Computation of homology groups and generators

S. Peltier, S. Alayrangues, L. Fuchs, J-O. Lachaud.

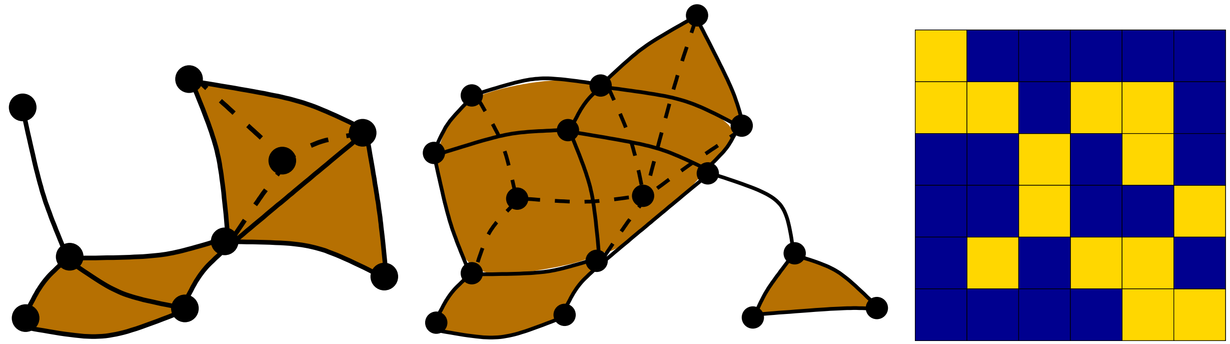
Université de Poitiers

Laboratoire SIC (FRE 2731 CNRS)

UFR SFA, Département d'Informatique

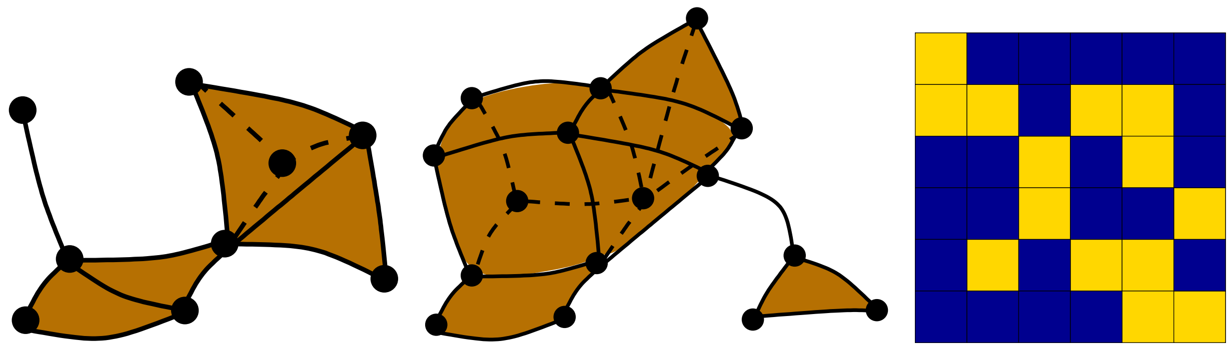
Introduction

- Computing topological properties :



Introduction

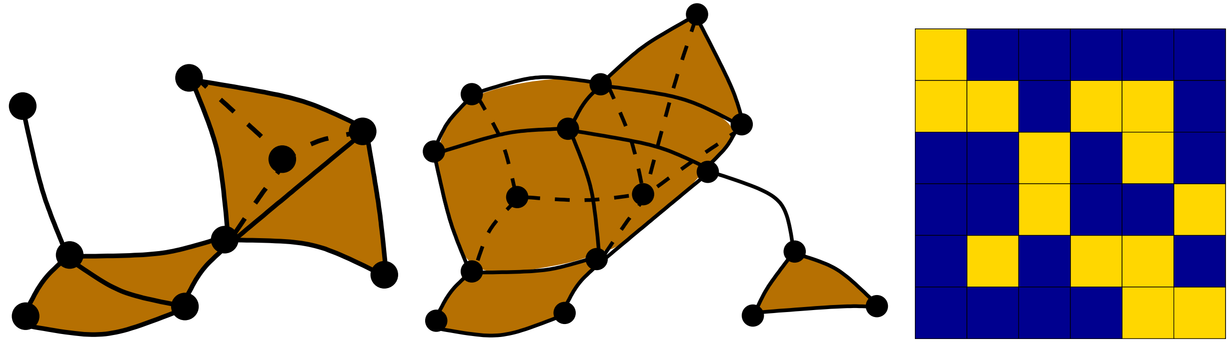
- Computing topological properties :



  - Homology,

Introduction

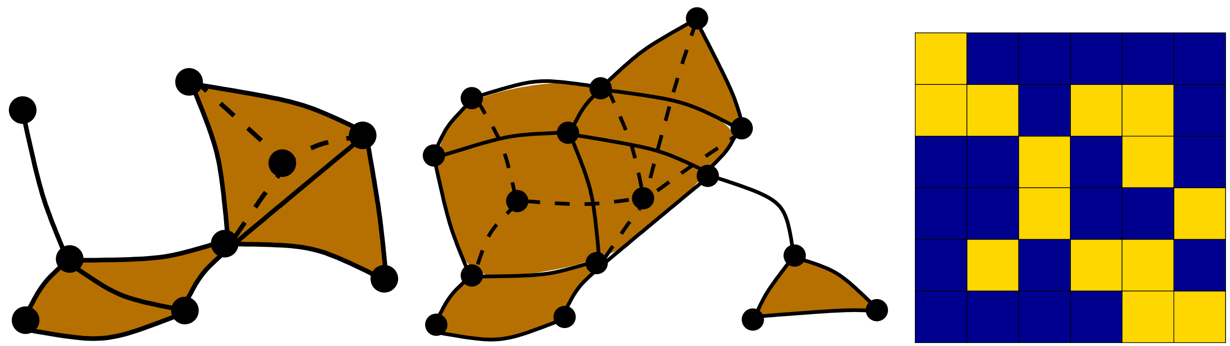
- Computing topological properties :



  - Homology,
  - Usually studied on simplicial objects,

Introduction

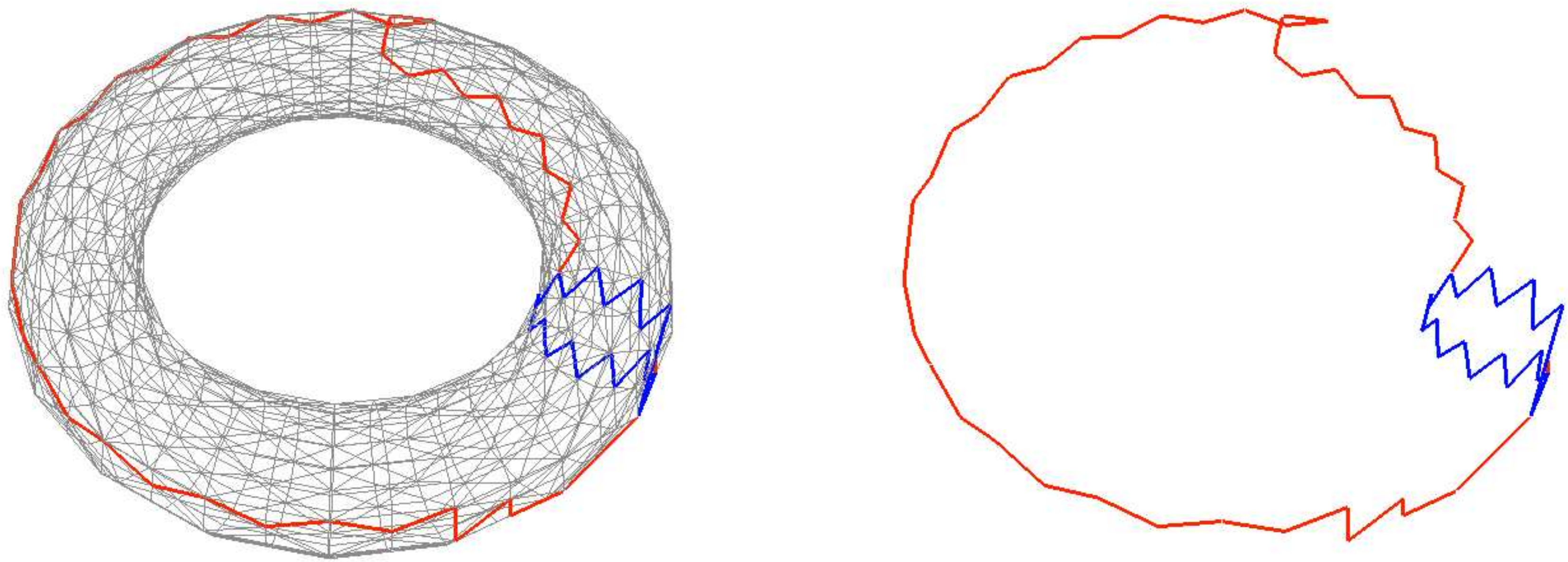
- Computing topological properties :



  - Homology,
  - Usually studied on simplicial objects,
  - Objective : Take advantage of various methods' qualities.

