Multiscale Analysis of Digital Segments by Intersection of 2D Digital Lines

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Abstract

A theory for the multiscale analysis of digital shapes would be very interesting for the pattern recognition community, giving a digital equivalent of the continuous scale-space theory. We focus here on providing analytical formulae of the multiresolution of Digital Straight Segments (DSS), which is a fundamental tool for describing digital shape contours.

1. Introduction

Digital Straight Lines (DSL) and Digital Straight Segments (DSS) are useful to describe the geometry of a digital shape (coding, geometric estimators, feature detection) and this explains why they have been so deeply studied (see the survey [7] or [6]). It is well known that shapes should be studied at different scales, and this has led to the development of regular and irregular pyramids for shape analysis and scene understanding (e.g. [1]). However there exists no analytical description of the multiresolution of a digital shape, contrary to the famous scale-space analysis in the continuous world [11, 8]. One of the contribution of this paper is to give new analytical results on the multiresolution of DSL and DSS. A byproduct is new results about digital line intersection.

Figueiredo [4] first provided an analytical description of the multiresolution of 8-connected DSL. Recently, Said *et al.* [9] have presented analogous results for *standard* DSL (4-connected DSL). They have also proved that its multiresolution is also a standard DSL. For DSS, they have given a sublinear algorithm to extract its characteristics, but no analytical formula.

Analytical formulae for DSS appear to be a much harder problem: since DSS are finite parts of DSL, they are more sensitive to arithmetic peculiarites. We therefore follow an indirect path to DSS multiresolution. In Section 2, given a DSS, we build two DSL whose intersection contains it and whose main connected part has the same arithmetic characteristics as well as the same number of patterns. We note here that we propose new results about the combinatorics of such digital line intersections, that are complementary to the results of Sivignon *et al.* [10]. Section 3 determines the multiresolution of DSS by examining the multiresolution of the intersection of these two DSL. We give a new analytical description of this set with arithmetic inequalities.

This paper is a first step toward the multiresolution of a DSS in constant time by analytical formulae. Proofs are omitted due to space limitations.

2. Standard Digital Lines Intersection

We recall here some definitions and properties about DSL, their relations with rational fractions, and a combinatoric definition of a DSS. We restrict our study of DSS to the main connected part (say S) of the intersection of two well-chosen standard DSL. These two lines are related to the downward moves in the Stern-Brocot tree during a DSS recognition. We finally show that S can be built so that it has the characteristics of any given DSS. Note that intersection of digital lines can be complex and may not be connected.

DSS, patterns and continued fractions. A standard line D of characteristics (a, b, μ) in the fourth quadrant is the set $\{(x, y) \in \mathbb{Z}^2, \mu \leq ax+by < \mu+a+b\}$. A *pattern* of characteristics (a, b) is the succession of Freeman moves between any two consecutive lower leaning points of D. We recall that a *simple continued fraction* is an expression:

$$z = \frac{a}{b} = [u_0, u_1, u_2, ..., u_i, ..., u_n] = u_0 + \cfrac{1}{u_1 + \cfrac{1}{\dots + \cfrac{1}{u_{n-1} + \cfrac{1}{u_n}}}},$$

whose intermediate fractions z_k are called *partial quotients* and n is its complexity.

Consider E a mapping from the set of positive rational number smaller than one onto Freeman code words defined as follows. First terms are stated as $E(z_0) = 0$ and $E(z_1) = 0^{u_1}3$ and others are expressed recursively:

$$\begin{aligned} E(z_{2i+1}) &= E(z_{2i})^{u_{2i+1}} E(z_{2i-1}) \\ E(z_{2i}) &= E(z_{2i-2}) E(z_{2i-1})^{u_{2i}} \end{aligned} \qquad (1)$$

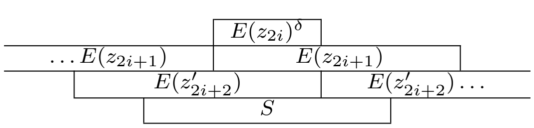
The role of partial quotients can be visualized with a structure called *Stern-Brocot tree* (see [5] for a complete definition and [12] for a link with continued fractions). In [3], Debled and Réveillès introduced the link between this tree and the recognition of digital line. Recognizing a piece of digital line is like going down the Stern-Brocot tree up to the directional vector of the line. Their online recognition algorithm **DR95** [3] (reported in [6]) updates the DSS slope when adding a point just exterior to the current line (weak exterior points). The slope evolution is analytically given by next property.

