Analytical Description of Digital Intersections : Minimal parameters and Multiscale representation ☆

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Abstract

The paper contributes to a multiscale theory of digital shapes by presenting novel methods for a multiscale representation of digital lines and their intersections according to the Stern-Brocot tree. We give a new definition of the intersection (main conntected part) of two specific digital straight lines on the same quadrant (First quadrant). More precisely, we give some new results about the minimal set of parameters (i.e. slope (a,b), shift (μ), parity (even or odd), and the coordinates of the upper leaning points) for each line and their intersections.
keywords: multiscale geometry, standard lines, digital straight segment recognition, Stern-Brocot tree, Digital Intersection

1. Introduction

Methods for recognizing a digital straight segment are known since the early 1980s, with basic ideas dating back to the late 1960s. Digital Straight Lines (DSL) and Digital Straight Segments (DSS) have been known for many years to be interesting tools for digital curve and shape analysis. When a straight line is digitized on a square grid, they obtain a sequence of grid points defining a digital straight segment. Methods of recognizing digital straight segments are known since long. In one of the first methods, Freeman [8] suggested to analyze the regularity in the pattern of the directions in the chain code [7] of a digital curve. Anderson and Kim [1] have presented an analysis of the properties of the DSS's and suggested a different algorithm based on calculating the convex hull of the points of digital curves to be analyzed.

In [15], Reveilles proposed an arithmetical definition that allows the representation of naive digital lines as well as thicker and thiner line.

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In [13], Kovalevsky presented a new classification of digital curves into boundary curves. Boundary curves and lines are a useful mean for fast drawing of regions defined by their boundaries.

Discrete geometry is different from Euclidean geometry in many ways, and the differences between the intersection of two Euclidean lines and two digital lines is often used to illustrate this difference. Indeed, while the intersection of two non parallel Euclidean lines is a Euclidean point, the intersection of two digital lines can be a discrete point (pixel), a set of discrete points or even empty on regular grids. Examples of digital lines intersection are depicted on Figure 1.

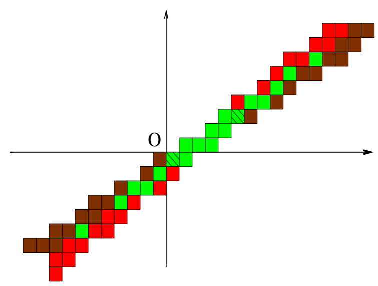


Figure 1: Intersection of $D_1(3, 4, 3)$ drawn as red boxes and $D_2(3, 5, 2)$ drawn as brown boxes, their intersection is drawn by green boxes. The endpoints of the main connected part are drawn by hatched green boxes. These two lines are in the same quadrant.

In [15], Reveilles presented a criterion to analyze the connectivity of the intersection of two digital naive lines with slopes between 0 and 1. But, he did not give any information about the intersection of any two digital naive lines.

In [4], Debled *et al.* presented a definition of the set of intersection pixels of two digital lines using a unimodular matrix. This definition enables the design of an efficient algorithm to determine all the pixels of an intersection, given the parameters of two lines. Sivignon *et al.* [17, 18] studied the geometrical and arithmetical properties of the intersection of two digital lines or planes. More precisely, some results about the connectivity, periodicity and minimal parameters of this intersection have been reported. They have proposed a characterization and an algorithm to find the minimal parameters of the intersection of any two digital naive lines using two different methods (Preimage study and Geometrical method) and emphasizing the links between them. They used the method derived from the solution proposed in the paper of Harel and Tarjan [9], for searching the nearest common ancestor of two nodes in a binary tree.

We recall in Section 2 some definitions and properties about rational fractions, more particularly the relation between the rational fractions and the Stern-Brocot tree. In Section 3, we calculate the intersection of two specific digital straight lines D_1 and D_2 in which these two lines belong to the same quadrant such that their intersection contains S. Moreover, we determine the coordinates of the upper leaning points and the position μ of each DSL. We note here that we propose new results about the combinatorics of such digital line intersections. We further show in Proposition 2, 3 (even and odd cases) that the computational complexity is constant time $O(1)$ to calculate the slope of the two DSL. In Section 4, we calculate the intersection of any two digital straight lines by using the arithmetic method. Section 5 determines the multiscale of digital curve that is extracted from a digital shape.

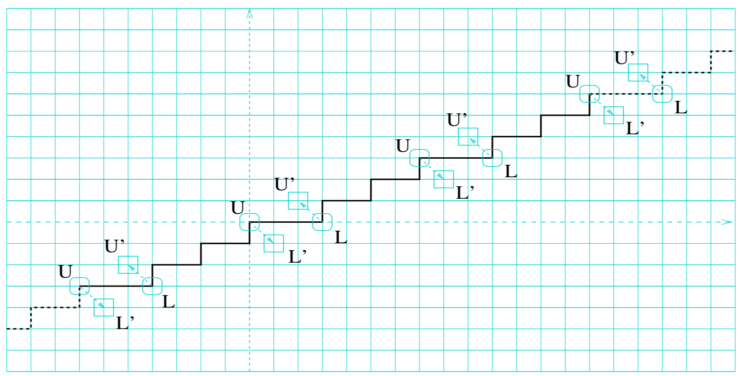


Figure 2: Positions of weakly exterior points on a digital straight line of characteristics $(3, 7, 0)$. Weakly exterior points are boxed and upper/lower leaning points are rectangular boxed with rounded corners.

2. Digital straightness and continued fractions

We recall the definition of Digital Straight Line in the first quadrant. the interior points and the weakly exterior points of a digital straight segment and the relation between a DSL and a simple continued fraction.

Definition 1. *[15] The set of points (x, y) of the digital plane verifying $\mu \leq ax - by < \mu + |a| + |b|$, with a and b are relatively prime integer numbers and μ is an integer number, is called standard line with slope a/b ($0 < a \leq b$ and $\gcd(a, b) = 1$) and shift μ (e.g. see Fig. 2).*

The standard lines are the 4-connected discrete lines. The quantity $r_{(a,b)}(P) = ax - by$ is the remainder of the points $P = (x, y)$ in the digital line of characteristics (a, b, μ). The points whose remainder is μ (resp. $\mu + |a| + |b| - 1$) are called upper (resp. lower) leaning points.

The original **DR95** [5] (reported in Klette and Rosenfeld [11]) algorithm recognizes naïve digital straight lines but it is easily adapted to standard lines. It extracts the characteristics (a, b, μ), with minimal $a + b$. The evolution of the characteristics is based on a simple test: each time we try to add a new point $4-$connected to the current digital straight segment, we compute its remainder with respect to the DSS parameters. According to this value the point can be added or not. If it is greater than or equal to $\mu + a + b + 1$ or less than or equal to

$\mu - 2$ the point is said to be exterior to the digital straight segment and cannot be added. Otherwise the point can be added to the segment to form a longer DSS and falls into two categories:

- interior points, with a remainder between μ and $\mu+a+b-1$ both included;
- weakly exterior points, with a remainder of $\mu-1$ for upper weakly exterior points and $\mu+a+b$ for lower weakly exterior points. Only in this case are the characteristics updated.

We also recall a few properties about *patterns* composing DSS and their close relations with continued fractions. They constitute a powerful tool to describe discrete lines with rational slopes (see Berstel and de Luca [2] for more details). All definitions and propositions stated below hold for DSS with slopes in the first quadrant. We can also transform this work to any quadrant.

