$

Convex and concave visibility



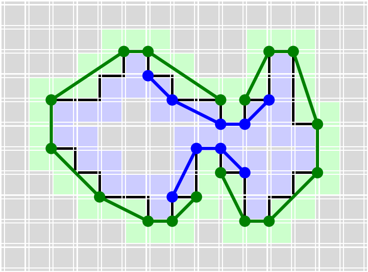
Cover of $Y \subset \mathbb{R}^d$

$\text{Cover}\,(Y) := \{c \in \mathcal{C}^d, c \cap Y \neq \emptyset\}$

Convex K-visibility

$A := \{p_1, \ldots, p_n\} \subset Z_K$ is **convex K-visible** iff $\text{Cover}\,(\text{Cvxh}\,(A)) \subset \text{Out}\,(K) \cup \text{Bd}\,(K)$

Convex and concave visibility



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Convex K-visibility

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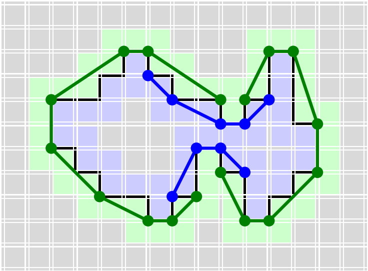
Convex and concave visibility



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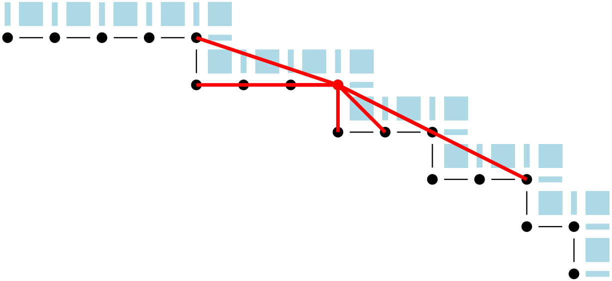
From now on, focus on convex visibility since concave visibility is entirely symmetric.

Convex visibility cone

$C_K(p)$

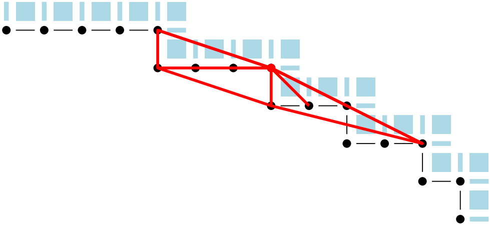
The **convex** K-**visibility cone** $C_K(p)$ of p is the set of points $q \in Z_K$ with $\{p, q\}$ convex K-visible.



Locally convex point, edge, face, ...

Locally convex point

Point $p \in Z_K$ is **locally convex** in K iff it is a vertex of $\mathrm{Cvxh}\,(C_K(p))$



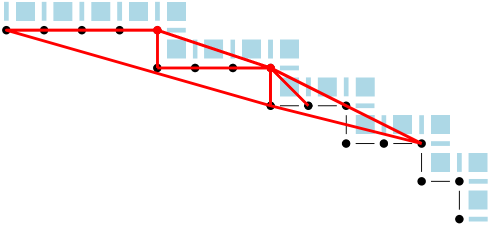
Locally convex point, edge, face, . . .

Locally convex point

Point $p \in Z_K$ is **locally convex** in K iff it is a vertex of $\mathrm{Cvxh}\,(C_K(p))$

Locally convex edge, face, . . .

Face $\{p_i\} \subset Z_K$ is **locally convex** in K iff it is a face of $\mathrm{Cvxh}\,(\cup_{p_i} C_K(p_i))$.



Locally convex point, edge, face, ...

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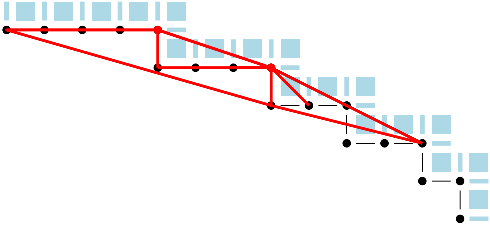

Lemma (Consistency of local convexity)

If A is locally convex in K, then any $A' \subset A$ is locally convex in K.

Full convexity implies local convexity

Full convexity [L. 2021]

The digital set $X \subset \mathbb{Z}^d$ is **fully convex** iff $\mathrm{Star}\,(\mathrm{Cvxh}\,(X)) \subset \mathrm{Star}\,(X)$.

- full convexity implies classical digital convexity
- full convexity implies connectedness

Full convexity implies local convexity

Full convexity [L. 2021]

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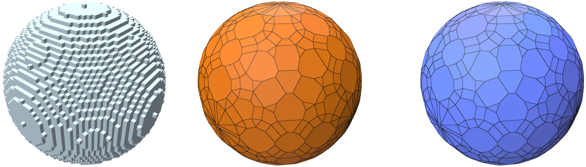
Theorem

Let $K \subset \mathcal{C}_d^d$ and X_K fully convex. The vertices and the faces of $\mathrm{Cvxh}\,(Z_K)$ are locally convex vertices and locally convex faces of K.



input K — $\mathrm{Cvxh}\,(X_K)$ — locally convex faces

Algorithm to extract locally convex zones

Input a set of n-dimensional cells K

- compute the boundary pointels Z_K of K
- compute the visibility cones $C_K(p)$, for all $p \in Z_K$
- for all point $p \in Z_K$
  - check if p is locally convex by computing $\mathrm{Cvxh}\,(C_K(p))$
  - collect incident edges in E if it is the case
  - store p in V if it is case
- for all edge $e := (p_1, p_2) \in E$
  - check if e is locally convex by computing $\mathrm{Cvxh}\,(\cup_i C_K(p_i))$
  - collect incident faces in F if it is the case
- for all face $f := (p_1, \ldots, p_k) \in F$
  - check if f is locally convex by computing $\mathrm{Cvxh}\,(\cup_i C_K(p_i))$
  - and store it in G if it is the case
- return locally convex points V and faces G

Algorithm to extract locally convex zones in 3d

Input a set of n-dimensional cells K

- compute the boundary pointels Z_K of K
- compute the visibility cones $C_K(p)$, for all $p \in Z_K$
- for all point $p \in Z_K$
  - check if p is locally convex by computing $\mathrm{Cvxh}\,(C_K(p))$
  - collect incident edges in E if it is the case
  - store p in V if it is case
- for all edge $e := (p_1, p_2) \in E$
  - check if e is locally convex by computing $\mathrm{Cvxh}\,(\cup_i C_K(p_i))$
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- return locally convex points V and faces G

More than 95% of the time is spent in computing visibility cones.
$\Rightarrow$ We have to prune the candidate locally convex points.