Proposition 1 *[2] The slope evolution in* ***DR95*** *depends on the parity of the complexity of its slope, the type of weakly exterior point added to the right (UWE and LWE stands respectively for upper and lower weakly exterior). This is summed up in the table below, where the slope is $[0, u_1, ..., u_n]$, $n = 2i$ even or $n = 2i+1$ odd, δ pattern(s) and δ' reversed pattern(s):*

	Even n	*Odd n*
UWE	$[0, u_1, ..., u_{2i}, \delta]$	$[0, u_1, ..., u_{2i+1} - 1, 1, \delta]$
LWE	$[0, u_1, ..., u_{2i} - 1, 1, \delta']$	$[0, u_1, ..., u_{2i+1}, \delta']$

Segments by digital line intersection. In order to study a DSS composed of δ patterns of slope z_n, we build a very similar DSS which includes it as the intersection of two DSL with carefully chosen slopes. Their patterns are placed so that one starts at the first lower leaning point and the other ends at the last lower leaning point (see Fig. 1).

Proposition 2 *The main connected part S of the intersection between $E(z_{2i+1})$ with $z_{2i+1} = [0, u_1, ..., u_{2i}, \delta]$ and $E(z'_{2i+2})$ with $z'_{2i+2} = [0, u_1, ..., u_{2i} - 1, 1, \delta]$ is defined as their common part as placed below:*



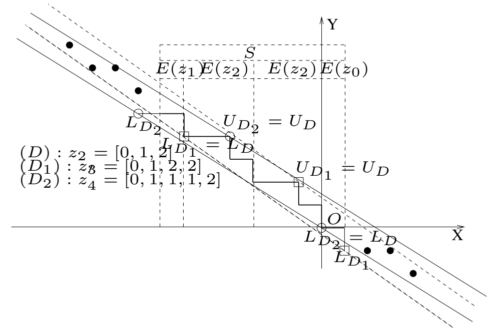
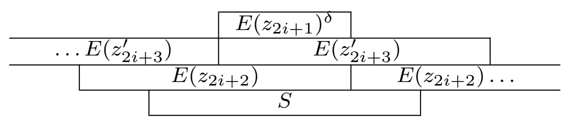


Figure 1. Intersection of two patterns $E(z_3)$ and $E(z'_4)$, where S is the main connected part of their intersection.

The word S is exactly $w_1 E(z_{2i})^\delta w_2$, with $w_1 = E(z_1)^{u_2} \cdots E(z_{2i-2k-1})^{u_{2i-2k}} \cdots E(z_{2i-3})^{u_{2i-2}} E(z_{2i-1})^{u_{2i}-1}$ and $w_2 = E(z_{2i-2})^{u_{2i-1}} \cdots E(z_{2i-2k})^{u_{2i-2k+1}} \cdots E(z_2)^{u_3} E(z_0)^{u_1}$.

We remark that it contains the pattern $E(z_{2i}) = E(z'_{2i})$ repeated δ times (Figure 1 exemplifies the construction of this intersection).

Proposition 3 *The main connected part S of the intersection between $E(z_{2i+2})$ with $z_{2i+2} = [0, u_1, ..., u_{2i+1}, \delta]$ and $E(z'_{2i+3})$ with $z'_{2i+3} = [0, u_1, ..., u_{2i+1} - 1, 1, \delta]$ is defined as their common part as placed below:*



The word S is exactly $w_1 E(z_{2i+1})^\delta w_2$, with $w_1 = E(z_1)^{u_2} E(z_3)^{u_4} \cdots E(z_{2i-2k-1})^{u_{2i-2k}} \cdots E(z_{2i-3})^{u_{2i-2}} E(z_{2i-1})^{u_{2i}}$. and $w_2 = E(z_{2i})^{u_{2i+1}-1} E(z_{2i-2})^{u_{2i-1}} E(z_{2i-4})^{u_{2i-3}} \cdots E(z_{2i-2k})^{u_{2i-2k+1}} \cdots E(z_2)^{u_3} E(z_0)^{u_1}$.