Given a standard line (a, b, μ), we call *pattern* of characteristics (a, b) the succession of Freeman moves between any two consecutive upper leaning points. The Freeman moves defined between any two consecutive lower leaning points is the previous word read from back to front and is called the *reversed pattern*. As noted by several authors (e.g. see Reveilles [15], Klette and Rosenfeld [11], Voss 1991 [19], the work of deVieilleville and Lachaud reported in [3] or Kiryati *et al.* 1991 [10]), the pattern of any slope can be constructed from the continued fraction of the slope. We recall that a *simple continued fraction* is an expression:

$$z = \frac{a}{b} = [u_0, u_1, u_2, ..., u_i, ..., u_n] = u_0 + \frac{1}{u_1 + \frac{1}{... + \frac{1}{u_{n-1} + \frac{1}{u_n}}}},$$

where n is the *depth* of the fraction, and u_0, u_1, etc, are all integers and called the *partial quotients*. We call *k-th convergent* to the simple continued fraction formed of the k first partial quotients: $z_k = \frac{p_k}{q_k} = [u_0, u_1, u_2, ..., u_k]$.
We recall a few more relations regarding the way convergents are related:

$$\forall k \geq 1 \quad p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1} \tag{1}$$

$$p_0 = 0 \;\; p_{-1} = 1 \quad \forall k \geq 1 \quad p_k = u_k p_{k-1} + p_{k-2} \tag{2}$$

$$q_0 = 1 \;\; q_{-1} = 0 \quad \forall k \geq 1 \quad q_k = u_k q_{k-1} + q_{k-2} \tag{3}$$

Continued fractions can be finite or infinite, we focus on the case of rational slopes of lines in the first quadrant, that is finite continued fractions between 0 and 1. Then for each i, u_i is a strictly positive integer. In order to have a unique writing we consider that the last *partial quotient* is greater or equal to two except for slope $1 = [0, 1]$.
Let us now explain how to compute the *pattern* associated with a rational slope z in the first quadrant.
Consider E a mapping from the set of positive rational number smaller than one onto Freeman code words defined as follows.The function E takes a continued fraction z as input to build recursively the pattern of a DSS of slope z in the first quadrant.

$$E(z_{-2}) = 0, E(z_{-1}) = 1, \quad \text{and}, \forall i \geq 0, \quad \begin{cases} E(z_{2i+1}) = E(z_{2i})^{u_{2i+1}} E(z_{2i-1}), \\ E(z_{2i}) = E(z_{2i-2}) E(z_{2i-1})^{u_{2i}}. \end{cases} \tag{4}$$

Let us take for example the fraction $\frac{5}{17} = [0; 3, 2, 2]$. The pattern of any DSS with this slope is thus:

$$\begin{array}{rrll}
E(z_3) = & E([0;3,2,2]) = & E([0;3,2])^2 \cdot E([0;3]) & 000010001000010001 \cdot 0001 \\
E(z_2) = & E([0;3,2]) = & E([0]) \cdot E([0;3])^2 & 0 \cdot 00010001 \\
E(z_1) = & E([0;3]) = & 0001 & 0001 \\
E(z_0) = & E([0]) = & 0 & 0
\end{array}$$

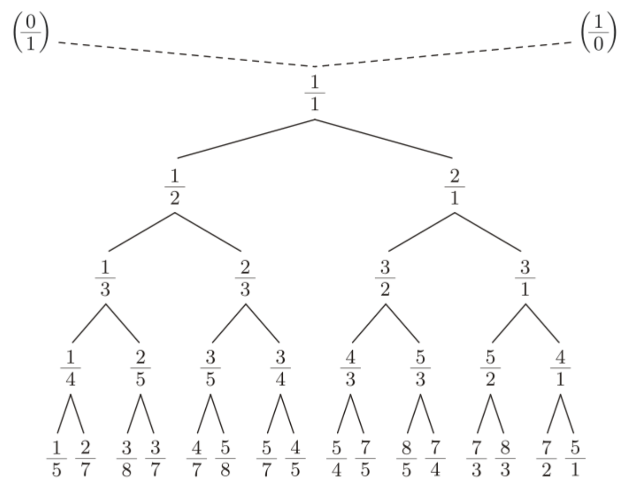


Figure 3: Stern-Brocot tree

The idea under its construction is to begin with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$ and to repeat the insertion of the mediant of these two fractions as follows: insert the median $\frac{m+m'}{n+n'}$ between $\frac{m}{n}$ and $\frac{m'}{n'}$. The sequence of partial quotients defines the sequence of right and left moves down the tree. Many works deal with the relations between irreducible rational fractions and digital lines (see Dorst and Smeulders [14] for characterization with Farey series, and Yaacoub [21] for a link with decomposition into continuous fractions). In [5], Debled first introduced the link between this tree and the recognition of digital line. Recognizing a piece of a digital line is like going down the Stern-Brocot tree up to the directional vector of the line. To sum up, the classical online DSS recognition algorithm **DR95** updates the DSS slope when adding a point that is just exterior to the current line (weak exterior points). The slope evolution is analytically given by next property.

Proposition 1. *[3] The slope evolution in **DR95** depends on the parity of the depth of its slope, the type of weakly exterior point added to the right or to the*

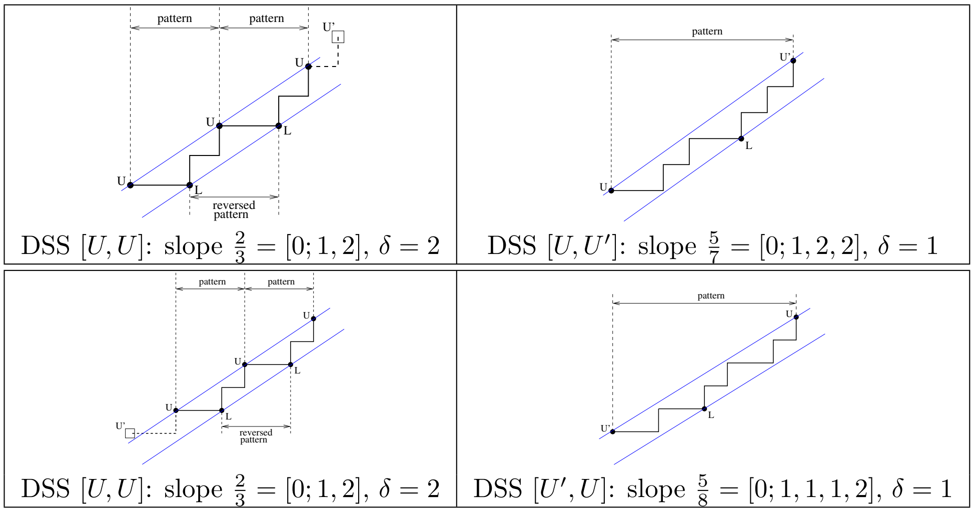


Figure 4: Slope evolution of a ULU DSS of slope $\frac{2}{3}$ with two patterns (and one reverse pattern). Top row: add the upper weakly exterior point U' to the right of the DSS, slope becomes $\frac{5}{7}$. Bottom row: add the upper weakly exterior point U' to the left of the DSS, slope becomes $\frac{5}{8}$.

left (*UWE and LWE stands respectively for upper and lower weakly exterior*) *and the number of patterns or reversed patterns in the current DSS. This is summed up in the table below, where the slope is* $[0, u_1, ..., u_n]$, $n = 2i$ *even or* $n = 2i + 1$ *odd,* δ *pattern(s) and* δ' *reversed pattern(s):*

- *Right side : an illustration is given in Fig. 4, top row*

	Even n	*Odd n*
UWE	$[0, u_1, ..., u_{2i}, \delta]$	$[0, u_1, ..., u_{2i+1} - 1, 1, \delta]$
LWE	$[0, u_1, ..., u_{2i} - 1, 1, \delta']$	$[0, u_1, ..., u_{2i+1}, \delta']$

- *Left side : an illustration is given in Fig. 4, bottom row*

	Even n	*Odd n*
UWE	$[0, u_1, ..., u_{2i} - 1, 1, \delta]$	$[0, u_1, ..., u_{2i+1}, \delta]$
LWE	$[0, u_1, ..., u_{2i}, \delta']$	$[0, u_1, ..., u_{2i+1} - 1, 1, \delta']$

We may look again at our example of fraction $\frac{5}{17}$. The path in the Stern-Brocot tree from the root $\frac{0}{1}$ to this fraction is the list of nodes $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{5}{17}$. Any *DSS* in a *DSL* of slope $\frac{5}{17}$ has a slope which is one of these fractions. We notice that the *k-th convergent* of $\frac{5}{17}$ is a fraction of the previous list.

3. Digital lines intersection.

In the previous section, we provide the slope evolution of a DSS S when we add a weakly exterior point (upper or lower) to the left or to the right of a DSS.

