Geometry of Gauss digitized convex shapes

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Context: geometric inference on digitized data

Gauss digitization at gridstep h (Klette & Rosenfeld '04)

Let $X \subset \mathbb{R}^d$ be some shape, its **Gauss digitization** is $X_h := h\mathbb{Z}^d \cap X$.

Questions:

- How can we infer the geometry of X only from its digitization X_h ?
- Can we get better results if the sampling grid step h tends to 0 ?

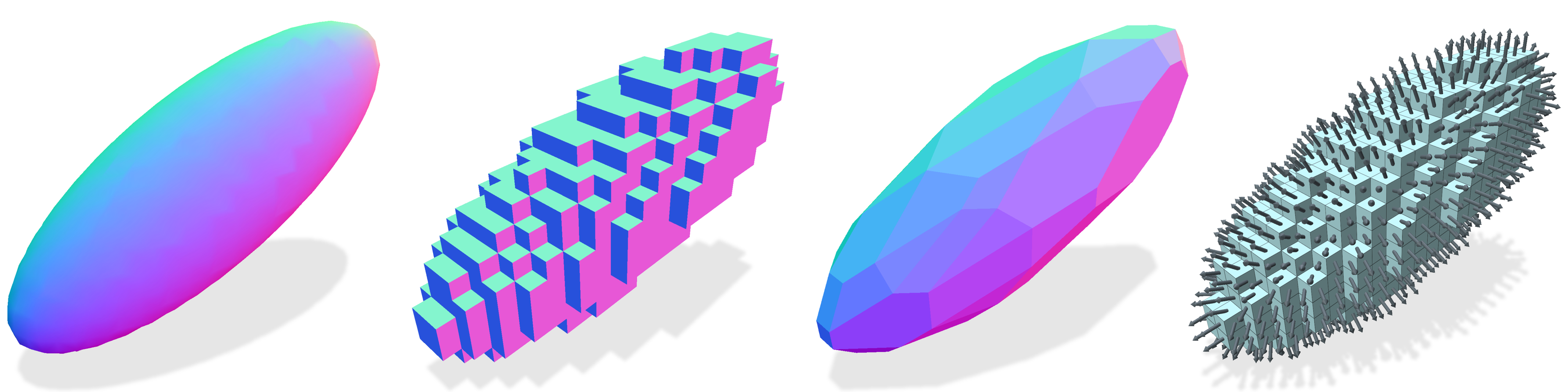
Answers:

- **multigrid convergence** results: same topology (Stelldinger et al. '06), area and moments estimators (Klette & Žunić '00), Hausdorff distance (Lachaud & Thibert '16), 2D derivatives (Esbelin & Malgoures '09, Provot & Gerard '11), 2D tangency with MDSS (Lachaud et al. '07), 3D normal estimations (Cuel et al '14, Lachaud et al '17), 3D curvatures (Coeurjolly et al '14, Lachaud et al '22), etc.
- **holy grail**: recover convex geometry in (locally) convex parts of X_h

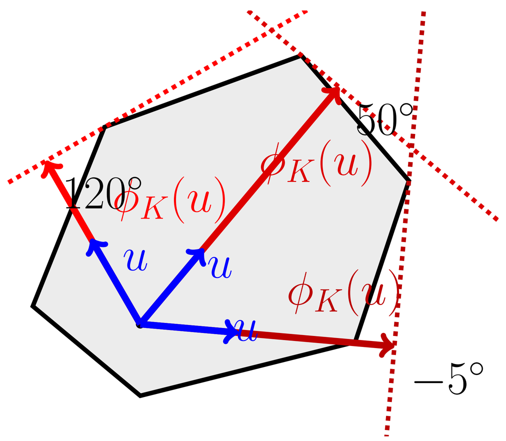
Our objective

Can the **convex hull** of X_h be used as a proxy for **geometric inference** of X.

Notations and tools



input shape X | X_h as voxels (visible surface is $\partial_h X$) | digitized convex hull Y_h | $\partial_h X$ with facet normals



Support function ϕ_S of convex set S in $\mathbb{R}^d$

$\phi_S : \mathbf{w} \in \mathbb{R}^d \mapsto \max_{\mathbf{x}\in S} \mathbf{w}\cdot\mathbf{x} \in \mathbb{R}$.