We remark that it contains the pattern $E(z_{2i+1}) = E(z'_{2i+1})$ repeated δ times.

Theorem 1 *[2] If the DSL D of even slope $\frac{p_{2i}}{q_{2i}} = [0, u_1, ..., u_{2i}]$ (or of odd slope $\frac{p_{2i+1}}{q_{2i+1}} = [0, u_1, ..., u_{2i+1}]$) is the common part of two standard digital lines D_1 and D_2, then their slopes are:*

	D_1	D_2
D has an even slope	$\frac{\delta p_{2i} + p_{2i-1}}{\delta q_{2i} + q_{2i-1}}$	$\frac{(\delta+1) p_{2i} - p_{2i-1}}{(\delta+1) q_{2i} - q_{2i-1}}$
D has an odd slope	$\frac{\delta p_{2i+1} + p_{2i}}{\delta q_{2i+1} + q_{2i}}$	$\frac{(\delta+1) p_{2i+1} - p_{2i}}{(\delta+1) q_{2i+1} - q_{2i}}$

Lemma 1 *On the arithmetic straight lines D_1 and D_2, If $D(a, b, \mu)$ has an even complexity with the remainder $\mu = a(x - x_0) + b(y - y_0)$ where (x, y) is the first point*

of D and (x_0, y_0) defines the origin of the pixels in $\mathbb{Z}^2$, then the points of D_1 and D_2 of remainders $\delta\mu + \mu - \delta$ and $\delta\mu$ respectively (If D has an odd complexity, then the points of D_1 and D_2 of remainders $\delta\mu$ and $\delta\mu + \mu - \delta$ respectively) have their coordinates given by (An illustration of this lemma is given in Figure 1) (top row: even complexity, bottom row: odd complexity):

D_1	D_2
$(\mu(b - q_{2i-1}) - \delta q_{2i}, \mu(p_{2i-1} - a) + \delta p_{2i}) + k(-\delta q_{2i} - q_{2i-1}, \delta p_{2i} + p_{2i-1})$	$(\mu(b - q_{2i-1}), \mu(p_{2i-1} - a)) + k(-(\delta + 1)q_{2i} + q_{2i-1}, (\delta + 1)p_{2i} - p_{2i-1})$
$(\mu q_{2i}, -\mu p_{2i}) + k(-\delta q_{2i+1} - q_{2i}, \delta p_{2i+1} + p_{2i})$	$(\mu q_{2i} - \delta q_{2i+1}, -\mu p_{2i} + \delta p_{2i+1}) + k(-(\delta+1)q_{2i+1} + q_{2i}, (\delta + 1)p_{2i+1} - p_{2i})$

Proposition 4 *Let $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ be two standard DSL of slopes $\frac{a_1}{b_1} = [0, u_1, ..., u_{2i}, \delta]$ and $\frac{a_2}{b_2} = [0, u_1, ..., u_{2i} - 1, 1, \delta]$ with $\mu_1 = \delta\mu + \mu - \delta$ and $\mu_2 = \delta\mu$. Then their main connected part is a DSS of slope z_{2i} with δ patterns and shift μ.*

Proposition 5 *Let $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ be two standard DSL of slopes $\frac{a_1}{b_1} = [0, u_1, ..., u_{2i+1}, \delta]$ and $\frac{a_2}{b_2} = [0, u_1, ..., u_{2i+1} - 1, 1, \delta]$ with $\mu_1 = \delta\mu$ and $\mu_2 = \delta\mu + \mu - \delta$. Then their main connected part is a DSS of slope z_{2i+1} with δ patterns and shift μ.*

3. Multiscale of digital lines intersection

We are now in position to study the multiresolution of a DSS defined by the intersection of two DSL, as specified in Proposition 4. We again denote these two DSL by $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ with $\mu_1 = \delta\mu + \mu - \delta$ and $\mu_2 = \delta\mu$.