Around global and local convexity in digital spaces

Context: digital geometry and convexity

Geometry of Gauss digitized convex shapes

Locally convex or concave digital shapes

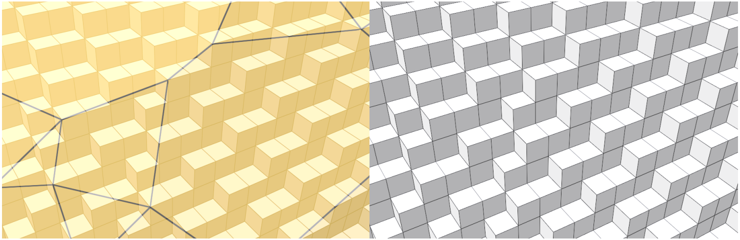
Fast extremal points identification with plane probing

joint work with

T. Roussillon, INSA Lyon

Conclusion

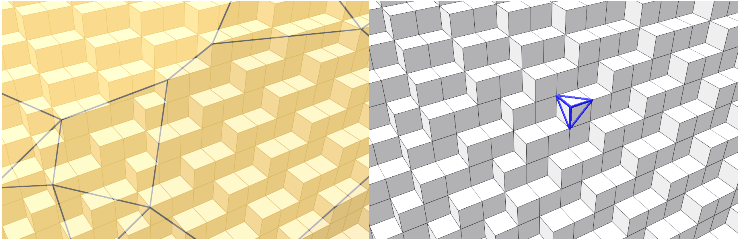
Finding extremal points



Objective

Find quickly the locally convex points of Z_K without computing their visibility cone.

Finding extremal points

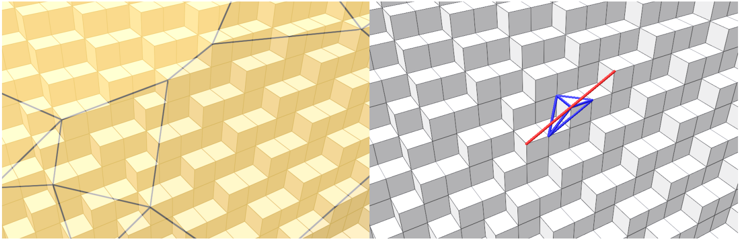


Objective

Find quickly the locally convex points of Z_K without computing their visibility cone.

- keep only the salient corners of Z_K

Finding extremal points

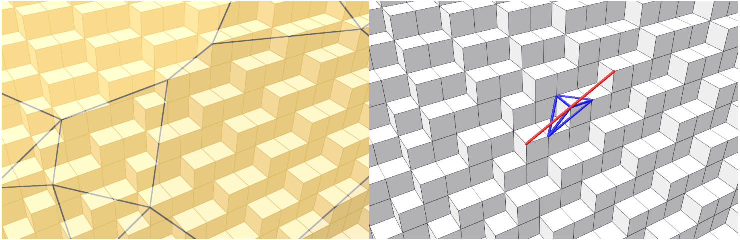


Objective

Find quickly the locally convex points of Z_K without computing their visibility cone.

- keep only the salient corners of Z_K
- eliminate corners c that are in-between two points of Z_K, i.e. $\exists \mathbf{v} \in \mathbb{Z}^d$ with $c \pm \mathbf{v} \in Z_K$

Finding extremal points



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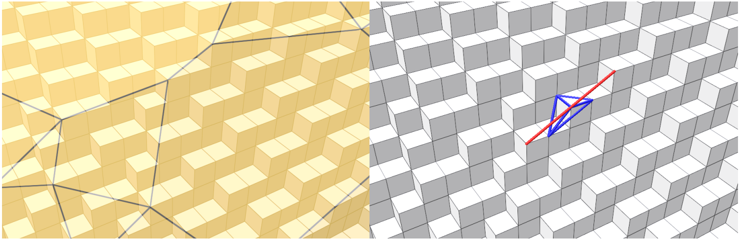
- method to find $\mathbf{v}$: a variant of plane probing, normally used for plane recognition [L., Provençal, Roussillon 16]

Finding extremal points



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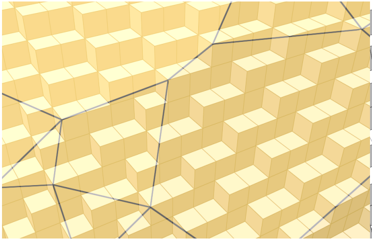
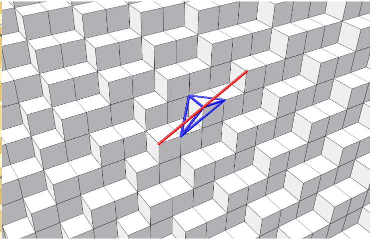
- symmetric approach for locally concave points

Finding extremal points

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Find quickly the locally convex points of Z_K without computing their visibility cone.