Introduction

- Characterize « holes » :

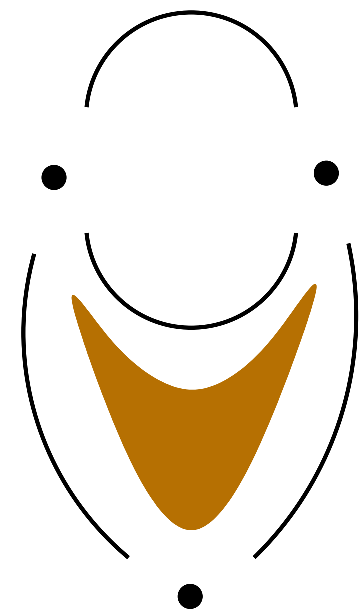


Plan

- Semi-simplicial sets
- Definition of homology
- Computing homology
- A method for moduli generators
- Experimentations - Perspectives

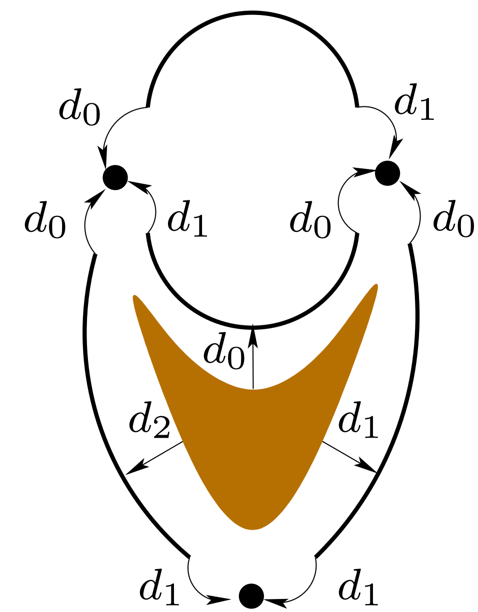
Semi-simplicial sets

- Set of abstract simplices



Semi-simplicial sets

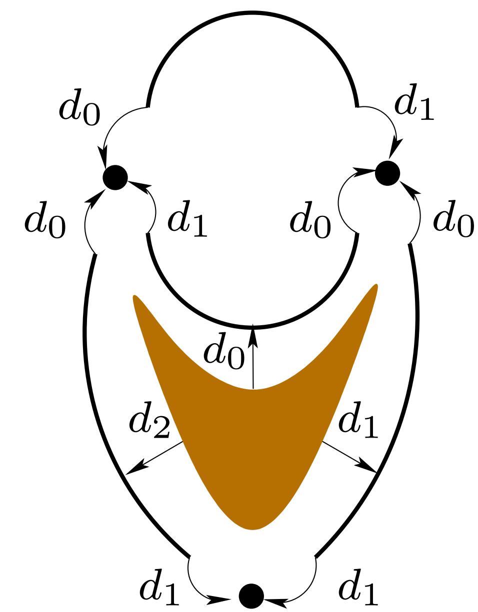
- Set of abstract simplices
- boundary operators



Semi-simplicial sets

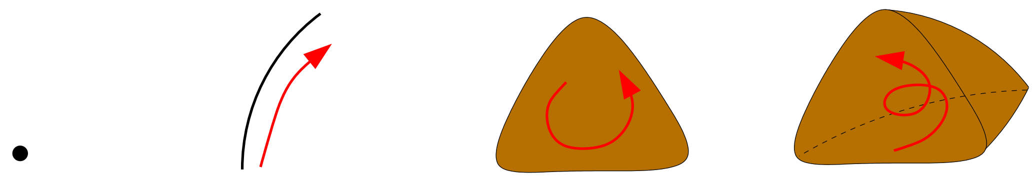
- Set of abstract simplices
- boundary operators
- identity relation