This section shows how to, from a given DSS S, define two digital straight lines $D_1(S)$ and $D_2(S)$ such that $S \subseteq MCP(D_1(S), D_2(S))$ (MCP stands for "main connected part" and is defined in definition 2) and study the precise structure of $MCP(D_1(S), D_2(S))$ (Proposition 2 and 3). These two lines are related to the downward moves in the Stern-Brocot tree during a DSS recognition. Their patterns are placed so that one starts at the first upper leaning point and the other ends at the last upper leaning point (see Fig. 4, or the paper of deVieilleville and Lachaud [3] for more details about the patterns).

These two digital straight lines $D_1(S)$ and $D_2(S)$ are built using Theorem 1, by adding an upper (or lower) weakly exterior point at the front or at the back of S. Lemma 1 gives the remainders and the coordinates of the upper leaning points of these two DSLs.

Theorem 1. *[3] Given a DSS S with even slope $z_{2i} = \frac{p_{2i}}{q_{2i}} = [0, u_1, ..., u_{2i}]$ (or with odd slope $z_{2i+1} = \frac{p_{2i+1}}{q_{2i+1}} = [0, u_1, ..., u_{2i+1}]$), their exist two digital straight lines $D_1(S)$ and $D_2(S)$ with slopes as defined in the table below such that $S \subset D_1(S) \cap D_2(S)$.*

		Upper weakly exterior	*Lower weakly exterior*
S has an even slope	$D_1(S)$	$\frac{\delta p_{2i}+p_{2i-1}}{\delta q_{2i}+q_{2i-1}}$	$\frac{(\delta'+1)p_{2i}-p_{2i-1}}{(\delta'+1)q_{2i}-q_{2i-1}}$
	$D_2(S)$	$\frac{(\delta+1)p_{2i}-p_{2i-1}}{(\delta+1)q_{2i}-q_{2i-1}}$	$\frac{\delta' p_{2i}+p_{2i-1}}{\delta' q_{2i}+q_{2i-1}}$
S has an odd slope	$D_1(S)$	$\frac{\delta p_{2i+1}+p_{2i}}{\delta q_{2i+1}+q_{2i}}$	$\frac{(\delta'+1)p_{2i+1}-p_{2i}}{(\delta'+1)q_{2i+1}-q_{2i}}$
	$D_2(S)$	$\frac{(\delta+1)p_{2i+1}-p_{2i}}{(\delta+1)q_{2i+1}-q_{2i}}$	$\frac{\delta' p_{2i+1}+p_{2i}}{\delta' q_{2i+1}+q_{2i}}$

Proof. Let S has an even depth and slope $\dfrac{p_{2i}}{q_{2i}} = [u_0, u_1, ..., u_{2i}]$, then:

1. D_1 has slope $z^1_{2i+1} = \frac{p^1_{2i+1}}{q^1_{2i+1}} = [u_0, u_1, ..., u_{2i}, \delta]$ by adding an UWE to the right (Proposition 1):
$\frac{p^1_{2i+1}}{q^1_{2i+1}} = \frac{u^1_{2i+1}p^1_{2i}+p^1_{2i-1}}{u^1_{2i+1}q^1_{2i}+q^1_{2i-1}} = \frac{\delta p_{2i}+p_{2i-1}}{\delta q_{2i}+q_{2i-1}}$

2. D_2 has slope $z^2_{2i+2} = \frac{p^2_{2i+2}}{q^2_{2i+2}} = [u_0, u_1, ..., u_{2i} - 1, 1, \delta]$ by adding an UWE to the left (Proposition 1):
$\frac{p^2_{2i+2}}{q^2_{2i+2}} = \frac{u^2_{2i+2}p^2_{2i+1}+p^2_{2i}}{u^2_{2i+2}q^2_{2i+1}+q^2_{2i}} = \frac{u^2_{2i+2}(u^2_{2i+1}p^2_{2i}+p^2_{2i-1})+p^2_{2i}}{u^2_{2i+2}(u^2_{2i+1}q^2_{2i}+q^2_{2i-1})+q^2_{2i}} = \frac{\delta(1\times p^2_{2i}+p^2_{2i-1})+p^2_{2i}}{\delta(1\times q^2_{2i}+q^2_{2i-1})+q^2_{2i}} =$
$\frac{(\delta+1)p^2_{2i}+\delta p^2_{2i-1}}{(\delta+1)q^2_{2i}+\delta q^2_{2i-1}} = \frac{(\delta+1)[(u_{2i}-1)p_{2i-1}+p_{2i-2}]+\delta p_{2i-1}}{(\delta+1)[(u_{2i}-1)q_{2i-1}+q_{2i-2}]+\delta q_{2i-1}} = \frac{(\delta+1)(u_{2i}p_{2i-1}+p_{2i-2})-p_{2i-1}}{(\delta+1)(u_{2i}q_{2i-1}+q_{2i-2})-q_{2i-1}}$
$= \frac{(\delta+1)p_{2i}-p_{2i-1}}{(\delta+1)q_{2i}-q_{2i-1}}$

3. D_1 has slope $z^1_{2i+2} = \frac{p^1_{2i+2}}{q^1_{2i+2}} = [u_0, u_1, ..., u_{2i} - 1, 1, \delta']$ by adding an LWE to the left (Proposition 1):
$\frac{p^1_{2i+2}}{q^1_{2i+2}} = \frac{u^1_{2i+2}p^1_{2i+1}+p^1_{2i}}{u^1_{2i+2}q^1_{2i+1}+q^1_{2i}} = \frac{u^1_{2i+2}(u^1_{2i+1}p^1_{2i}+p^1_{2i-1})+p^1_{2i}}{u^1_{2i+2}(u^1_{2i+1}q^1_{2i}+q^1_{2i-1})+q^1_{2i}} = \frac{\delta'(1\times p^1_{2i}+p^1_{2i-1})+p^1_{2i}}{\delta'(1\times q^1_{2i}+q^1_{2i-1})+q^1_{2i}} =$

$\frac{(\delta'+1)p^1_{2i}+\delta' p^1_{2i-1}}{(\delta'+1)q^1_{2i}+\delta' q^1_{2i-1}} = \frac{(\delta'+1)[(u_{2i}-1)p_{2i-1}+p_{2i-2}]+\delta' p_{2i-1}}{(\delta'+1)[(u_{2i}-1)q_{2i-1}+q_{2i-2}]+\delta' q_{2i-1}} = \frac{(\delta'+1)(u_{2i}p_{2i-1}+p_{2i-2})-p_{2i-1}}{(\delta'+1)(u_{2i}q_{2i-1}+q_{2i-2})-q_{2i-1}}$
$= \frac{(\delta'+1)p_{2i}-p_{2i-1}}{(\delta'+1)q_{2i}-q_{2i-1}}$

4. D_2 has slope $z^2_{2i+1} = \frac{p^2_{2i+1}}{q^2_{2i+1}} = [u_0, u_1, ..., u_{2i}, \delta']$ by adding an LWE to the right (Proposition 1):
$\frac{p^2_{2i+1}}{q^2_{2i+1}} = \frac{u^2_{2i+1}p^2_{2i}+p^2_{2i-1}}{u^2_{2i+1}q^2_{2i}+q^2_{2i-1}} = \frac{\delta' p_{2i}+p_{2i-1}}{\delta' q_{2i}+q_{2i-1}}$

Definition 2. *The main connected part $S_m = MCP(D_1(S), D_2(S))$ of two specific digital straight lines $D_1(S)$ and $D_2(S)$ denotes the centered connected region of the intersection of both DSLs and contains S. S_m is defined by $w_1 E(z_n)^\delta w_2$, where w_1 is the prefix of S_m before the first upper leaning point of S, w_2 is the suffix of S_m after the last upper leaning point of S, and δ is the number of patterns of S. w_1 (resp. w_2) is also the suffix (resp. prefix) of S. The slope of S_m is equal to the slope of S.*

In the next lemma, we focus only on the calculation of the upper leaning points of two digital straight lines already denoted $D_1(S)$ and $D_2(S)$. We do not need to study the LUL case, because we can transform it to ULU case (see Fig. 8,a,b).