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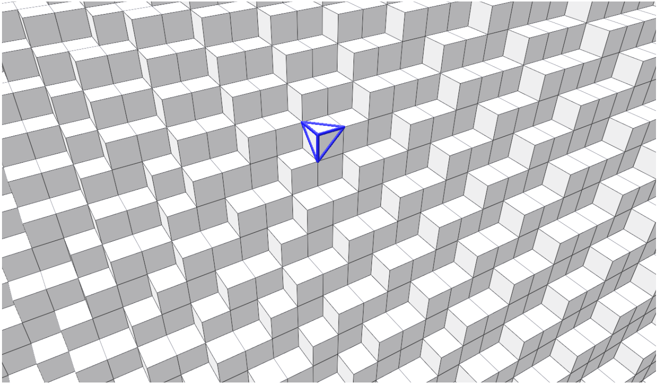
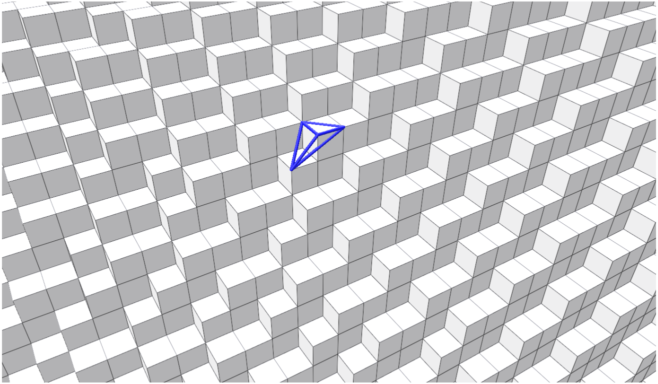
- eliminate corners c that are in-between two points of Z_K, i.e. $\exists \mathbf{v} \in \mathbb{Z}^d$ with $c \pm \mathbf{v} \in Z_K$
- method to find $\mathbf{v}$: a variant of plane probing, normally used for plane recognition [L., Provençal, Roussillon 16]
- symmetric approach for locally concave points
- formalized and illustrated in 3d, but easily extendable

Illustration of the plane probing algorithm



Start from a corner and deform the three basis vectors by probing Z_K. From now on, the corner is at $\mathbf{0}$ (just translate Z_k).

Illustration of the plane probing algorithm



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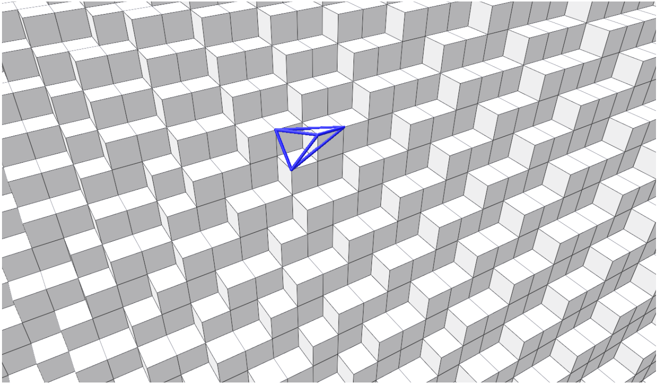

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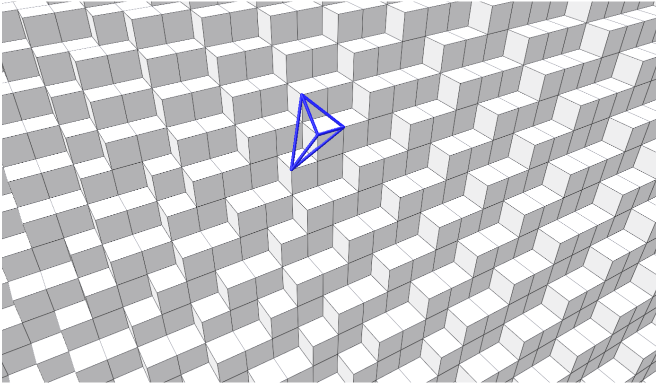

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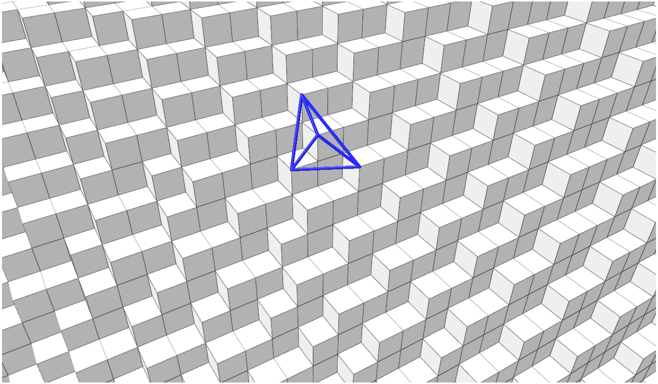

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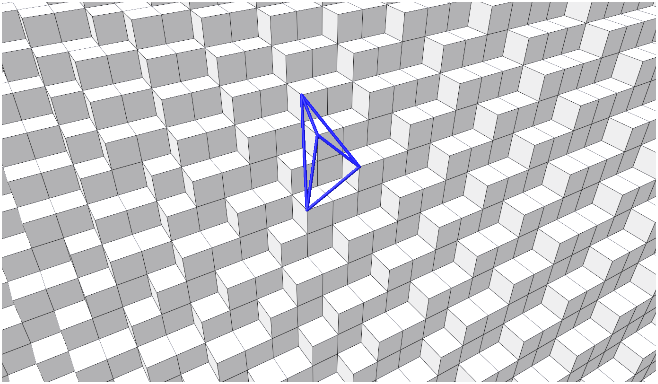

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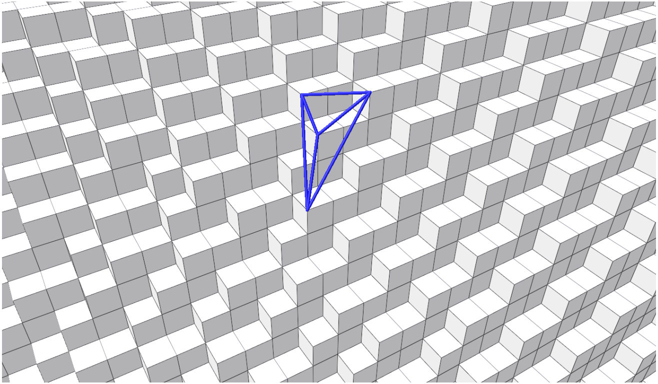