$$d_i d_j = d_j d_{i-1} \text{ with } j < i$$



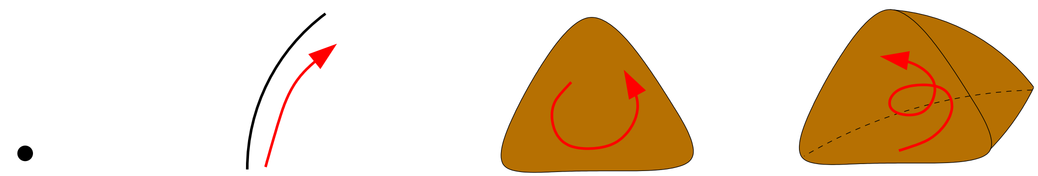
Orientation

- Oriented simplices



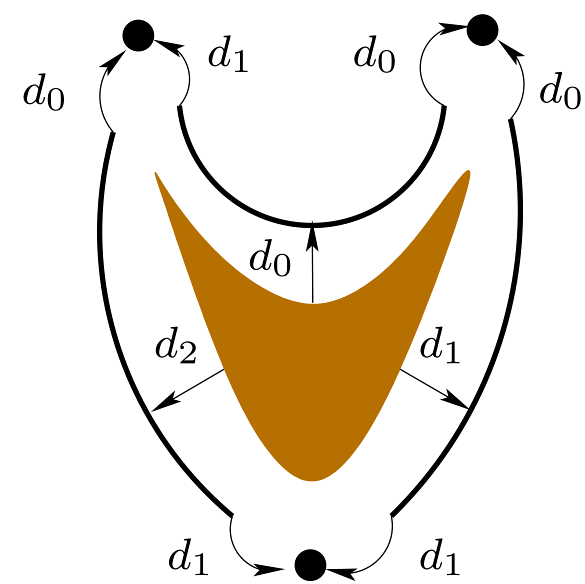
Orientation

- Oriented simplices



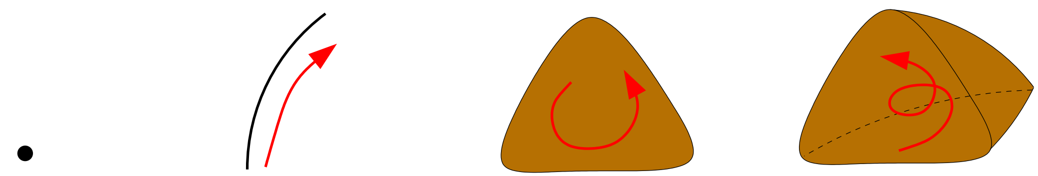
- Boundary of a simplex

$$\partial(\sigma) = \sum_{i=0}^{dim(\sigma)} (-1)^i \sigma d_i$$



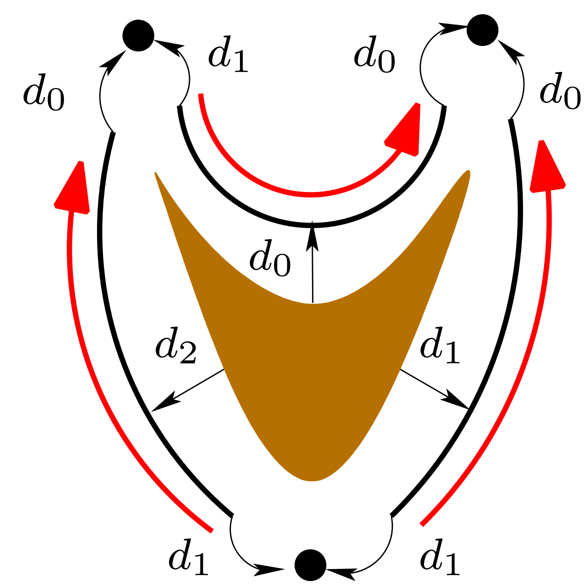
Orientation

- Oriented simplices



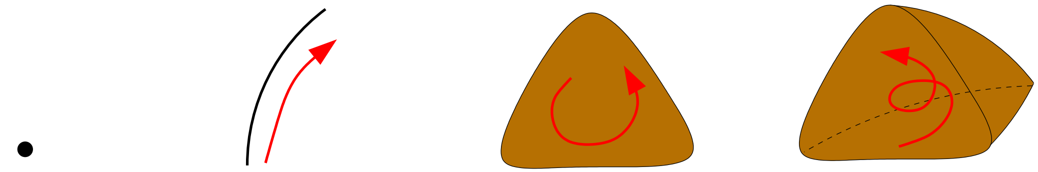
- Boundary of a simplex

$\partial(\sigma) = \sum_{i=0}^{dim(\sigma)} (-1)^i \sigma d_i$



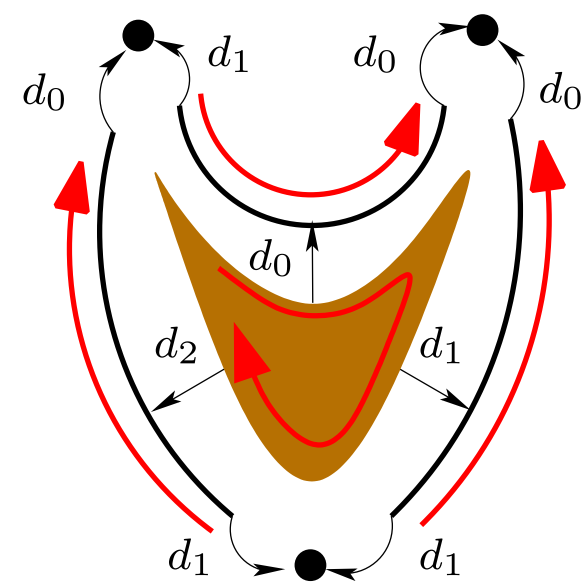
Orientation

- Oriented simplices



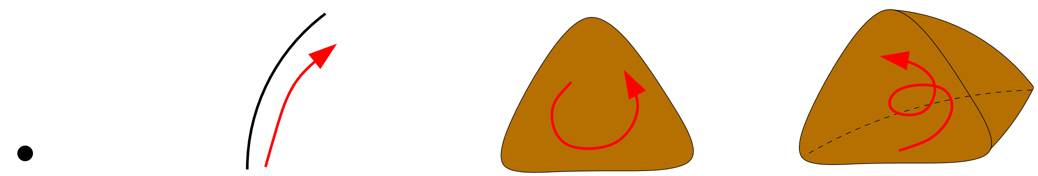
- Boundary of a simplex

$$\partial(\sigma) = \sum_{i=0}^{dim(\sigma)} (-1)^i \sigma d_i$$



Orientation

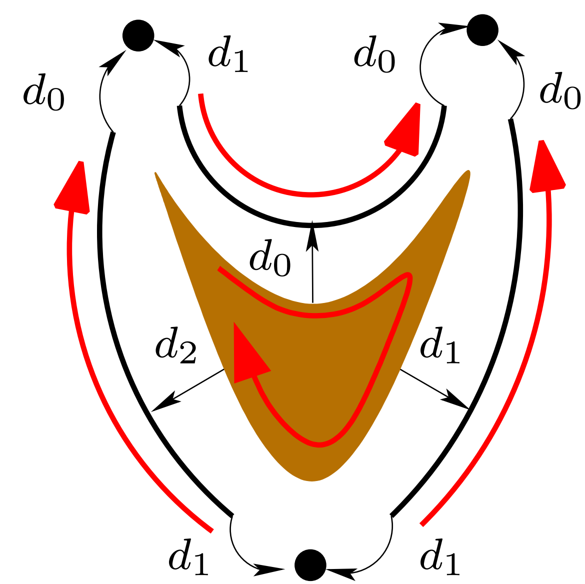
- Oriented simplices



- Boundary of a simplex

$$\partial(\sigma) = \sum_{i=0}^{dim(\sigma)} (-1)^i \sigma d_i$$

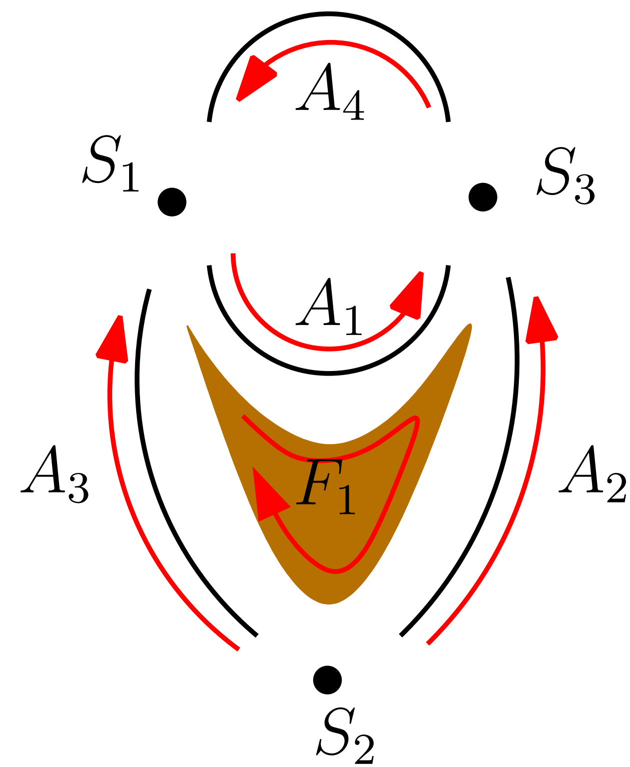
$$\boxed{\partial(\partial(\sigma)) = 0}$$



Free chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial \circ \partial = 0$



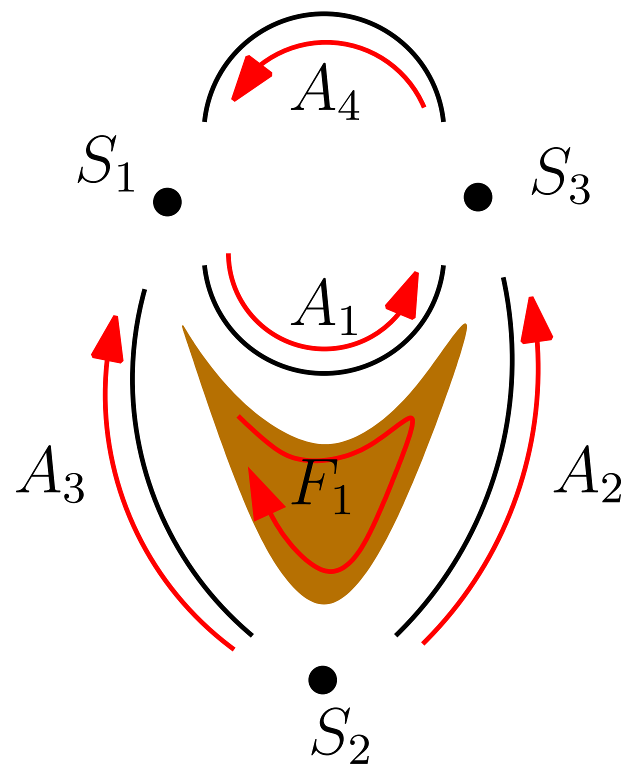
Free chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial \circ \partial = 0$

example :

$$\begin{aligned}\partial(A_1 + 2A_2) &= \partial(A_1) + 2\partial(A_2)\\ &= (S_3 - S_1) + 2(S_3 - S_2)\\ &= 3S_3 - S1 - 2S_2\end{aligned}$$