Lemma 1. *Let $D_1(S)$ and $D_2(S)$ be two digital straight lines. Assume that the main connected part $MCP(D_1(S), D_2(S)) = S_m(a, b, \mu)$ has an even complexity (or an odd complexity) with the intercept $\mu = a(x - x_0) - b(y - y_0)$ where (x, y) is the coordinate of the leftmost upper leaning point of S_m and (x_0, y_0) defines the origin of the pixels in $\mathbb{Z}^2$. Then the remainders of $D_1(S)$ and $D_2(S)$ are respectively $\delta\mu + \mu_p$ and $(\delta + 1)\mu - \delta - \mu_p$ (or the remainders of $D_1(S)$ and $D_2(S)$ are respectively $\delta\mu - \delta + \mu_p$ and $(\delta + 1)\mu - \mu_p$), where $\mu_p = p_{n-1}(x - x_0) - q_{n-1}(y - y_0)$ and $\frac{p_{n-1}}{q_{n-1}}$ is the $(n-1)$-th* convergent *of $\frac{p_n}{q_n}$ ($\frac{p_n}{q_n} = \frac{a}{b}$), and the coordinates of the upper leaning points are given by (An illustration of this lemma is given in Figure 5).*

	$D_1(S)$	$D_2(S)$
S_m has an even slope	$(x - x_0, y - y_0) + k(\delta q_{2i} + q_{2i-1}, \delta p_{2i} + p_{2i-1})$	$(x - x_0 + \delta q_{2i}, y - y_0 + \delta p_{2i}) + k((\delta + 1)q_{2i} - q_{2i-1}, (\delta + 1)p_{2i} - p_{2i-1})$
S_m has an odd slope	$(x - x_0 + \delta q_{2i+1}, y - y_0 + \delta p_{2i+1}) + k(\delta q_{2i+1} + q_{2i}, \delta p_{2i+1} + p_{2i})$	$(x - x_0, y - y_0) + k((\delta + 1)q_{2i+1} - q_{2i}, (\delta+1)p_{2i+1} - p_{2i})$

Proof. If S_m has an even depth, then the slope of $D_1(S)$ is $\frac{\delta p_{2i}+p_{2i-1}}{\delta q_{2i}+q_{2i-1}}$ and the slope of $D_2(S)$ is $\frac{(\delta+1)p_{2i}-p_{2i-1}}{(\delta+1)q_{2i}-q_{2i-1}}$. The remainders of the upper leaning points in $D_1(S)$ and $D_2(S)$ are calculated as follows:

$r_{D_1}((x-x_0, y-y_0)+k(\delta q_{2i}+q_{2i-1}, \delta p_{2i}+p_{2i-1}))$

$=r_{D_1}((x-x_0, y-y_0))+kr_{D_1}((\delta q_{2i}+q_{2i-1}, \delta p_{2i}+p_{2i-1}))$
$=r_{D_1}((x-x_0, y-y_0))+k[(\delta p_{2i}+p_{2i-1})(\delta q_{2i}+q_{2i-1})-(\delta q_{2i}+q_{2i-1})(\delta p_{2i}+p_{2i-1})]$
$=r_{D_1}((x-x_0, y-y_0))+k\times 0$
$=(\delta p_{2i}+p_{2i-1})(x-x_0)-(\delta q_{2i}+q_{2i-1})(y-y_0)$
$=\delta[p_{2i}(x-x_0)-q_{2i}(y-y_0)]+(p_{2i-1}(x-x_0)-q_{2i-1}(y-y_0))$
$=\delta\mu+\mu_p$

and
$r_{D_2}((x-x_0+\delta q_{2i}, y-y_0+\delta p_{2i})+k((\delta+1)q_{2i}-q_{2i-1}, (\delta+1)p_{2i}-p_{2i-1}))$

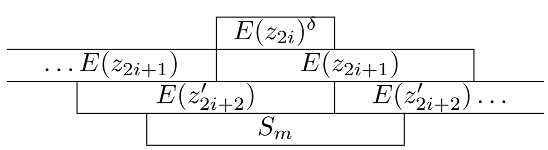
$=r_{D_2}((x-x_0+\delta q_{2i}, y-y_0+\delta p_{2i}))+kr_{D_2}(((\delta+1)q_{2i}-q_{2i-1}, (\delta+1)p_{2i}-p_{2i-1}))$
$=r_{D_2}((x-x_0+\delta q_{2i}, y-y_0+\delta p_{2i}))+k[((\delta+1)p_{2i}-p_{2i-1})[(\delta+1)q_{2i}-q_{2i-1}]-$
$((\delta+1)q_{2i}-q_{2i-1})[(\delta+1)p_{2i}-p_{2i-1}]]$
$=r_{D_2}((x-x_0+\delta q_{2i}, y-y_0+\delta p_{2i}))+k\times 0$
$=(x-x_0+\delta q_{2i})[(\delta+1)p_{2i}-p_{2i-1}]-(y-y_0+\delta p_{2i})[(\delta+1)q_{2i}-q_{2i-1}]$
$=(\delta+1)[p_{2i}(x-x_0)-q_{2i}(y-y_0)]+\delta(p_{2i}q_{2i-1}-q_{2i}p_{2i-1})-(p_{2i-1}(x-x_0)-$
$q_{2i-1}(y-y_0))$
$=(\delta+1)\mu-\delta\times 1-\mu_p$
$=(\delta+1)\mu-\delta-\mu_p.$

The proof of the second case is analogous to the first one. □

3.1. Slopes

We propose here two propositions (Proposition 2 and 3) that give the exact combinatorial structure (slope and repetition of the pattern) of the main connected part of $D_1(S)$ and $D_2(S)$, which is denoted by S_m.

Proposition 2. *Let S be a digital straight segment of even slope $z_{2i} = [0, u_1, u_2, \ldots, u_{2i}]$ and let $D_1(S)$ and $D_2(S)$ be two specific digital straight lines. We have $D_1(S)$ has an odd slope z_{2i+1} with $z_{2i+1} = [0, u_1, \cdots, u_{2i}, \delta]$ and $D_2(S)$ has an even slope z'_{2i+2} with $z'_{2i+2} = [0, u_1, \cdots, u_{2i}-1, 1, \delta]$, with the slope of $D_1(S)$ is greater than the slope of $D_2(S)$ (from Theorem 1). Then the intersection S_m (main connected part) of $D_1(S)$ and $D_2(S)$ is exactly $w_1 E(z_{2i})^\delta w_2$, with $w_1 = E(z_1)^{u_2} \cdots E(z_{2i-2k-1})^{u_{2i-2k}} \cdots E(z_{2i-3})^{u_{2i-2}} E(z_{2i-1})^{u_{2i}-1} = \#_{k=i-1}^{1} E(z_{2i-2k-1})^{u_{2i-2k}} E(z_{2i-1})^{u_{2i}-1}$ and $w_2 = E(z_{2i-2})^{u_{2i-1}} \cdots E(z_{2i-2k})^{u_{2i-2k+1}} \cdots E(z_2)^{u_3} E(z_0)^{u_1} = \#_{k=1}^{i} E(z_{2i-2k})^{u_{2i-2k+1}}$ (# is the concatenation of $E(.)$) The parity of the depth of S_m is also even.*



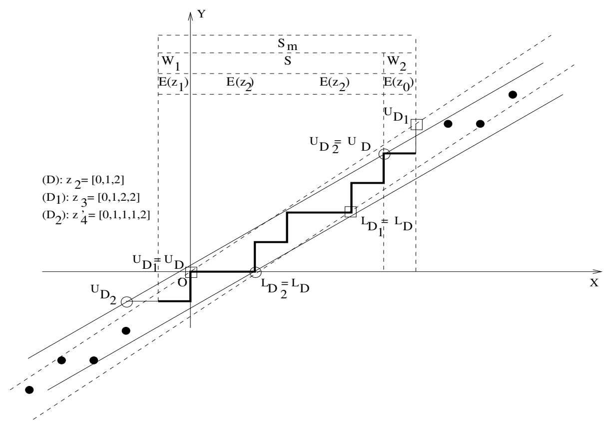


Figure 5: Intersection of two patterns $E(z_3)$ and $E(z_4')$, where S_m is the main connected part of their intersection. The leaning points of $D_1(S)$ (resp. of $D_2(S)$) are drawn as boxes (resp. as circles). The black line represents the word of the segment S_m. S is a DSS of two patterns $\delta = 2$ and characteristics $(\frac{2}{3}, \mu = 0)$ included in S_m.