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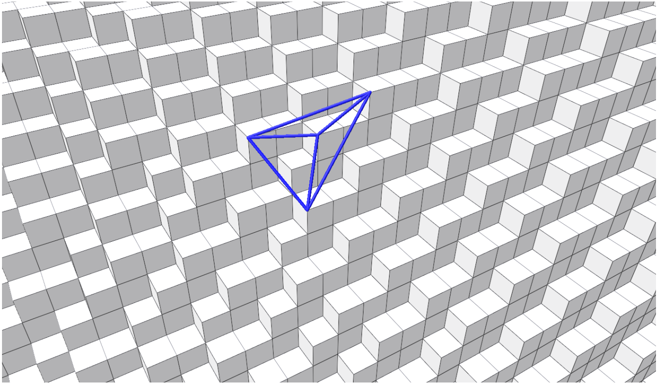

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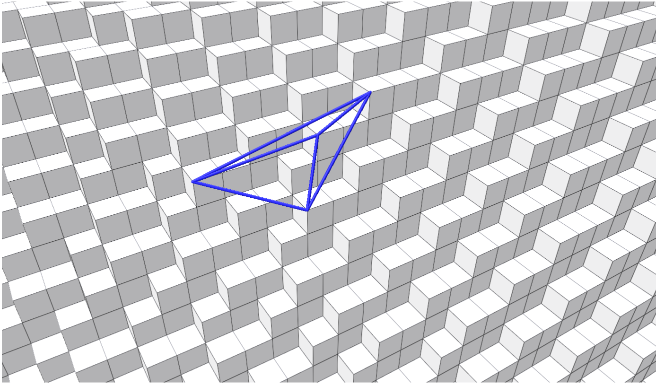
Illustration of the plane probing algorithm



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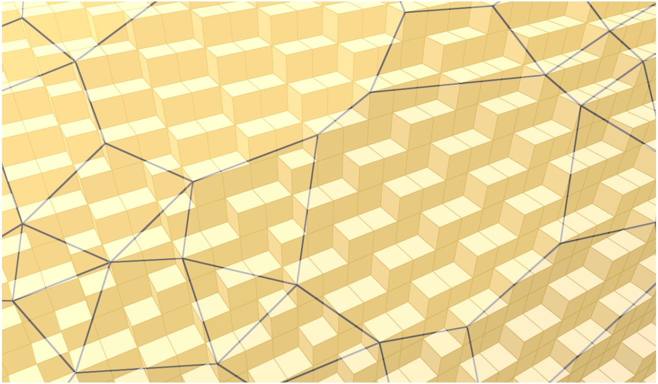

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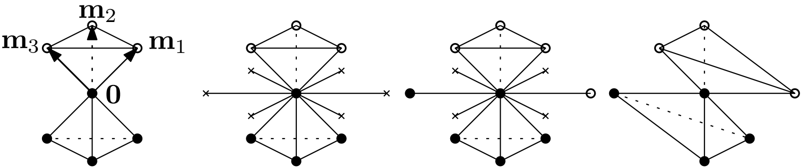
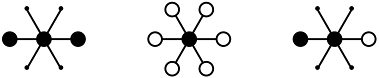
Probing algorithm

- **start** at corner $\mathbf{0} \in \mathbb{Z}_K$, with directions $\mathbf{M} = [\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3]$
- **Invariant**: "valid tetrahedron $\mathbf{M} = [\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3]$" with $-\mathbf{m}_k \in Z_K, \mathbf{m}_k \notin Z_K$
- Loop these steps



vectors $\mathbf{m}_k$ go up | probe $\mathbf{m}_k - \mathbf{m}_{k\pm1}$ in Z_K ? | choose a direction i going up | move $\mathbf{m}_i$

- possible configurations at the six points $\mathbf{m}_k - \mathbf{m}_{k\pm1}$



NO | YES | PROBING

movement is some $\mathbf{MN}_\sigma$

with $\mathbf{N}_\sigma = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Algorithm termination

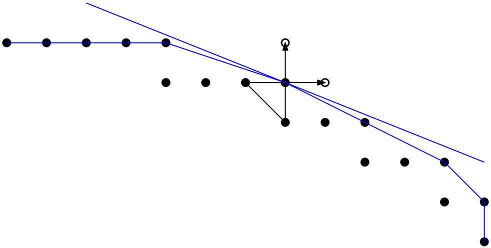
Theorem

Let $Z \subset \mathbb{Z}^3$ with $\mathbf{0}$ a salient corner. If $\mathbf{0}$ is a vertex of $\mathrm{Cvxh}\,(Z)$, the probing algorithm returns Yes *after at most n iterations, with $n \leqslant 2\sqrt{3}A$ and A the total area of the facets of $\mathrm{Cvxh}\,(Z)$ incident to $\mathbf{0}$.*

Theorem

Let $Z \subset \mathbb{Z}^3$ be a finite digital set containing $\mathbf{0}$ and let $\mathbf{M}$ be a valid initial tetrahedron. Then the probing algorithm terminates after a finite number of iterations.