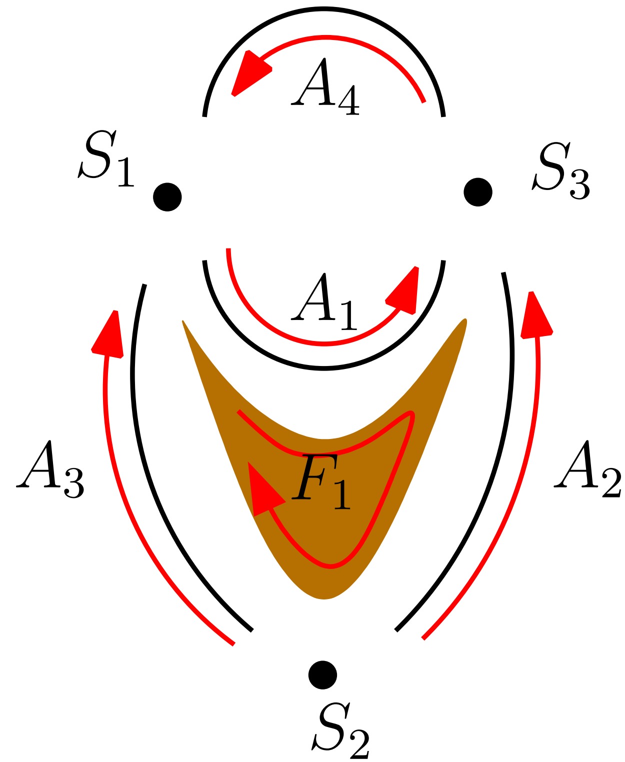
UNIVERSITÉ de Poitiers

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

$p-$cycles

c is a $p-$cycle $\Leftrightarrow \partial_p(c) = 0$



CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

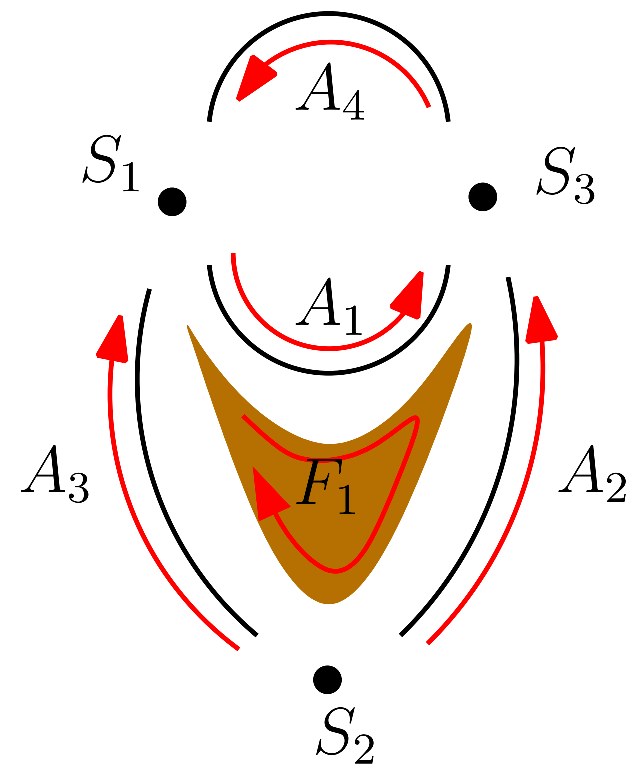
SIC SIGNAL IMAGE COMMUNICATIONS

$p-$cycles

c is a $p-$cycle $\Leftrightarrow \partial_p(c) = 0$

examples :

Any $0-$chain is a cycle



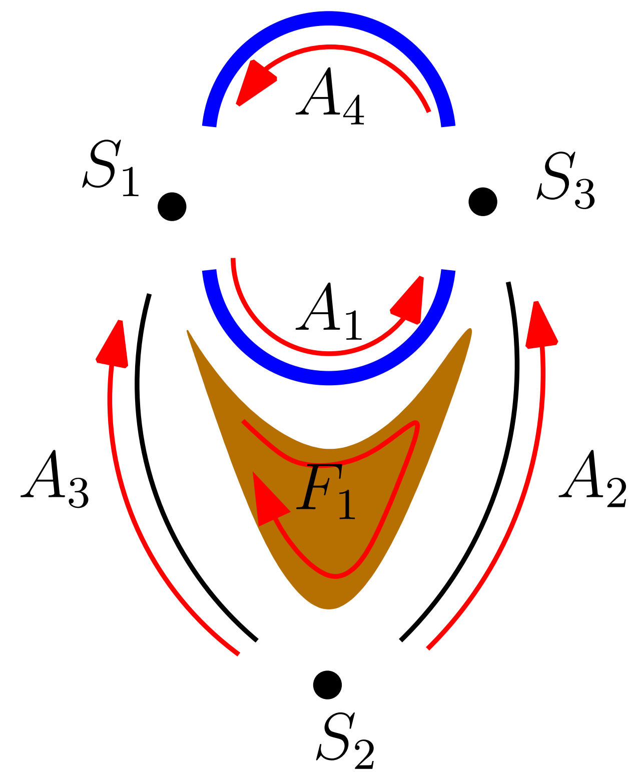
$p-$cycles

c is a $p-$cycle $\Leftrightarrow \partial_p(c) = 0$

examples :

Any $0-$chain is a cycle

$\partial(A_1 + A_4) = 0$



$p-$cycles

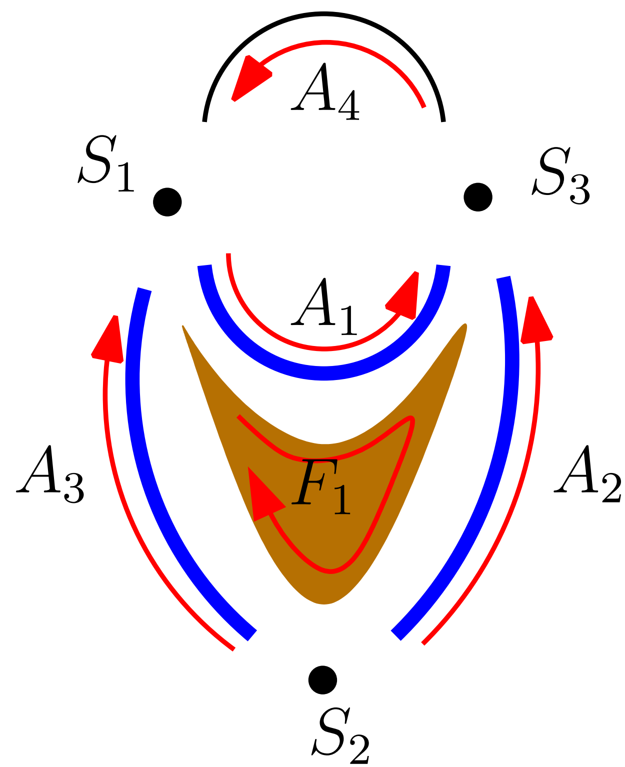
c is a $p-$cycle $\Leftrightarrow \partial_p(c) = 0$

examples :

Any $0-$chain is a cycle

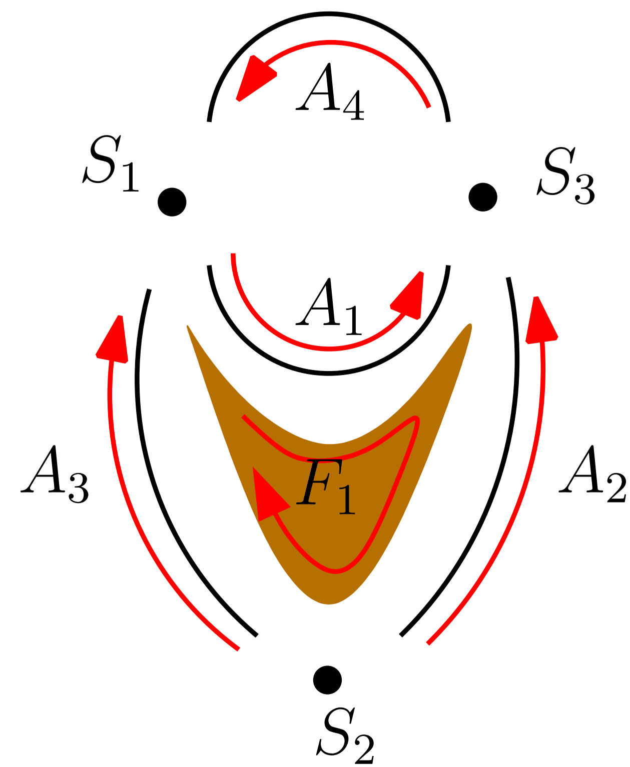
$\partial(A_1 + A_4) = 0$

$\partial(A_1 - A_2 + A_3) = 0$



$p-$boundaries

c is a $p-$boundary $\Leftrightarrow \partial_{p+1}(c') = c$

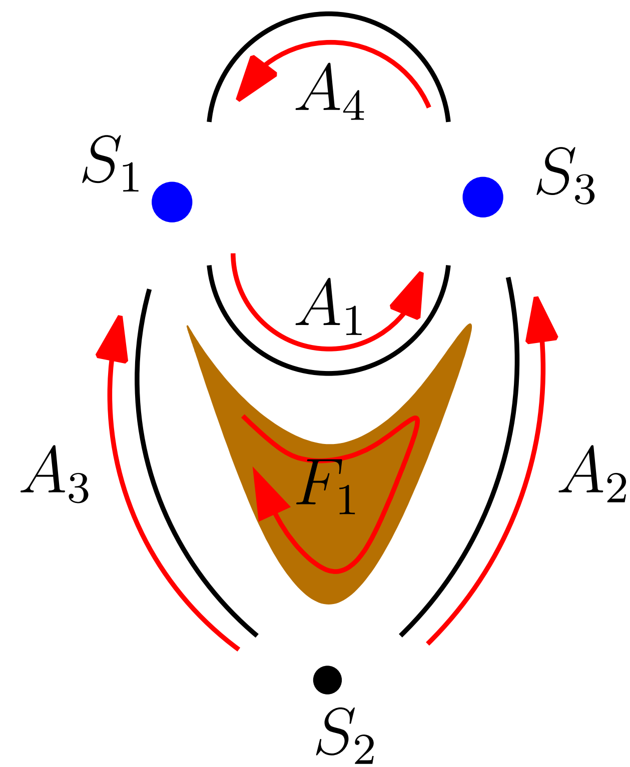


$p-$boundaries

c is a $p-$boundary $\Leftrightarrow \partial_{p+1}(c') = c$

examples :