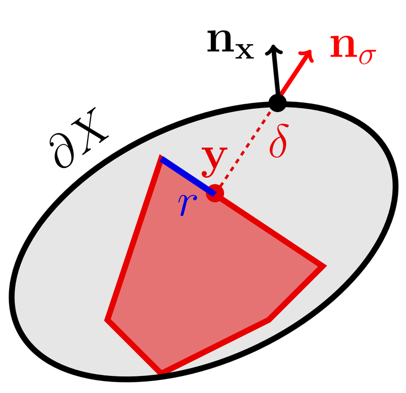
Theorem. For $S \subset T$ two convex sets, $\forall \mathbf{w} \in \mathbb{R}^d, \phi_S(\mathbf{w}) \leqslant \phi_T(\mathbf{w})$.

Results on arbitrary convex set $X \subset \mathbb{R}^d$

Lemma 1 (normal approximation)

Let Y be *any* convex polyhedron inside X. Let $\mathbf{x} \in \partial X$, and $\mathbf{n_x}$ its normal vector. Let $\mathbf{y} \in \partial Y$, closest to $\mathbf{x}$, σ the facet containing $\mathbf{y}$ with normal vector $\mathbf{n}_\sigma$. Then it holds:

$$\mathbf{n_x}\cdot\mathbf{n}_\sigma \geqslant 0, \quad \sin^2\angle(\mathbf{n_x},\mathbf{n}_\sigma) \leqslant \frac{\delta^2}{\delta^2+r^2}, \quad \text{with} \quad \begin{cases}\delta := \|\mathbf{x}-\mathbf{y}\|,\\ r := \mathrm{d}_{\mathbf{E}}(\mathbf{y},\partial\sigma).\end{cases}$$

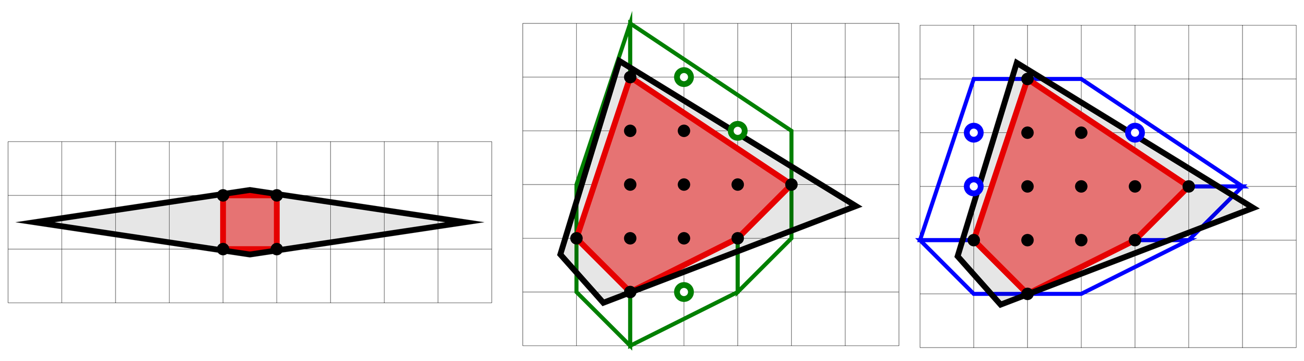


Proof. Use the support function of X and Y. □

Theorem 1 (close exterior lattice points)

Any 2D edge σ of ∂Y_h such that $\|\sigma\|_\infty \geqslant 2h$, and any 3D triangle with one projected area greater or equal to $\frac{7}{6}h^2$, have at least one *exterior lattice point* at distance $\leqslant h$ not in X that projects inside along some axis.