Idea of the proof of first Theorem



- $\exists$ supporting plane with normal $\mathbf{v} \in \mathbb{Z}_+^2$ in $\mathbf{0}$, where $\|\mathbf{v}\|_1$ bounded by incident edge lengths (areas in dD)
- $\mathbf{v}^\top \mathbf{M} = (c_1, c_2)$, $\mathbf{v}^\top \mathbf{M}' = (c_1', c_2') = (c_1 - c_2, c_2)$ or $(c_1, c_2 - c_1)$, since $\mathbf{M}' = \mathbf{M}\mathbf{N}_\sigma$
- c_1, c_2 are positive integers, strictly decreasing

$$(2, 5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (2, 5)$$

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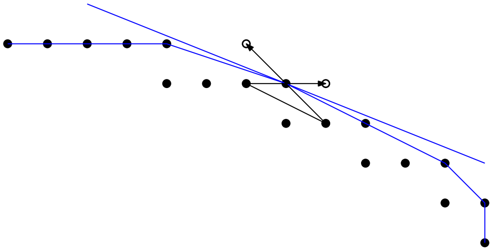

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$$(2, 5) \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = (2, 3)$$

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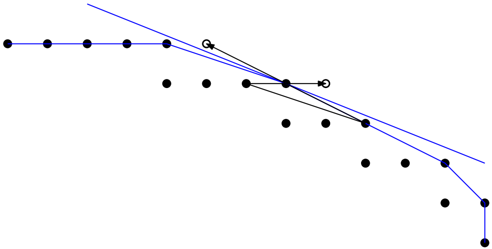

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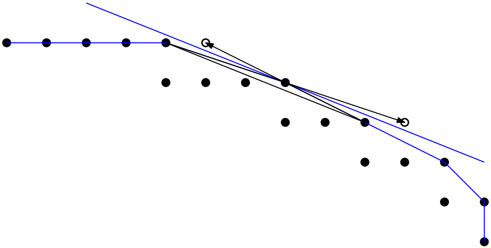

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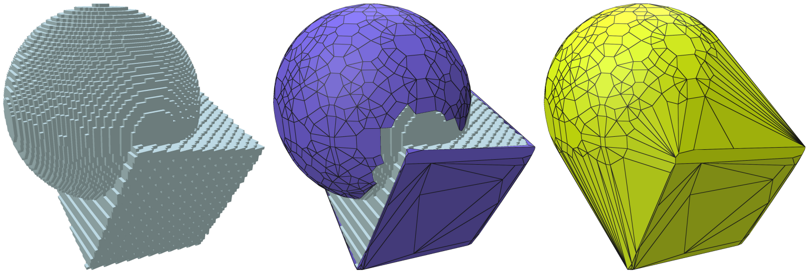
$$(2,5)\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = (1,1)$$

How good is the probing algorithm as a filter ?

- efficient on digitizations of smooth shapes (here ellipsoid) with gridstep h
- n_{init}: number of salient corners
- n_{final}: corners labeled as extremal by probing algorithm
- $n_{\mathrm{Cvxh}(Z)}$: expected number of vertices of $\mathrm{Cvxh}\,(Z)$

grid step	$\#Z$	n_{init}	n_{final}	$n_{\mathrm{Cvxh}(Z)}$
0.5	984	112	112	112
0.1	24.808	2.032	1.128	1.128
0.05	99.448	7.784	3.064	3.064
0.01	2.488.104	186.664	33.864	33.784

A few results (convex zones)



shape	$\#Z$	n_{init}	n_{final}	#facets	time(ms)
cps	34036	3681	991	959	2529
torus-knot-128	96622	15196	2924	2752	29321
sharpsphere129	119846	16715	3099	2542	40492

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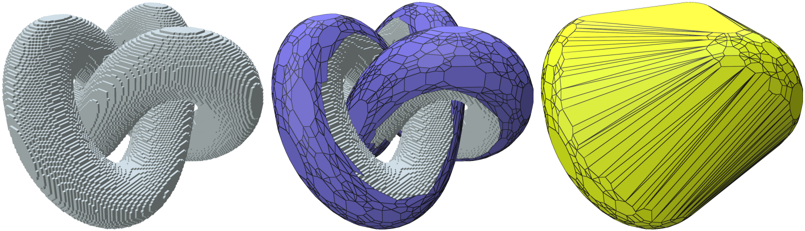

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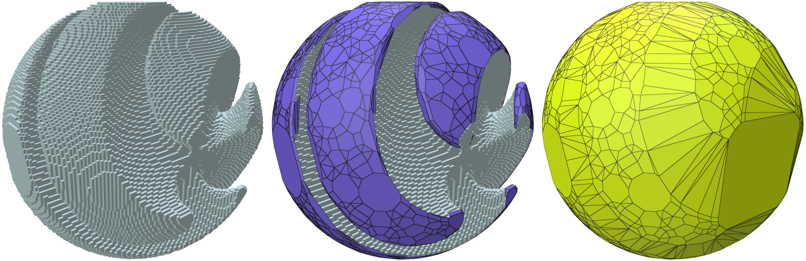

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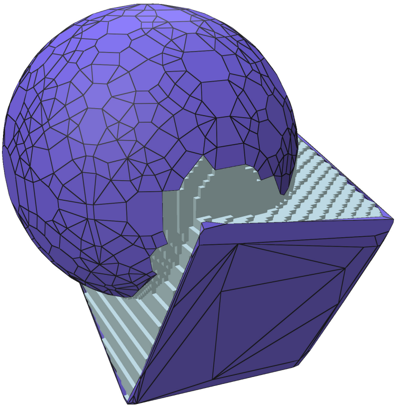
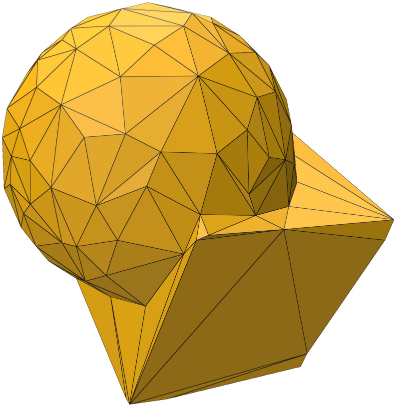
Comparison with greedy triangulation



our method



greedy triangulaton

Both methods are at Hausdorff distance 1 from digital surface.