$S_1 - S_3 = \partial(A_3 - A_2)$



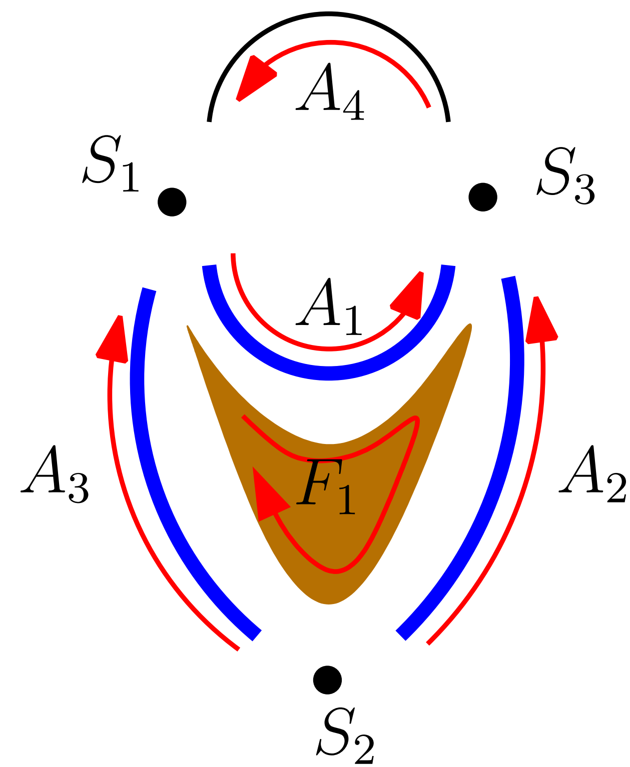
$p-$boundaries

c is a $p-$boundary $\Leftrightarrow \partial_{p+1}(c') = c$

examples :

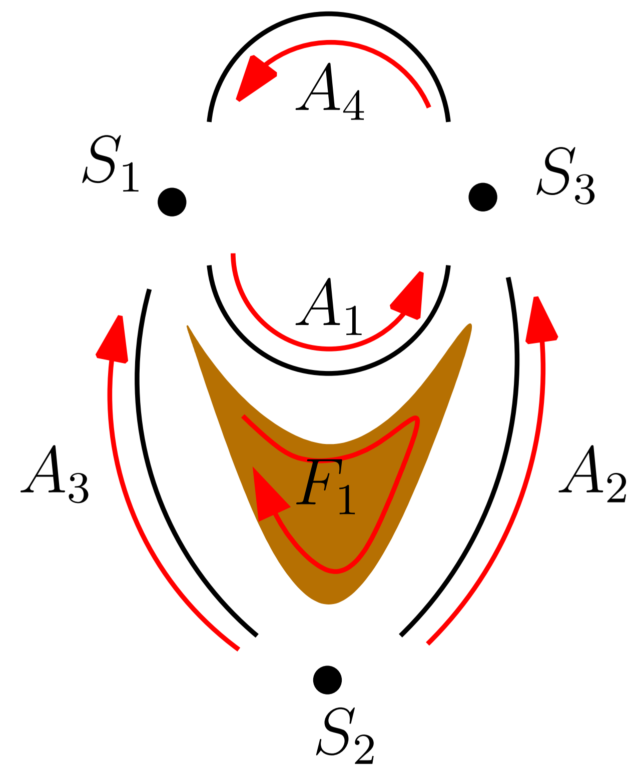
$S_1 - S_3 = \partial(A_3 - A_2)$

$A_1 - A_2 + A_3 = \partial(F_1)$



Homology groups H_p

z_1 homologous to z_2 if $z_1 = z_2 + \partial(c)$

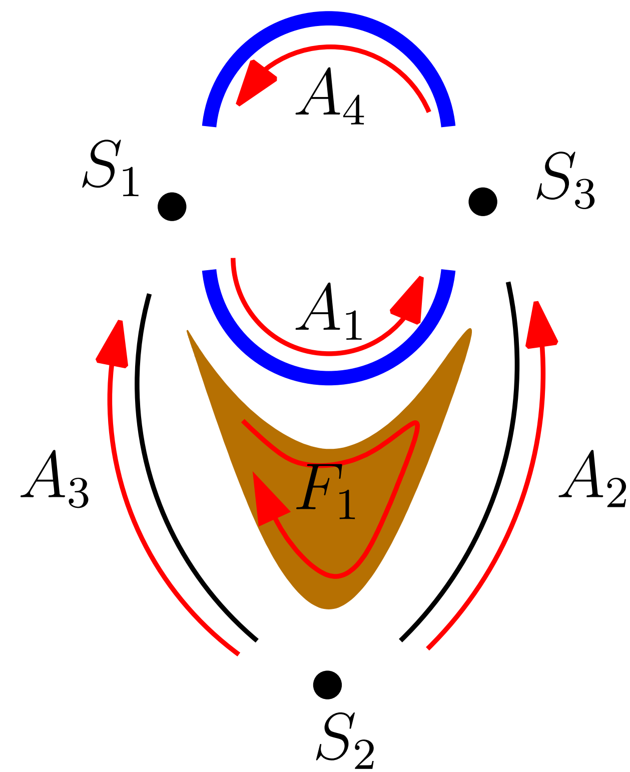


Homology groups H_p

z_1 homologous to z_2 if $z_1 = z_2 + \partial(c)$

examples :

$z_1 \quad = A_1 + A_4$

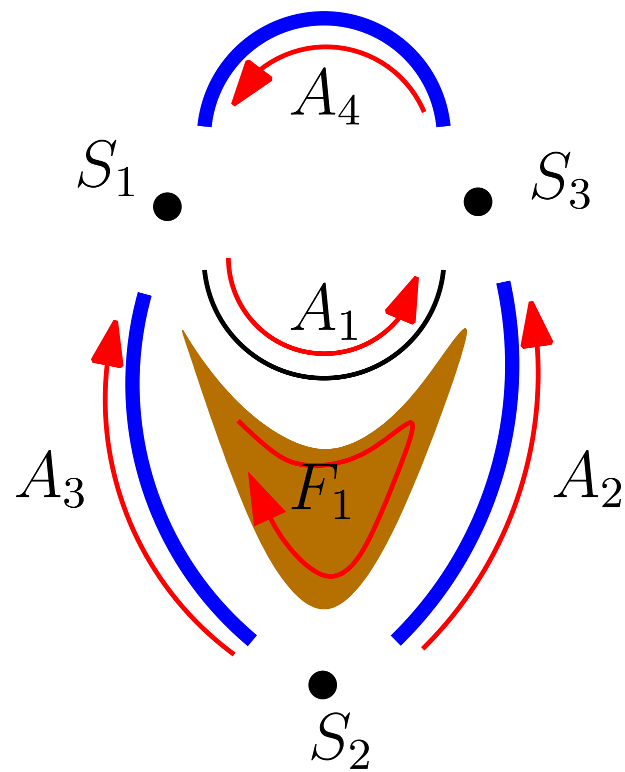


Homology groups H_p

z_1 homologous to z_2 if $z_1 = z_2 + \partial(c)$

examples :

$$
\begin{aligned}
z_1 &= A_1 + A_4 \\
z_2 &= A_2 + A_4 - A_3 \\
&= z_1 + \partial(-F_1)
\end{aligned}
$$