Proof. D_1 of odd slope z_{2i+1} and D_2 of even slope $z^{'}_{2i+2}$, then from (4), we have:

$$E(z_{2i+1}) = E(z_{2i})^{u_{2i+1}} E(z_{2i-1}) \quad = \quad E(z_{2i})^{\delta} E(z_{2i-1}).$$

and

$$\begin{aligned}
z_{2i} &= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i}] \\
z_{2i+1} &= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i}, u_{2i+1}] \\
&= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i}, \delta] \\
z^{'}_{2i+2} &= \left[0, u^{'}_1, u^{'}_2, \cdots, u^{'}_{2i-2}, u^{'}_{2i-1}, u^{'}_{2i}, u^{'}_{2i+1}, u^{'}_{2i+2}\right] \\
&= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i} - 1, 1, \delta] \\
z^{'}_{2i-2} &= [0, u_1, u_2, \cdots, u_{2i-2}] = z_{2i-2} \\
z^{'}_{2i-1} &= [0, u_1, u_2, \cdots, u_{2i-1}] = z_{2i-1} \\
z^{'}_{2i+1} &= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i} - 1, 1] \\
&= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i} - 1 + 1] \\
&= [0, u_1, u_2, \cdots, u_{2i-2}, u_{2i-1}, u_{2i}] = z_{2i}.
\end{aligned}$$

$$\begin{aligned}
E(z^{'}_{2i+2}) &= E(z^{'}_{2i}) E(z^{'}_{2i+1})^{u_{2i+2}} = E(z^{'}_{2i}) E(z^{'}_{2i+1})^{\delta} \\
&= E(z'_{2i-2}) E(z'_{2i-1})^{u'_{2i}} E(z'_{2i+1})^{\delta} = E(z_{2i-2}) E(z_{2i-1})^{u_{2i}-1} E(z_{2i})^{\delta}.
\end{aligned}$$

It is clear that $E(z_{2i})^{\delta}$ is the first common part between $E(z_{2i+1})$ and $E(z^{'}_{2i+2})$ (Figure 5 exemplifies the construction of this intersection) and the slope of S_m is equal to the slope of S. In the following, we calculate the largest prefix intersection between $E(z_{2i-1})$ of $E(z_{2i+1})$ and $E(z_{2i-2})$ $E(z_{2i-1})^{u_{2i}-1}$ of $E(z^{'}_{2i+2})$ in

the right of $E(z_{2i})^\delta$ (resp. the largest suffix intersection between $E(z_{2i})\ E(z_{2i-1})$ and $E(z_{2i-2})\ E(z_{2i-1})^{u_{2i}-1}$ in the left of $E(z_{2i})^\delta$).
In the first case, we have:

$$
\begin{aligned}
E(z_{2i-1}) &= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-3}) = E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})^{u_{2i-3}}E(z_{2i-5})\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})^{u_{2i-3}}E(z_{2i-6})^{u_{2i-5}}E(z_{2i-7})\\
&= \ldots\\
&= \#_{k=1}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_3)\\
&= \#_{k=1}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)^{u_3}E(z_1)\\
&= \#_{k=1}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)^{u_1}E(z_{-1})\\
&= \#_{k=1}^{i}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_{-1})
\end{aligned}
$$

$$
\begin{aligned}
E(z_{2i-2})E(z_{2i-1})^{u_{2i}-1} &= E(z_{2i-2})E(z_{2i-1})E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})E(z_{2i-2})^{u_{2i-1}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-2})E(z_{2i-3})E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})E(z_{2i-3})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})E(z_{2i-4})^{u_{2i-3}}E(z_{2i-5})E(z_{2i-3})^{u_{2i-2}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})^{u_{2i-3}}E(z_{2i-4})E(z_{2i-5})E(z_{2i-3})^{u_{2i-2}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \ldots\\
&= \#_{k=1}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)E(z_3)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)E(z_2)^{u_3}E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)^{u_3}E(z_2)E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_1)^{u_2}E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_1)E(z_1)^{u_2}\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_0)^{u_1}E(z_{-1})\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)^{u_1}E(z_0)E(z_{-1})\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}\\
&= \#_{k=1}^{i}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_{-1})\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad E(z_{2i-1})^{u_{2i}-2}
\end{aligned}
$$

Then their prefix intersection is $\#_{k=1}^{i} E(z_{2i-2k})^{u_{2i-2k+1}}$.
In the second case, we have:

$$
\begin{aligned}
E(z_{2i})E(z_{2i-1}) &= E(z_{2i-2})E(z_{2i-1})^{u_{2i}}E(z_{2i-1}) \\
&= E(z_{2i-2})E(z_{2i-1})E(z_{2i-1})^{u_{2i}} \\
&= E(z_{2i-2})E(z_{2i-1})E(z_{2i-1})E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_{2i-2})E(z_{2i-1})E(z_{2i-2})^{u_{2i-1}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_{2i-2})E(z_{2i-1})E(z_{2i-2})^{u_{2i-1}-1}E(z_{2i-2})E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1} \\
& \quad (Let\ L = E(z_{2i-2})E(z_{2i-1})E(z_{2i-2})^{u_{2i-1}-1}) \\
&= LE(z_{2i-4})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1} \\
&= LE(z_{2i-4})E(z_{2i-3})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1} \\
&= LE(z_{2i-4})E(z_{2i-4})^{u_{2i-3}}E(z_{2i-5})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1} \\
&= LE(z_{2i-4})^{u_{2i-3}}E(z_{2i-4})E(z_{2i-5})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1} \\
&= \ldots \\
&= L\#_{k=2}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_4)E(z_3)\#_{k=i-3}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)E(z_3)^{u_4}E(z_3)\#_{k=i-3}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)E(z_3)E(z_3)^{u_4}\#_{k=i-3}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)E(z_2)^{u_3}E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-2}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_2)^{u_3}E(z_2)E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_1)^{u_2}E(z_1)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1} \\
&= L\#_{k=2}^{i-1}E(z_{2i-2k})^{u_{2i-2k+1}}E(z_0)E(z_1)\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}} \\
& \quad E(z_{2i-1})^{u_{2i}-1}
\end{aligned}
$$

$$
\begin{aligned}
E(z_{2i-2})E(z_{2i-1})^{u_{2i}-1} &= E(z_{2i-4})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_{2i-6})E(z_{2i-5})^{u_{2i-4}}E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1} \\
&= \ldots \\
&= E(z_4)\#_{k=i-3}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_2)E(z_3)^{u_4}\#_{k=i-3}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_2)\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_0)E(z_1)^{u_2}\#_{k=i-2}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1} \\
&= E(z_0)\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1}
\end{aligned}
$$

Then their suffix intersection is $\#_{k=i-1}^{1}E(z_{2i-2k-1})^{u_{2i-2k}}E(z_{2i-1})^{u_{2i}-1}$ □

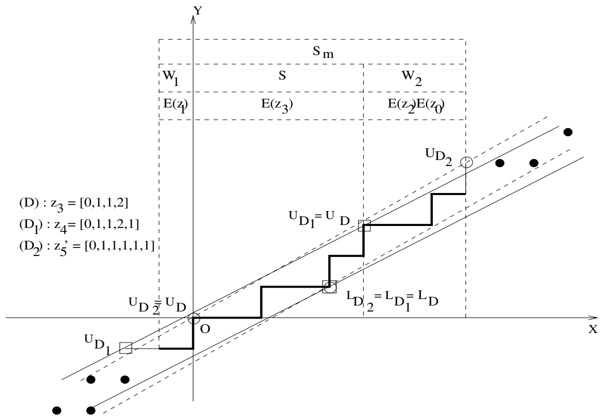
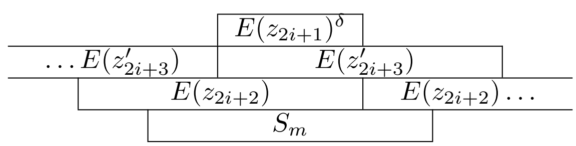


Figure 6: Intersection of two patterns $E(z_4)$ and $E(z'_5)$, where S_m is the main connected part of their intersection. The leaning points of $D_1(S)$ (resp. of $D_2(S)$) are drawn as boxes (resp. as circles). The black line represents the word of the segment S_m. S is a DSS of one pattern $\delta = 1$ and characteristics $(\frac{3}{5}, \mu = 0)$ included in S_m.