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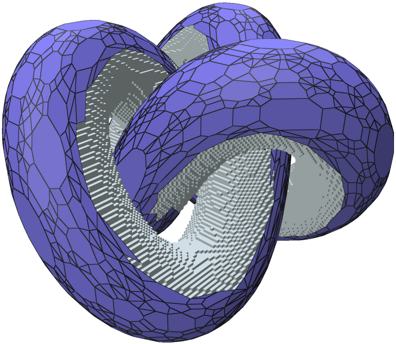

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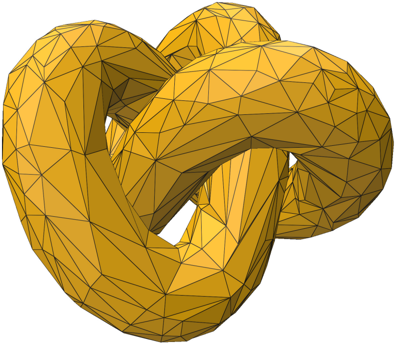

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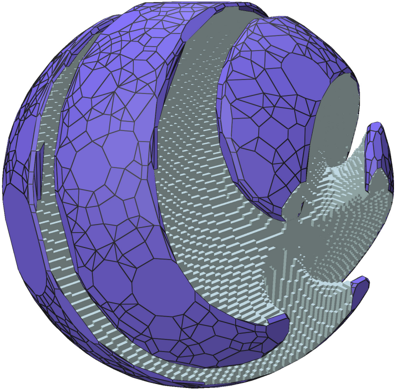

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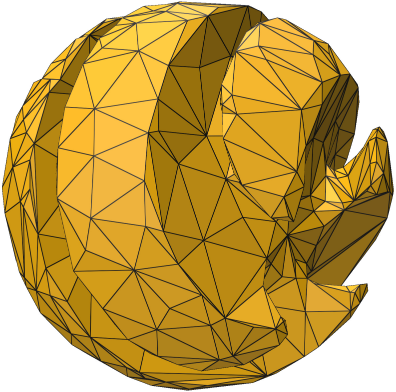

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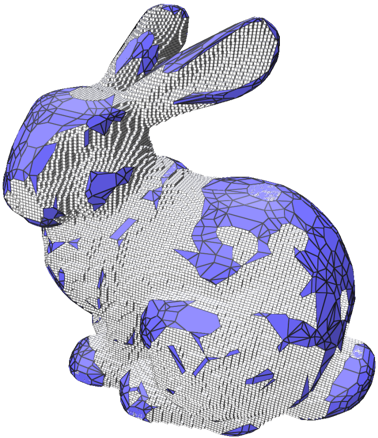

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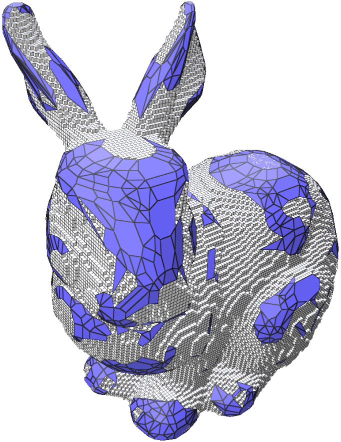

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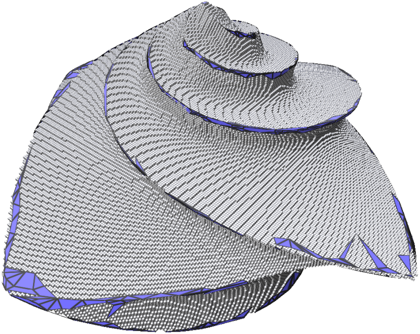

Around global and local convexity in digital spaces

Conclusion

convex hull is a good estimator in convex parts

- Hausdorff distance to shape is $\sqrt{d}h$;
- its vertices are even much closer to shape boundary $O\left(h^{\frac{2d}{d+1}}\right)$;
- its normals are close to shape normals as $\Theta(\sqrt{h})$,
- but not better in practice than classical estimators (II or VCM)

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a sound definition of local convexity / concavity

- cone of visibility between either outer or inner corners
- local definition of convexity through convex hulls of visibility cones
- fast probing algorithm to identify at 99% extremal points

Perspectives

How to speed up visibility computations ?

- restrict visibility computations to related extremal points
- identify compatibility between corners (difficulty is at orthant changes)
- compute visibility in a coarse-to-fine way with a pyramid descent

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- use other probing variants to identify meaningfull saddle vertices/edges
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Thank you for your attention !

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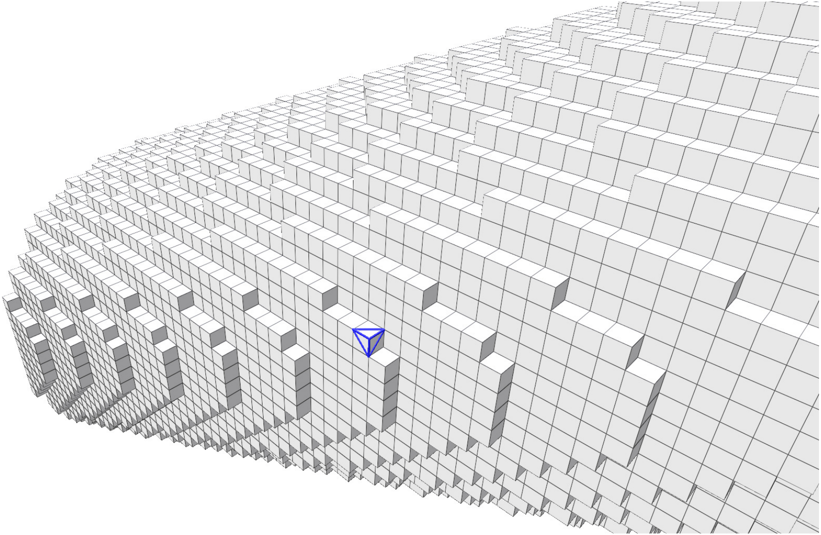