UNIVERSITÉ de Poitiers

CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

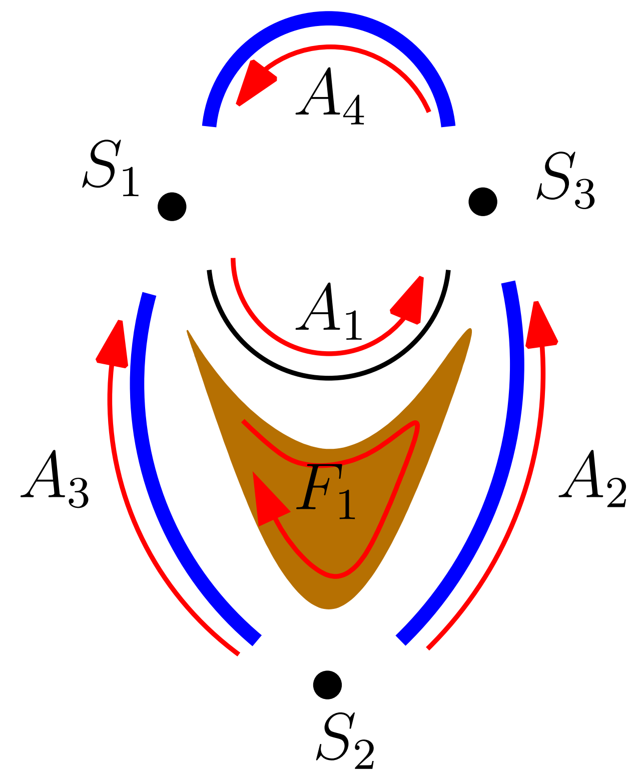
Homology groups H_p

z_1 homologous to z_2 if $z_1 = z_2 + \partial(c)$

examples :

$$\begin{aligned} z_1 &= A_1 + A_4 \\ z_2 &= A_2 + A_4 - A_3 \\ &= z_1 + \partial(-F_1) \end{aligned}$$

H_p : Equivalence classes

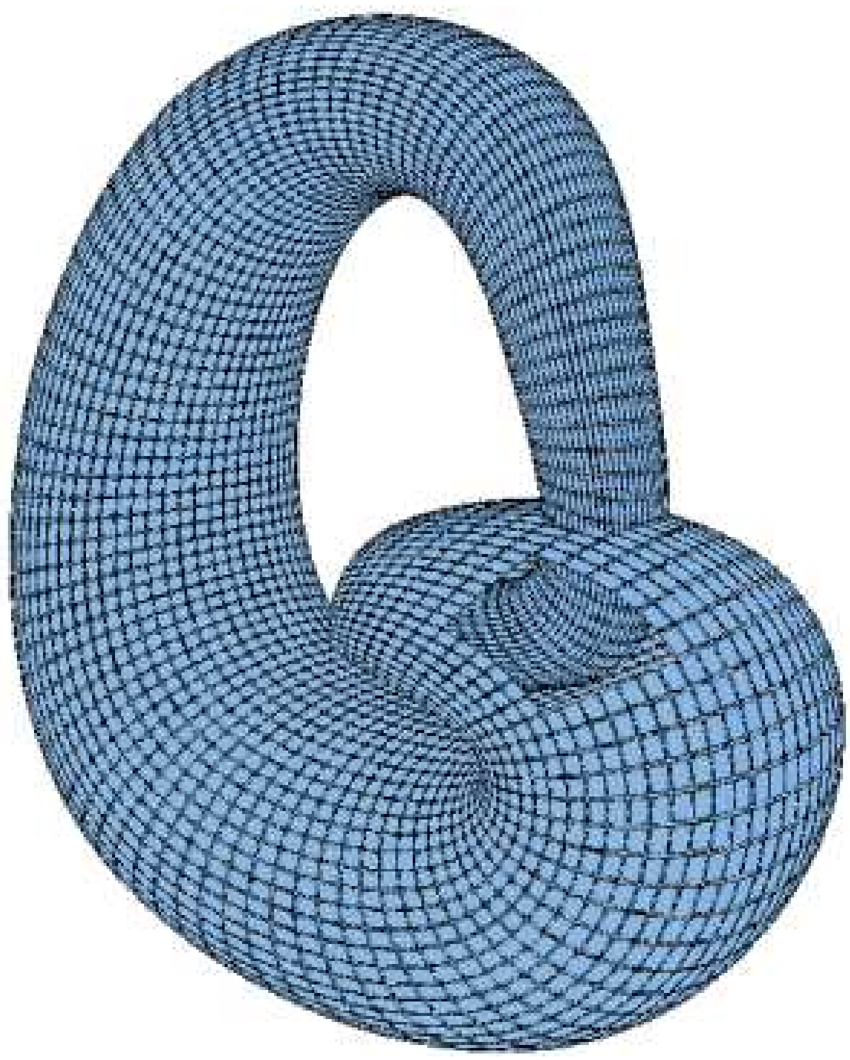


Homology groups H_p

- Z_p is a subgroup of C_p
- B_p is a subgroup of C_p
- B_p is a subgroup of Z_p due to $\partial \circ \partial = 0$
- H_p is the quotient group Z_p / B_p
- $H_p \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\beta} \oplus \mathbb{Z}/t_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/t_n\mathbb{Z}$

Klein bottle

Klein Bottle



Computing homology groups means...

- Getting Betti numbers

  [DE93] An Incremental Algorithm for Betti numbers of simplicial complexes

  [KMM04] Computational homology

Computing homology groups means...

- Getting Betti numbers

  [DE93] An Incremental Algorithm for Betti numbers of simplicial complexes

  [KMM04] Computational homology

- Getting Betti numbers and torsion coefficients

  [Mun84] Elements of algebraic topology

Computing homology groups means...

- Getting Betti numbers

  [DE93] An Incremental Algorithm for Betti numbers of simplicial complexes

  [KMM04] Computational homology

- Getting Betti numbers and torsion coefficients

  [Mun84] Elements of algebraic topology

- Getting representants of homology classes

  [Ago76] Algebraic Topology, a first course

Computing homology groups means...

- Getting Betti numbers

  [DE93] An Incremental Algorithm for Betti numbers of simplicial complexes

  [KMM04] Computational homology

- Getting Betti numbers and torsion coefficients

  [Mun84] Elements of algebraic topology

- Getting representants of homology classes

  [Ago76] Algebraic Topology, a first course

- Getting homology groups in low dimensions

  [DG96] Computing Homology Groups of Simplicial Complexes in R3

  [Z04] Topology for computing

UNIVERSITÉ de Poitiers

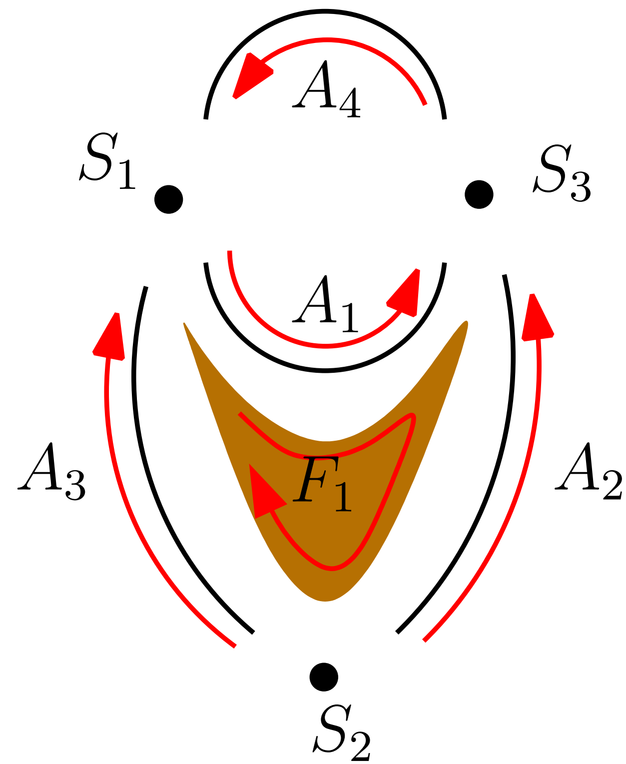
CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

Incidence matrices

$$\mathbf{E}^0 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

$$\mathbf{E}^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} F_1 \\ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \end{array}$$



Smith normal form

$$\left(\begin{array}{ccc|c}
\lambda_1 & & 0 & \\
 & \ddots & & \mathbf{0} \\
0 & & \lambda_k & \\
\hline
 & \mathbf{0} & & \mathbf{0}
\end{array}\right)$$