The tiling generated by $S(h, v)$ on $\mathbb{Z}^2$ induces a new coordinate system where coordinates (X, Y) are related to the canonical coordinates of $\mathbb{Z}^2$ by the obvious relations $X = \left[\frac{x}{h}\right]$ and $Y = \left[\frac{y}{v}\right]$, where $\left[\frac{x}{h}\right]$ is the quotient of the euclidean division of x by h. Furthermore we denote by $\left\{\frac{x}{h}\right\}$ the remainder of this division. An (h, v)-covering of a set of points of $\mathbb{Z}^2$ is the set of tiles of $S(h, v)$ which intersect it. Let $\Delta_1(a'_1, b'_1, \mu'_1)$ and $\Delta_2(a'_2, b'_2, \mu'_2)$ be the two digital straight lines that are the (h, v)-covering of D_1 and D_2 respectively. Theorem 1 of [9] states that these two lines are standard and gives their arithmetic inequalities:

$$-p^1 + Q_2^1 - Q_1^1 + SI^1 \le a'_1X + b'_1Y < Q_3^1 - Q_2^1 + SS^1 \quad (\Delta_1)$$
$$-p^2 + Q_2^2 - Q_1^2 + SI^2 \le a'_2X + b'_2Y < Q_3^2 - Q_2^2 + SS^2 \quad (\Delta_2)$$

where, for $i \in \{1, 2\}, p^i = a'_i + b'_i$, $g^i = gcd(a_ih, b_iv)$, $a'_i = \dfrac{a_ih}{g^i}$, $b'_i = \dfrac{b_iv}{g^i}$, $\mu_i = \delta\mu + (2 - i)(\mu - \delta)$, for $k = 1, 2, 3$, $Q_k^i = \left[\frac{(k-1)\mu_i+k(a_i+b_i)-1}{g^i}\right]$, $R_k^i = \left\{\frac{(k-1)\mu_i+k(a_i+b_i)-1}{g^i}\right\}$ and

$$SI^i = \begin{cases} 0 & \text{if } R_2^i \le R_1^i \\ 1 & \text{otherwise} \end{cases} \qquad SS^i = \begin{cases} 0 & \text{if } R_3^i \le R_2^i \\ 1 & \text{otherwise} \end{cases}$$

To simplify equations, we set $A^i = -p^i + Q_2^i - Q_1^i + SI^i$ and $B^i = Q_3^i - Q_2^i + SS^i$ for $i = 1, 2$. To find their intersection we have thus to solve the following system of equations:

$$\begin{array}{l} A^1 \le a'_1X + b'_1Y < B^1 \\ A^2 \le a'_2X + b'_2Y < B^2 \end{array} \tag{2}$$

Since a'_1 and b'_1 are relatively prime, there exist u_1 and v_1 such that $a'_1u_1 + b'_1v_1 = 1$. We introduce $U = \begin{pmatrix} u_1 & -b'_1 \\ v_1 & a'_1 \end{pmatrix}$ and the change of coordinates $\begin{pmatrix} X' \\ Y' \end{pmatrix} = U^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}$. Thus Equation (2) can be rewritten as:

$$\begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ u_1a'_2 + v_1b'_2 & a'_1b'_2 - a'_2b'_1 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} < \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \tag{3}$$

Let $\lambda_1 = u_1a'_2 + v_1b'_2$ and $\lambda_2 = a'_1b'_2 - a'_2b'_1$. The solution of the intersection can be computed with a double loop in X' and Y' and formulated as expressions given below.

Theorem 2 *The digital intersection of two digital straight lines Δ_1 and Δ_2 of $S(h, v)$ covering respectively the two digital straight lines D_1 and D_2 of $\mathbb{Z}^2$ is defined by:*

$$A^1 \le X' < B^1 \tag{4}$$

The expression of the boundaries of Y' depends on the sign of λ_2:

$$\lambda_2 > 0, -\left[\frac{-A^2 + \lambda_1X'}{\lambda_2}\right] \le Y' < -\left[\frac{-B^2 + \lambda_1X'}{\lambda_2}\right] \tag{5}$$

$$\lambda_2 < 0, \left[\frac{B^2 - \lambda_1X'}{\lambda_2}\right] + 1 \le Y' < \left[\frac{A^2 - \lambda_1X'}{\lambda_2}\right] + 1 \tag{6}$$