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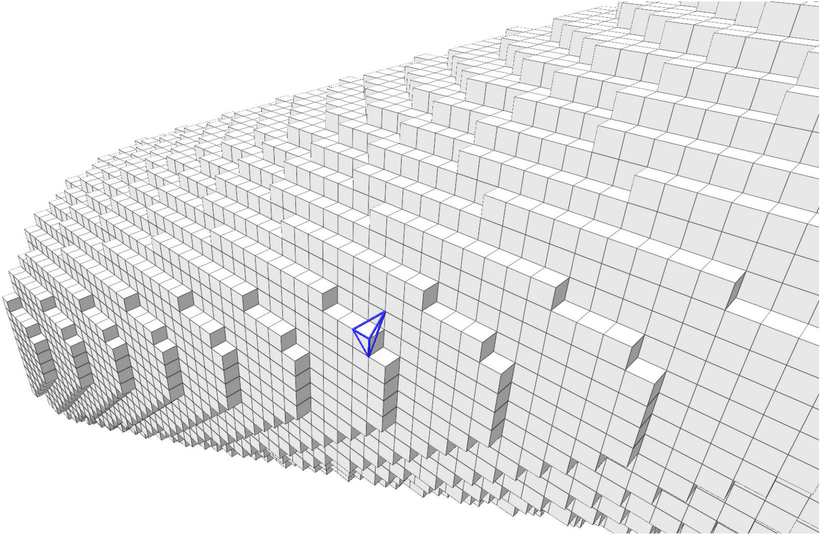

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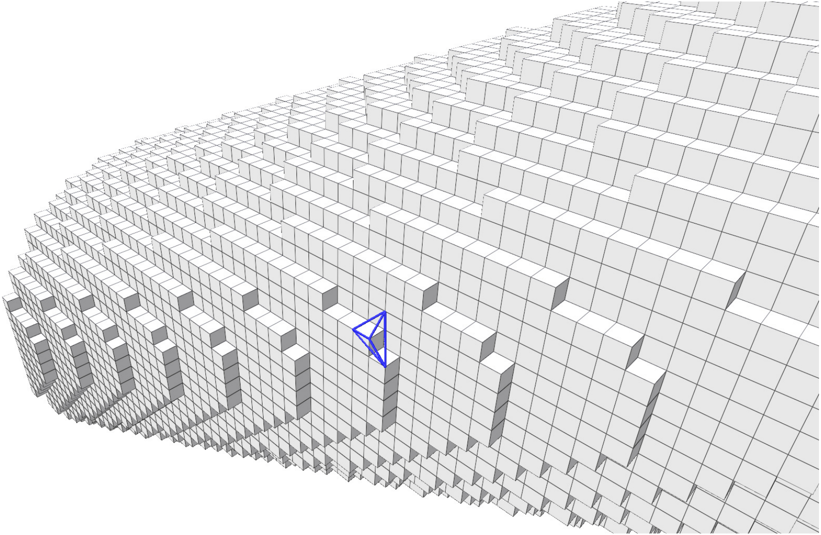

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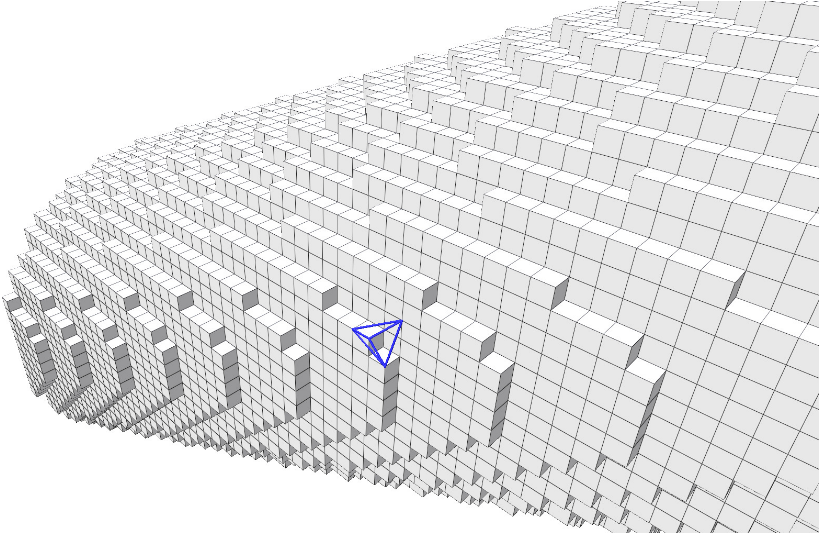

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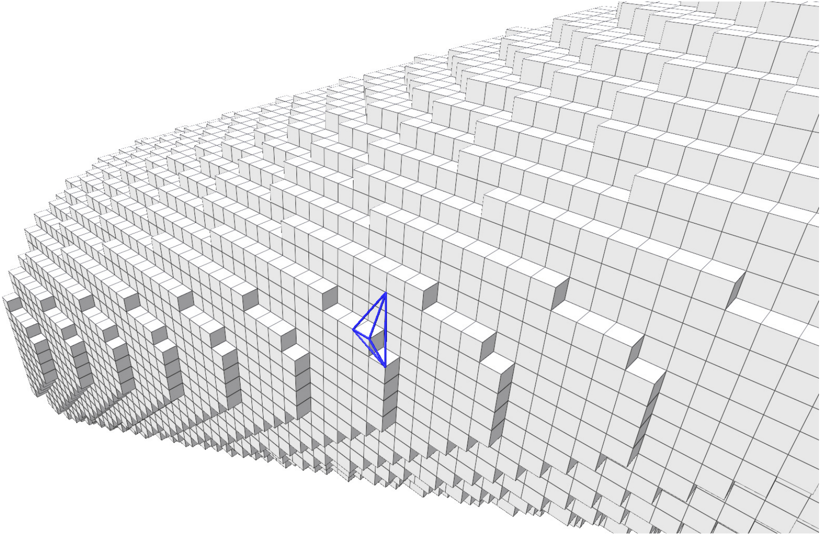