Where λ_1 divides $\lambda_2 \cdots$ divides λ_k

UNIVERSITÉ de Poitiers

CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

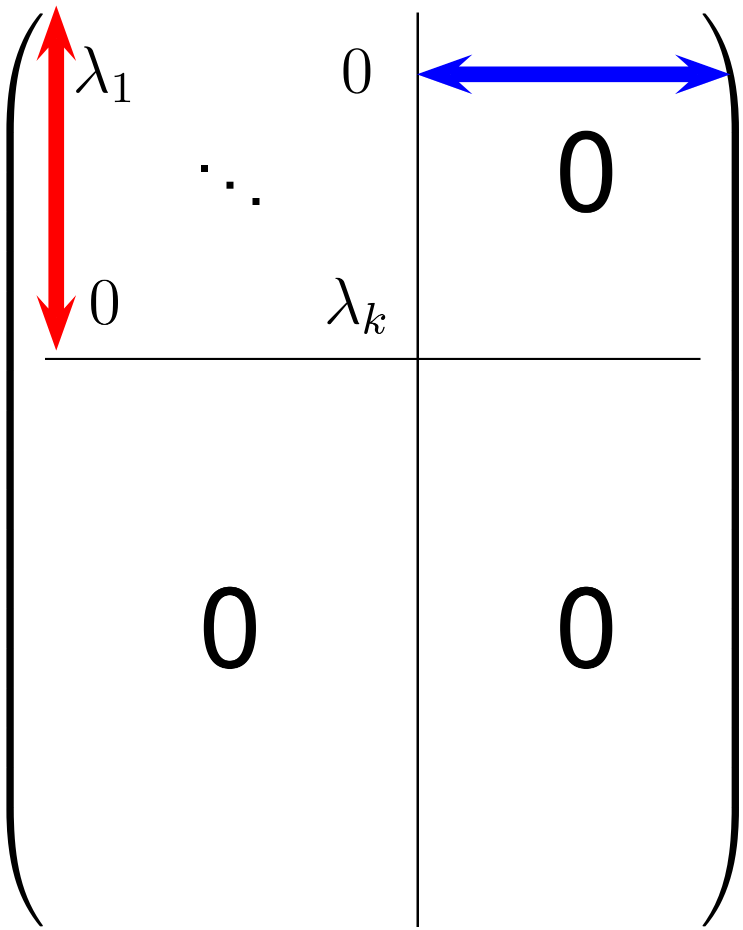
Smith normal form

$$
\left(\begin{array}{cc|c}
\lambda_1 & 0 & \\
\ddots & & 0 \\
0 & \lambda_k & \\
\hline
0 & & 0
\end{array}\right)
$$

$q-$cycles $\sharp Z_q$

Where λ_1 divides $\lambda_2 \cdots$ divides λ_k

Smith normal form



$q-$cycles $\sharp Z_q$

$(q-1)-$boundaries $\sharp B_{q-1}$

Where λ_1 divides $\lambda_2 \cdots$ divides λ_k

Smith normal form

$$\left(\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_k & \\ \hline & 0 & & 0 \end{array}\right)$$

$q-$cycles $\sharp Z_q$

$(q-1)-$boundaries $\sharp B_{q-1}$

$$\boxed{\beta_q = \sharp Z_q - \sharp B_q}$$

Where λ_1 divides $\lambda_2 \cdots$ divides λ_k

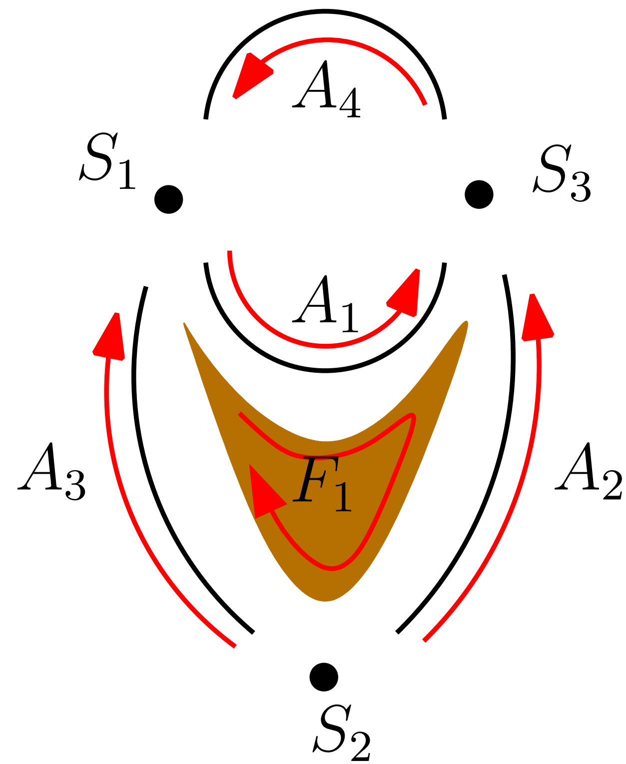
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

Smith normal form

$$\mathbf{N}^{\mathbf{0}}_{*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{N}^{\mathbf{1}}_{*} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



UNIVERSITÉ de Poitiers

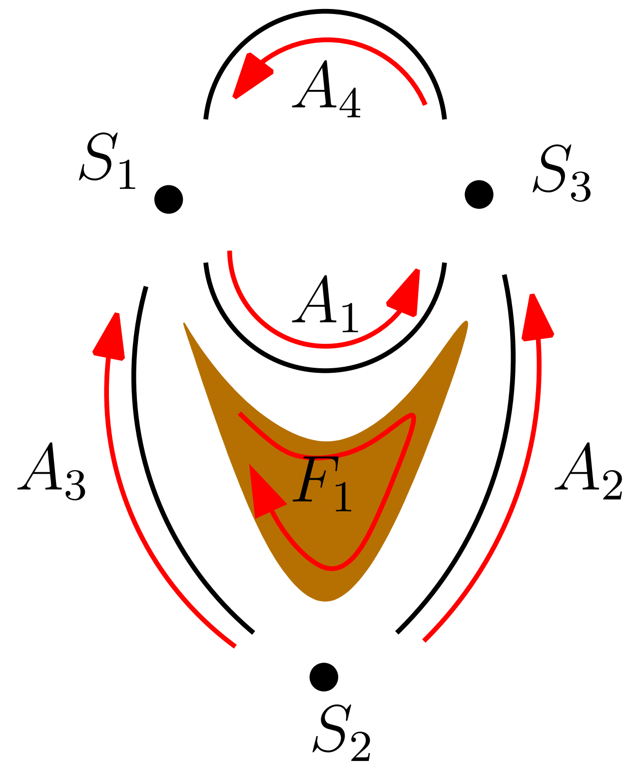
CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

Smith normal form

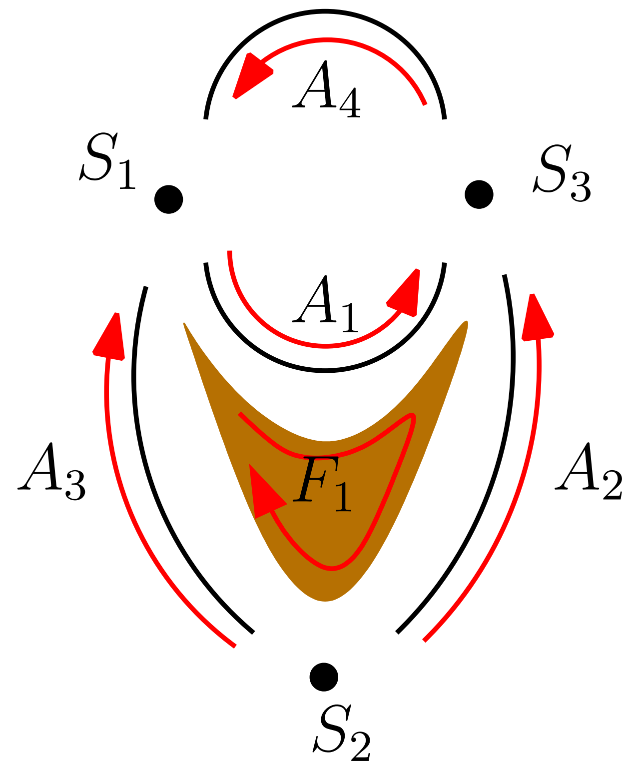
$$\mathbf{N}_*^{\mathbf{0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{N}_*^{\mathbf{1}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Smith normal form