Proposition 3. *Let S be a digital straight segment of odd slope $z_{2i+1} = [0, u_1, u_2, \dots, u_{2i+1}]$ and let $D_1(S)$ and $D_2(S)$ be two specific digital straight lines. We have $D_1(S)$ has an even slope z_{2i+2} with $z_{2i+2} = [0, u_1, \cdots, u_{2i+1}, \delta]$ and $D_2(S)$ has an odd slope z'_{2i+3} with $z'_{2i+3} = [0, u_1, \cdots, u_{2i+1} - 1, 1, \delta]$, with the slope of $D_1(S)$ is lower than the slope of $D_2(S)$. Then the intersection S_m (main connected part) of $D_1(S)$ and $D_2(S)$ is exactly $w_1 E(z_{2i+1})^{\delta} w_2$, with $w_1 = \#_{k=i-1}^{1} E(z_{2i-2k-1})^{u_{2i-2k}} E(z_{2i-1})^{u_{2i}}$ and $w_2 = E(z_{2i})^{u_{2i+1}-1} \#_{k=1}^{i} E(z_{2i-2k})^{u_{2i-2k+1}}$ The parity of the depth of S_m is also odd.*



Proof. The proof of this proposition is similar to the proof of Proposition 2 (Figure 6 exemplifies the construction of this intersection). □

3.2. Combinatorial Segment by digital lines intersection

We are now in position to study the slope and the shift of the main connected part of the intersection of two DSL, as specified in Proposition 4, even depth (or Proposition 5, odd depth).

Proposition 4. *Let S be a digital straight segment of even slope $z_{2i} = [0, u_1, u_2, \dots, u_{2i}]$ and let $D_1(S)(a_1, b_1, \mu_1)$ and $D_2(S)(a_2, b_2, \mu_2)$ be two specific digital straight lines of slopes $\frac{a_1}{b_1} = [0, u_1, \cdots, u_{2i}, \delta]$ and $\frac{a_2}{b_2} = [0, u_1, \cdots, u_{2i} - 1, 1, \delta]$ with $\mu_1 = \delta\mu + \mu_p$ and $\mu_2 = (\delta + 1)\mu - \delta - \mu_p$. Then the main connected part of $D_1(S)$ and $D_2(S)$ is a DSS S_m of slope z_{2i} of even depth with δ patterns and shift μ.*

Proof. Let us denote $w_1 = E(z_1)^{u_2}E(z_3)^{u_4}\cdots E(z_{2i-2k-1})^{u_{2i-2k}}\cdots E(z_{2i-3})^{u_{2i-2}}$ $E(z_{2i-1})^{u_{2i}-1}$ and $w_2 = E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})^{u_{2i-3}}\cdots E(z_{2i-2k})^{u_{2i-2k+1}}\cdots$ $E(z_2)^{u_3}E(z_0)^{u_1}$ two factors (Left and Right) of the main connected part S_m of a common intersection of $D_1(S)$ and $D_2(S)$.

$$\begin{aligned}
E(z_{2i}) &= E(z_{2i-2})E(z_{2i-1})^{u_{2i}}\\
&= E(z_{2i-2})E(z_{2i-1})E(z_{2i-1})^{u_{2i}-1}\\
&= E(z_{2i-2})E(z_{2i-2})^{u_{2i-1}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-2})E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})E(z_{2i-3})E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1}\\
&= E(z_{2i-2})^{u_{2i-1}}E(z_{2i-4})^{u_{2i-3}}\cdots E(z_{2i-2k})^{u_{2i-2k+1}}\cdots E(z_4)^{u_5}E(z_2)^{u_3}\\
&\quad E(z_0)^{u_1}E(z_0)E(z_{-1})E(z_1)^{u_2}E(z_3)^{u_4}E(z_5)^{u_6}\cdots E(z_{2i-2k-1})^{u_{2i-2k}}\\
&\quad \cdots E(z_{2i-5})^{u_{2i-4}}E(z_{2i-3})^{u_{2i-2}}E(z_{2i-1})^{u_{2i}-1}\\
&= w_2E(z_0)E(z_{-1})w_1.
\end{aligned}$$

According to the previous decomposition of $E(z_{2i})$, we further get that w_2 is a left factor of $E(z_{2i})$ and w_1 is a right factor of $E(z_{2i})$. The slope of S_m is defined from the slope of $E(z_{2i})$. This is due to the fact that the word w_2 is a strict left factor of $E(z_{2i})$ and hence does not modify the slope of S_m when concatenated to the right. Furthermore, the word w_1 is a strict right factor of $E(z_{2i})$ and it does not modify the slope of S when concatenated to the left. According to Lemma 1, the first upper leaning point of S_m is equal to a upper leaning point of D_1 ($U_D = U_{D_1}$) of coordinate $(x - x_0, y - y_0)$, then $r_S((x - x_0, y - y_0)) = p_{2i}(x - x_0) - q_{2i}(y - y_0) = \mu$. As $E(z_{2i})$ in S repeated δ times, therefore, S_m is a DSS of slope z_{2i} with δ patterns and shift μ.

Proposition 5. *Let S be a digital straight segment of odd slope $z_{2i+1} = [0, u_1, u_2, \ldots, u_{2i+1}]$ and let $D_1(S)(a_1, b_1, \mu_1)$ and $D_2(S)(a_2, b_2, \mu_2)$ be two specific digital straight lines of slopes $\frac{a_1}{b_1} = [0, u_1, \cdots, u_{2i+1}, \delta]$ and $\frac{a_2}{b_2} = [0, u_1, \cdots, u_{2i+1} - 1, 1, \delta]$ with $\mu_1 = \delta\mu - \delta + \mu_p$ and $\mu_2 = (\delta + 1)\mu - \mu_p$. Then the main connected part of $D_1(S)$ and $D_2(S)$ is a DSS S_m of slope z_{2i+1} of odd depth with δ patterns and shift μ.*

Proof. The proof of this proposition is similar to the proof of Proposition 4.

4. Arithmetical Segment by digital lines intersection

To find the intersection of $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ we have thus to solve the following system of equations:

$$\begin{aligned}
\mu_1 \le a_1x - b_1y < \mu_1 + w_1\\
\mu_2 \le a_2x - b_2y < \mu_2 + w_2
\end{aligned} \tag{5}$$

x'	3			4			5		6			7			8		9		
y'	-4	-3	-2	-4	-3	-2	-4	-3	-5	-4	-3	-5	-4	-3	-5	-4	-6	-5	-4
x	-7	-3	1	-4	0	4	-1	3	-2	2	6	1	5	9	4	8	3	7	11
y	-6	-3	0	-4	-1	2	-2	1	-3	0	3	-1	2	5	1	4	0	3	6

Table 1: Points of intersection of $D_1(3,4,3)$ and $D_2(3,5,2)$, where the green cells contain the points of its main connected part.