Exemple. Let $D(2, 3, 2)$ be a standard digital line of slope $\frac{2}{3} = [0, 1, 2]$. For instance, suppose $\delta = 1$ and $(h, v) = (2, 2)$, Proposition 4 gives $D_1(3, 4, 3)$ and $D_2(3, 5, 2)$. Their $(2, 2)$-covering is the two lines $\Delta_1(3, 4, -2)$ and $\Delta_2(3, 5, -3)$. We apply Theorem 2 to determine the set of points of their intersection.
Hence $-2 \le X' < 5$, since $\lambda_2 = 3 > 0$, then the value of Y' is given by the equation below:

$$-\left[\frac{3-X'}{3}\right] \quad \le \quad Y' \quad < \quad -\left[\frac{-5-X'}{3}\right] \tag{7}$$

Finally, we have applied the unimodular matrix $U = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ on (X', Y') to get the final result given in the table below and illustrated on Fig. 3.

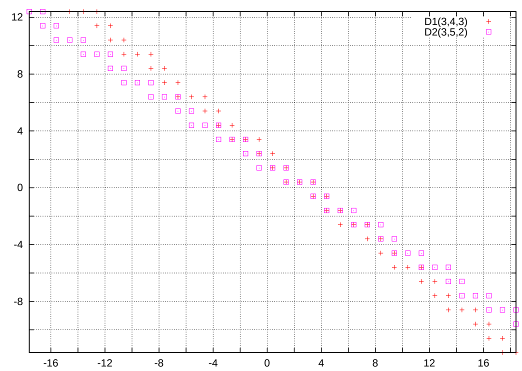


Figure 2. Intersection of $D_1(3,4,3)$ and $D_2(3,5,2)$, their intersection is drawn by $\boxplus$.

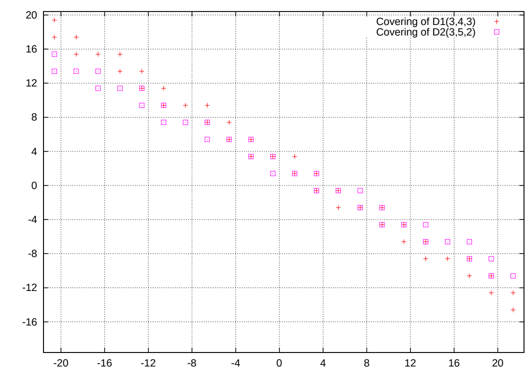


Figure 3. Intersection of $\Delta_1(3,4,-2)$ drawn as boxes and $\Delta_2(3,5,-3)$ drawn by symbols $+$, and their intersection is drawn by symbols $\boxplus$.

Points of $\Delta_1(3,4,-2) \cap \Delta_2(3,5,-3)$

X'	-2		-1			0			1	
Y'	-1	0	-1	0	1	-1	0	1	0	1
X	-2	-6	1	-3	-7	4	0	-4	3	-1
Y	1	4	-1	2	5	-3	0	3	-2	1

X'	2			3			4	
Y'	0	1	2	0	1	2	1	2
X	6	2	-2	9	5	1	8	4
Y	-4	-1	2	-6	-3	0	-5	-2

According to Lemma 1, the coordinates of the first and the last points of the pattern P (i.e. a subset of the main connected part S of the intersection of D_1 and D_2) of D are $(x_f, y_f) = (1,0)$ and $(x_l, y_l) = (4,-2)$ (see Figure 2). Let P' be a covering of P by the tiling $(h,v) = (2,2)$, i.e. a subset of the main connected part S' of the intersection of Δ_1 and Δ_2 (S' is a covering of S by the same tiling). Therefore, the first and last points of P' are $(X_f, Y_f) = (0,0)$ and $(X_l, Y_l) = (2,-1)$, and every point of P' is shadowed in the table.

Let Δ be a covering of D by the same tiling. We have calculated the characteristics (a, b, μ) of Δ that is equal to (3,4,-5) by using Theorem 1 of [9]. Finally, using Algorithm $SmartDSS$ of [9] to compute the characteristics of P' that is some subset of a DSL Δ, given a starting point (X_f, Y_f) and an ending point (X_l, Y_l) ($(X_f, Y_f), (X_l, Y_l) \in \Delta$), and are equal to $(1,1,0)$.

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