$$\mathbf{N}_*^{\mathbf{0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



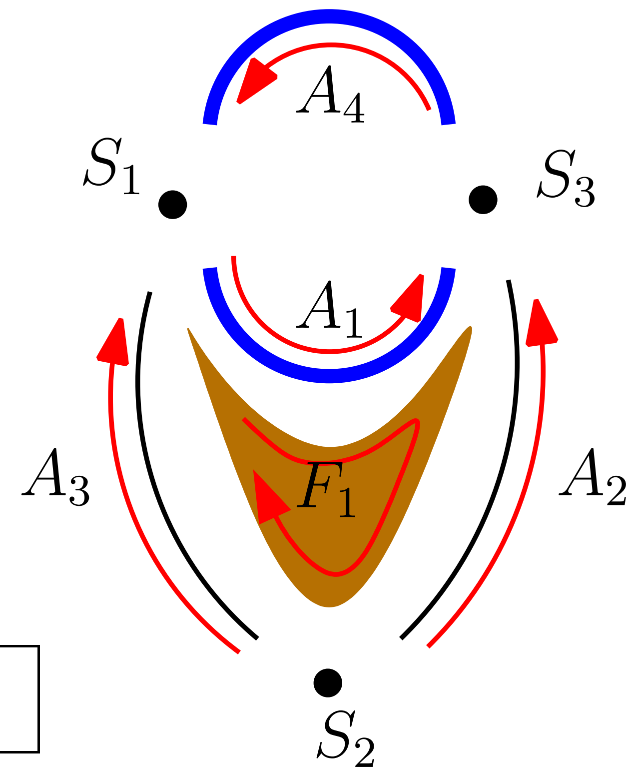
$$\mathbf{N}_*^{\mathbf{1}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Smith normal form

$$\mathbf{N}_*^{\mathbf{0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{N}_*^{\mathbf{1}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\beta_1 = 2 - 1 = 1}$$



Smith normal form

Storjohann

- Betti numbers
- Torsion coefficients
- Moduli operations

Agoston

- Betti numbers
- Torsion coefficients
- Generators

UNIVERSITÉ de Poitiers

CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SIC SIGNAL IMAGE COMMUNICATIONS

Smith normal form

Storjohann

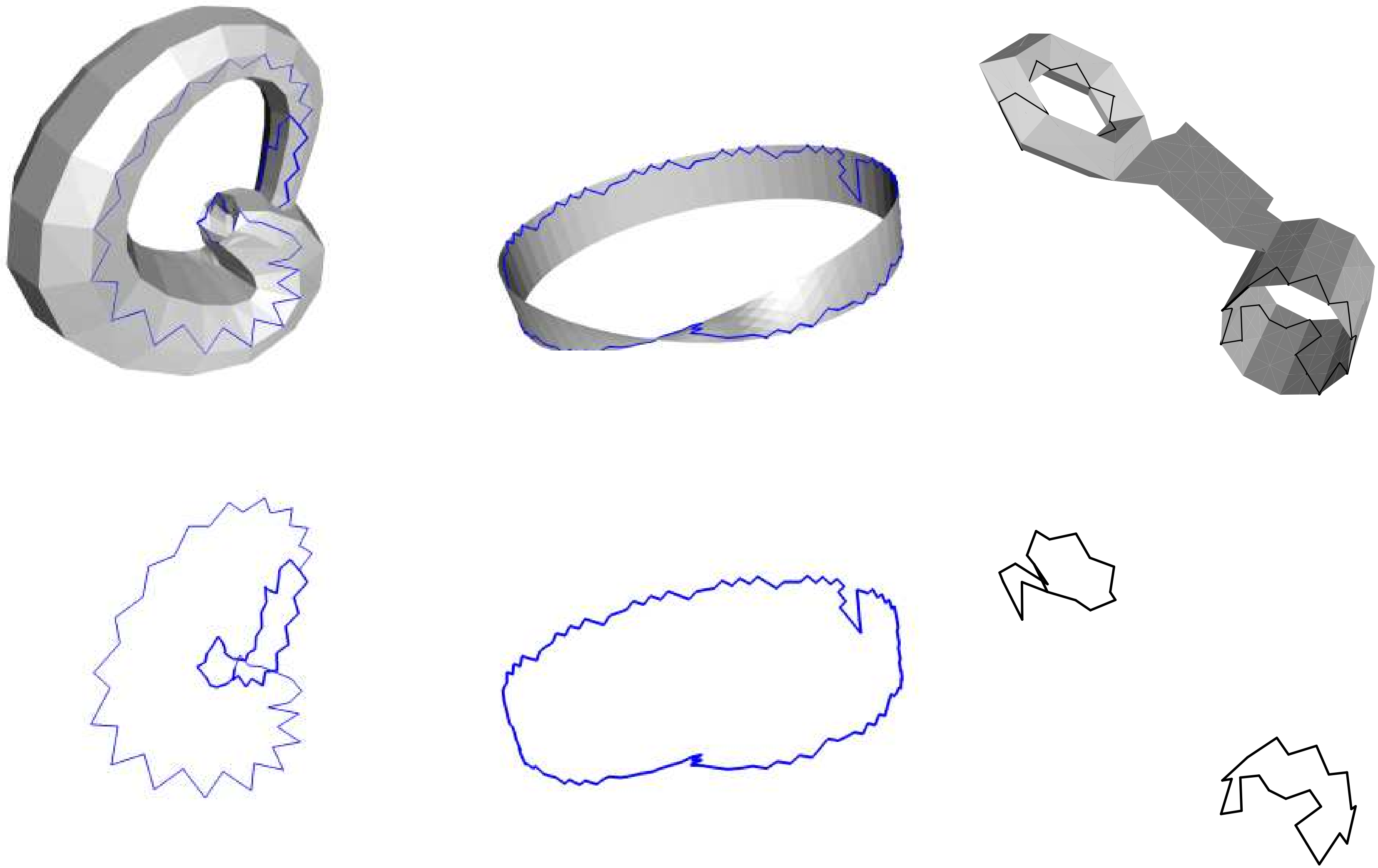
- Betti numbers
- Torsion coefficients
- Moduli operations

Agoston

- Betti numbers
- Torsion coefficients
- Generators

Moduli generators

Experimentation



Experimentation

- Computing time

Experimentation

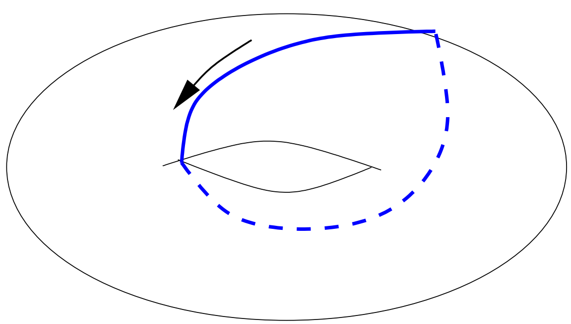
- Computing time

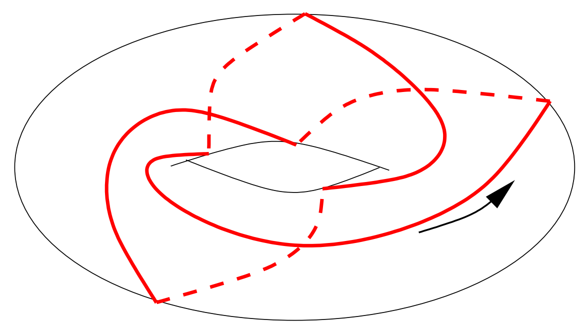
Discussion, perspectives

- Relation between moduli generators and classical generators ?

Discussion, perspectives

- Relation between moduli generators and classical generators ?
- A basis for $H_1(T, \mathbf{Z}/2)$ that is not a basis for $H_1(T, \mathbf{Z})$:





Perspectives

- Continuing the software

Perspectives

- Continuing the software
- Experimentations

Perspectives

- Continuing the software
- Experimentations
  - Less simple objects

Perspectives

- Continuing the software
- Experimentations
  - Less simple objects
  - Objects obtained from a segmentation

Perspectives

- Continuing the software
- Experimentations
  - Less simple objects
  - Objects obtained from a segmentation
  - Apparition of huge integers

Perspectives

- Continuing the software
- Experimentations
  - Less simple objects
  - Objects obtained from a segmentation
  - Apparition of huge integers
- Application for $3D$ and $4D$ images