Where $w_1 = |a_1| + |b_1|$ and $w_2 = |a_2| + |b_2|$ ($w_i, i = 1,2$ is called the thickness of the digital straight line)

Since a_1 and b_1 are relatively prime, there exist u_1 and v_1 such that $a_1u_1 - b_1v_1 = 1$. We introduce $U = \begin{pmatrix} u_1 & b_1 \\ v_1 & a_1 \end{pmatrix}$ and the change of coordinates $\begin{pmatrix} x' \\ y' \end{pmatrix} = U^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus Equation (5) can be rewritten as:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ u_1a_2 - v_1b_2 & a_2b_1 - a_1b_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} < \begin{pmatrix} \mu_1 + w_1 \\ \mu_2 + w_2 \end{pmatrix} \tag{6}$$

Let $\lambda_1 = u_1a_2 - v_1b_2$ and $\lambda_2 = a_2b_1 - a_1b_2$. The solution of the previous equation can be formulated as expressions given below.

Theorem 2. *[6] The digital intersection of two digital straight lines D_1 and D_2 of $\mathbb{Z}^2$ is defined by:*

$$\mu_1 \leq x' < \mu_1 + w_1 \tag{7}$$

The expression of the boundaries of y' depends on the sign of λ_2:

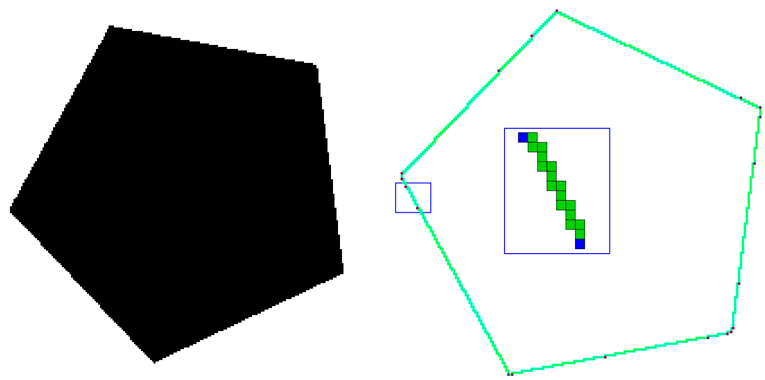
$$\lambda_2 > 0, -\left[\frac{-\mu_2 + \lambda_1 x'}{\lambda_2}\right] \leq y' < -\left[\frac{-\mu_2 - w_2 + \lambda_1 x'}{\lambda_2}\right] \tag{8}$$

$$\lambda_2 < 0, \left[\frac{\mu_2 + w_2 - \lambda_1 x'}{\lambda_2}\right] + 1 \leq y' < \left[\frac{\mu_2 - \lambda_1 x'}{\lambda_2}\right] + 1 \tag{9}$$

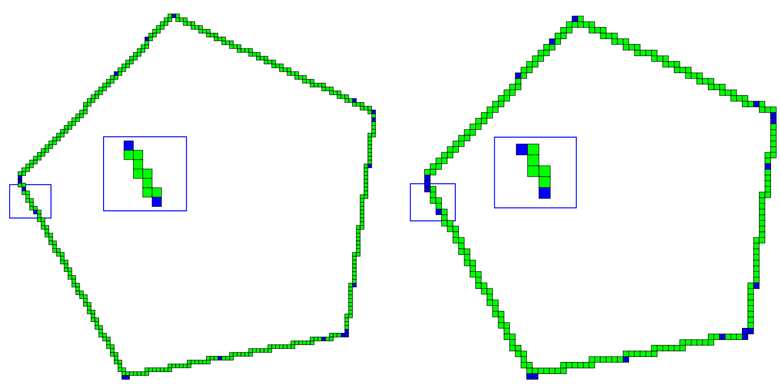
Example 1. *Let $D(2,3,2)$ be a standard digital line of even slope $\frac{2}{3} = [0,1,2]$. For instance, suppose $\delta = 1$, Proposition 4 gives $D_1(3,4,3)$ and $D_2(3,5,2)$. We apply Theorem 2 to determine the set of points of their intersection.*
Hence $3 \leq x' < 10$, since $\lambda_2 = -3 < 0$, then the value of y' is given by the equation below:

$$\left[\frac{10 + x'}{-3}\right] + 1 \quad \leq \quad y' \quad < \quad \left[\frac{2 + x'}{-3}\right] + 1 \tag{10}$$

Finally, we have applied the unimodular matrix $U = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ on (x', y') to get the final result given in the table 1 and illustrated on Fig. 1.



(a) source, 750 points (b) 19 DSS, h=v=1, 750 points



(c) h=v=2, 373 points (d) h=v=3, 249 points

Figure 7: Extract the digital curve from the digital shape. Results obtained from the covering of a polygon (digital curve,(b)) for $(h, v) \in \{(1, 1), (2, 2), (3, 3)\}$. For each shape, the endpoints of each covering segment are drawn by blue boxes. Before subsampling, the endpoints of each segment are upper/lower leaning points. But after subsampling, the endpoints are not necessarily upper/lower leaning points.

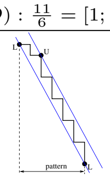
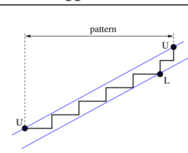
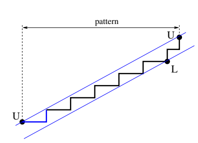
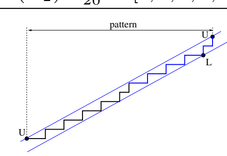
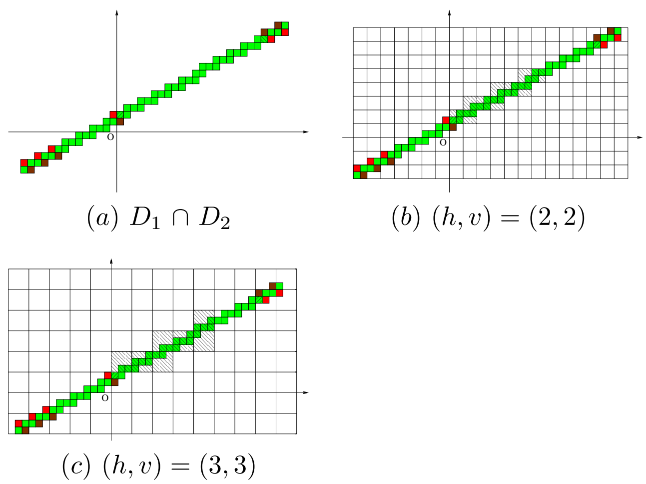
$(D) : \frac{11}{6} = [1; 1, 5]$	$(D) : \frac{6}{11} = [0; 1, 1, 5]$	$(D_1) : \frac{7}{13} = [0; 1, 1, 6]$	$(D_2) : \frac{11}{20} = [0; 1, 1, 4, 2]$
 (a)	 (b)	 (c)	 (d)

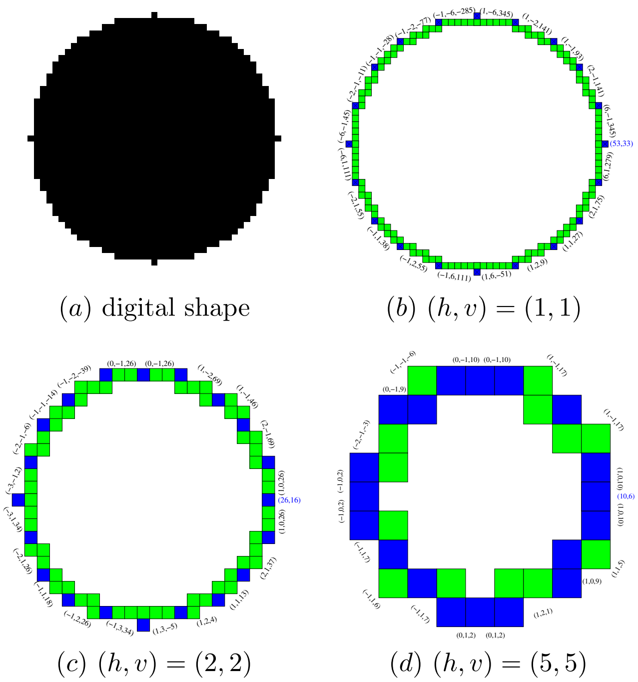
Figure 8: Slope evolution of a LUL DSS (b) of slope $\frac{6}{11}$ with one odd pattern (we choose the DSS (a) of slope $\frac{11}{6}$ that is inside the blue box in the Figure 7,b). The DSS (c) (resp. (d)) was obtained after adding some pixels (blue lines) to the left (resp. right).