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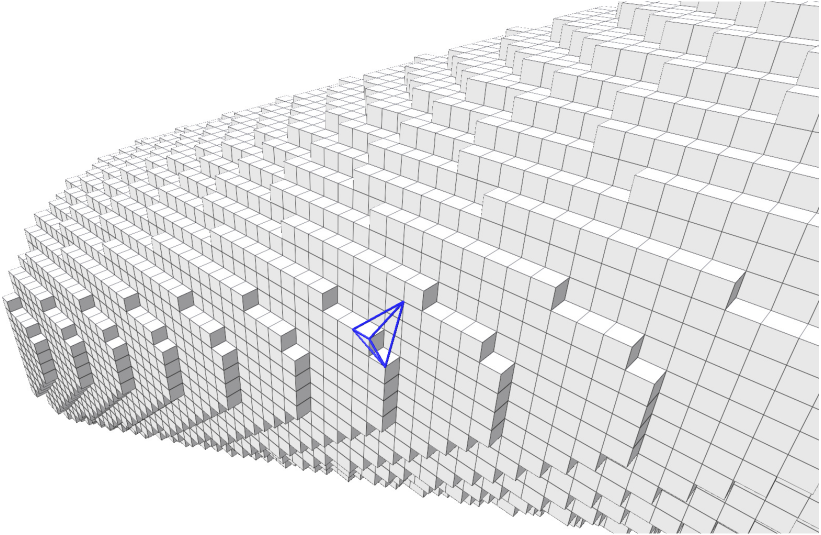

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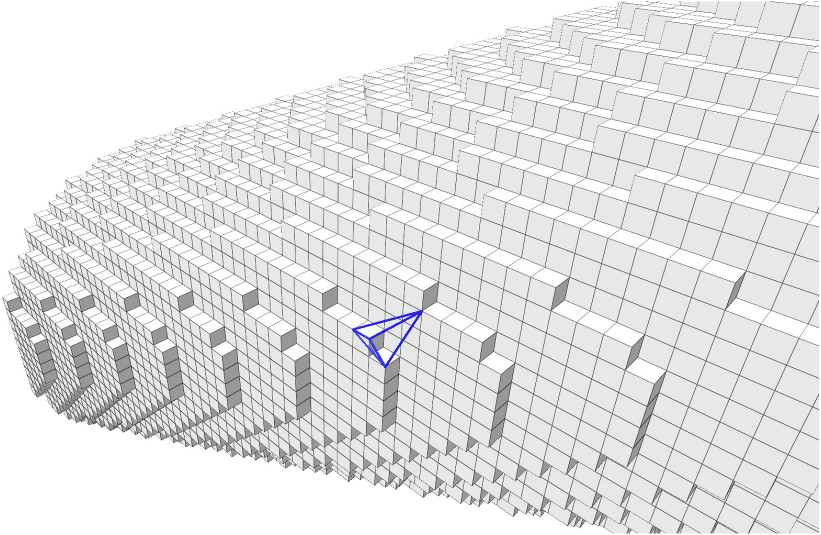

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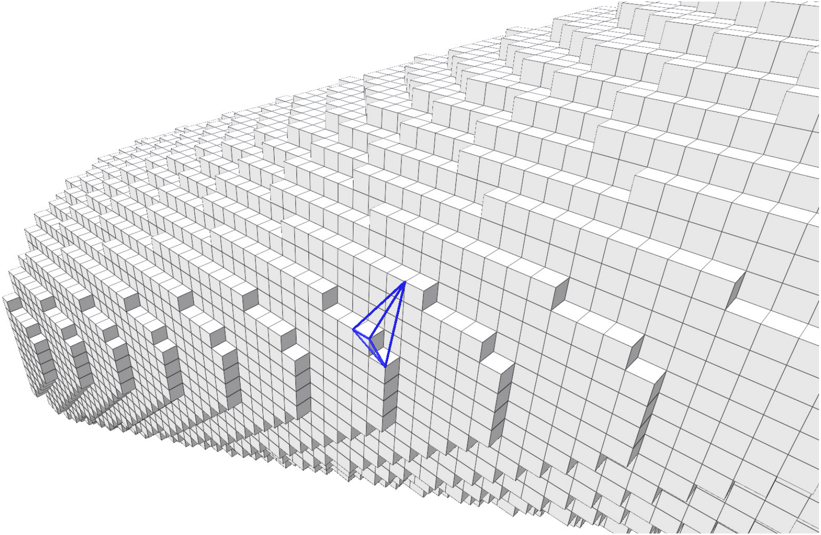

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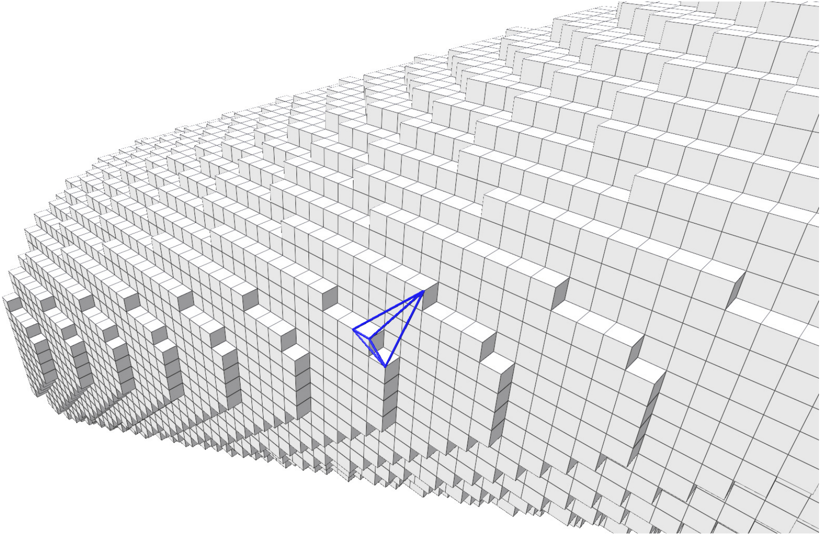

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