(a) $D_1 \cap D_2$ (b) $(h, v) = (2, 2)$

(c) $(h, v) = (3, 3)$

Figure 9: Illustration to the intersection of two digital straight line by using the tiling (h, v) in the first quadrant. $D_1(7, 13, -33)$ drawn as red boxes, $D_2(11, 20, -49)$ drawn as brown boxes, and their intersection is drawn by green boxes. The hitched boxes in (b,c) represent the (h, v)-covering of the initial segment S.



(a) digital shape (b) $(h, v) = (1, 1)$

(c) $(h, v) = (2, 2)$ (d) $(h, v) = (5, 5)$

Figure 10: Multiscale computation of a digital curve according to several tiling (h, v). The blue boxes represent the endpoints of the segments. The blue text represents the coordinate of the first point of the contour.

5. Applications to multiscale representation of digital curves

It is well known that shapes should be studied at different scales. However there exists no analytical description of the multiscale analysis of a digital shape, contrary to the famous scale-space analysis in the continuous world (see Witkin [20] and Koenderink [12]). One of the contribution of this paper is to give new analytical results on the multiscale analysis of DSL and DSS. Figueiredo [6] first provided an analytical description of the multiscale analysis of DSL (8-connected). Recently, Said *et al.* [16] have presented analogous results for DSL (4-connected). They have also proved that its multiscale is also a DSL. For DSS, they have given a sublinear algorithm to extract its characteristics, but no analytical formulae (see [16] for more details).

Analytical formulae to the multiresolution of a digital straight segment DSS seems out of reach at the moment. So, in this section, we focus on the multiscale analysis of a DSS defined by the intersection of two digital straight lines DSLs.

Let us recall that the tiling generated by $\mathbb{S}(h, v)$ on $\mathbb{Z}^2$ induces a new coordinate system where coordinates (X, Y) are related to the canonical coordinates of $\mathbb{Z}^2$ by the obvious relations $X = \left[\frac{x}{h}\right]$ and $Y = \left[\frac{y}{v}\right]$, where $\left[\frac{x}{h}\right]$ is the quotient of the Euclidean division of x by h. Furthermore we denote by $\left\{\frac{x}{h}\right\}$ the remainder of this division. An (h, v)-covering of a set of points of $\mathbb{Z}^2$ is the set of tiles of $\mathbb{S}(h, v)$ which intersect it.

We have implemented the presented methods on a digital curve (Fig. 7,b) that is extracted from a digital shape (Fig. 7,a). We simply choose one digital straight segment S of slope $\frac{11}{6}$ from the figure 7,b, that is inside the blue box. According to the previous propositions and $a \leq b$, we then exchange the values of a and b to obtain a DSS of slope $\frac{6}{11} = [0, 1, 1, 5]$ of one odd pattern (see Fig. 8,a,b). As the new segment has an odd depth repeated once ($\delta = 1$), then we obtain by using Proposition 5 the slope of two lines D_1 and D_2 in which their intersection contains S. Consider the case where the shift μ of S is equal to -27 and μ_p is equal to -5 (see Lemma 1). From Prop. 5 we get: $D_1(7, 13, -33)$ and $D_2(11, 20, -49)$ (see Fig. 8,c,d). Let Δ_1 and Δ_2 be the two digital straight lines that are the (h, v)-covering of D_1 and D_2 respectively. For example if $(h, v) = (2, 2)$, then we get by using Theorem 1 of [16] these two lines $\Delta_1(7, 13, -20)$ and $\Delta_2(11, 20, -30)$ (these two lines are also standard digital lines, see Figure 9).

To find their intersection we have thus to solve the following system of equations:

$$
\begin{aligned}
-20 \leq 7X - 13Y < 0 \quad (\Delta_1) \\
-30 \leq 11X - 20Y < 1 \quad (\Delta_2)
\end{aligned}
$$

We can apply Theorem 2 to determine the set of points of $\Delta_1 \cap \Delta_2$ (see Example 1). As no analytical formulae to calculate the characteristics of these points, we can use *SmartDSS* Algorithm to find the characteristics $(slope, \mu)$ of these points at different scales (see Figure 10).

6. Conclusion

A new concept of studying digital straight segments lying on the intersection of two standard digital lines was presented. The proposed method can be considered as combinatorial method and can be applied on discrete contour. Moreover, we have calculated the coordinates of the upper leaning points, all the characteristics (a, b, μ) of these lines and their intersection. From these results we have computed all the characteristics of the (h, v)-covering of these lines by using Theorem 1 of Said et al. [16]. The results are very interesting and open the door to calculate theoretically the covering of a segment by the tiling (h, v).

References

[1] T.A. Anderson and C.E. Kim. Representation of digital line segments and their preimages. *Computer Vision Graphics Image Processing*, 30(3):279–288, June 1985.

[2] J. Berstel and A. de Luca. Sturmian words, lyndon words and trees. *Theor. Comput. Sci.*, 178(1-2):171–203, 1997.

[3] F. de Vieilleville and J.-O. Lachaud. Revisiting digital straight segment recognition. In *Proc. DGCI*, vol. 4245 of LNCS, pages 355–366, 2006.

[4] I. Debled and J.P. Reveilles. A new approach to digital planes. In *Proc. Spie's Internat. Symposium on Photonics and Industrial Applications-Technical conference vision geometry 3*, Boston, 1994.

[5] I. Debled-Rennesson. *Etude et reconnaissance des droites et plans discrets.* PhD thesis, Université Louis Pasteur, Strasbourg, 1995.

[6] O. Figueiredo. *Advances in discrete geometry applied to the extraction of planes and surfaces from 3D volumes.* PhD thesis, EPFL, Lausanne, 1994.

[7] H. Freeman. On the encoding of arbitrary geometric configurations. *Theoretical Computer Science*, 10(2):260–268, June 1961.

[8] H. Freeman. Computer processing of line-drawing images. *ACM Comput. Surv.*, 6(1):57–97, 1974.

[9] D. Harel and R.E. Tarjan. Fast algorithms for finding nearest common ancestors. *SIAM J. Comput.*, 13(2):338–355, 1984.

[10] N. Kiryati, M. Lindenbaum, and A. M. Bruckstein. Digital or analog hough transform? *Pattern Recognition Letters*, 12(5):291–297, 1991.

[11] R. Klette and A. Rosenfeld. *Digital Geometry - Geometric Methods for Digital Picture Analysis.* Morgan Kaufmann, San Francisco, 2004.

[12] J. J. Koenderink. The structure of images. *Biol. Cyb.*, 50:363–370, 1984.

[13] V. Kovalevsky. Applications of digital straight segments to economical image encoding. In *Proc. DGCI*, LNCS, pages 51–62, London, UK, 1997.

[14] M. D. McIlroy. A note on discrete representation of lines. *AT&T Tech. J.*, 64:481–490, February 1985.

[15] Jean-Pierre Reveillès. *Géométrie discrète, calcul en nombres entiers et algorithmique*. 1991.

[16] M. Said, J.-O. Lachaud, and F. Feschet. Multiscale Discrete Geometry. In *Proc. DGCI*, volume 5810 of *LNCS*, pages 118–131, 2009.

[17] I. Sivignon, F. Dupont, and J-M Chassery. New results about digital intersections. In *Proc. DGCI*, pages 102–113, 2003.

[18] I. Sivignon, F. Dupont, and J.-M. Chassery. Digital intersections: minimal carrier, connectivity, and periodicity properties. *Graphical Models*, 66(4):226–244, 2004.

[19] K. Voss. Coding of digital straight lines by continued fractions. *Computers and Artificial Intelligence*, 10:75–80, 1991.

[20] A. P. Witkin. Scale-space filtering. In *Proc. IJCAI*, pages 1019–1022, 1983.

[21] J. Yaacoub. *Enveloppes convexes de réseaux et applications au traitement d'images*. PhD thesis, Université Louis Pasteur, Strasbourg, 1997.