Proof. Use Pick's theorem in 2D and (Reeves '57, Th. I) in 3D. □

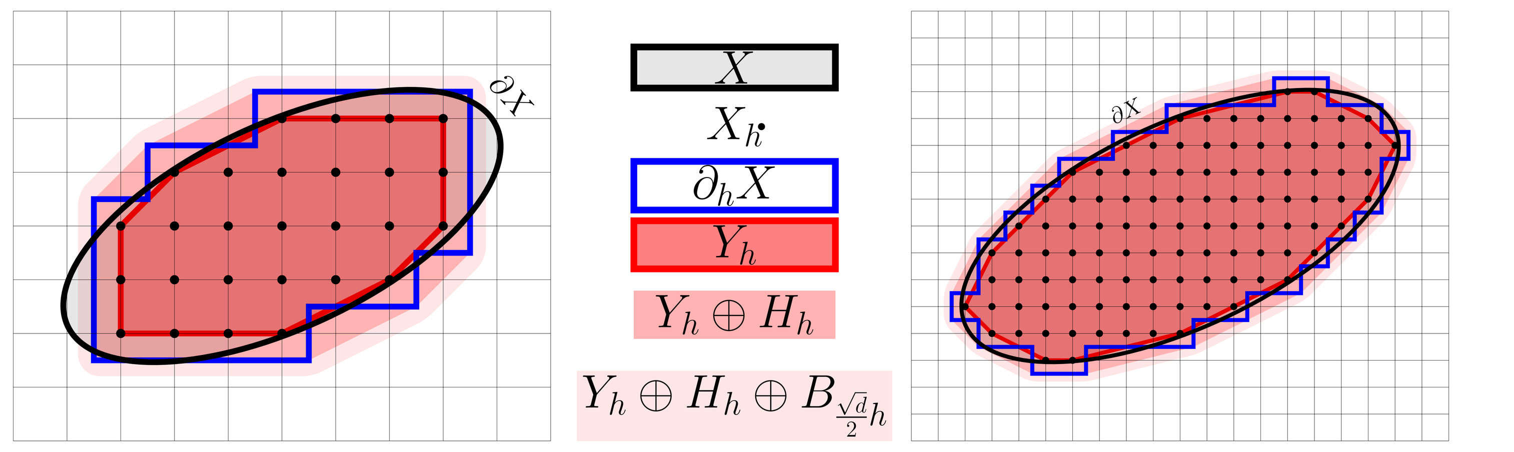


Corollary 1 (convergence of convex edges as tangents)

If X is convex and smooth enough, any edge σ having at least the average edge length of edges of ∂Y_h has a normal $\mathbf{n}_\sigma$ close the normal $\mathbf{n_x}$, where $\mathbf{x}$ is the closest point of ∂X to the edge center, and $\angle(\mathbf{n_x},\mathbf{n}_\sigma) \leqslant \Theta(h^{\frac{1}{3}})$.

Proof. Previous Lemma 1 and Theorem 1, combined with Balog and Barany's result, which tells that the number of vertices of Y_h is some $\Theta(h^{-\frac{2}{3}})$ (Balog & Bárány '91, Th. 2). □

Results on smooth convex set $X \subset \mathbb{R}^d$



Theorem 2 (boundary ∂X is sandwiched)

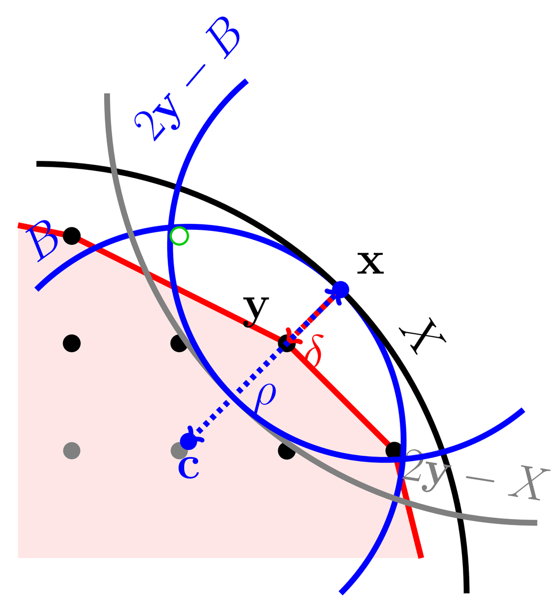
Assume that $\mathrm{reach}(\partial X) > \rho > 0$. For all gridsteps h, $0 < h < \frac{2\rho}{\sqrt{d}}$, we have:

$$Y_h \subset X \subset Y_h \oplus H_h \oplus B_{\frac{\sqrt{d}}{2}h}, \quad \mathrm{d}_H(X,Y_h) = \mathrm{d}_H(\partial Y_h, \partial X) \leqslant \sqrt{d}h.$$

Proof. Essentially, (Lachaud & Thibert '16, Th. 2), commutativity of Minkowski'sum with convex hull, support functions and (Wills '07, Th. 20). □

Theorem 3 (vertices of Y_h are close to ∂X) (see (Balog & Bárány '91))

Assume that $\mathrm{reach}(\partial X) > \rho > 0$. Let $\mathbf{y}$ be a vertex of Y_h. For gridsteps h, $0 < h \leqslant \rho$, $\delta := \mathrm{d}_{\mathbf{E}}(\mathbf{y},\partial X) < \alpha_d \rho^{-\frac{d-1}{d+1}} h^{\frac{2d}{d+1}}$, where constant α_d depends on dim.



Proof.
- let $\mathbf{x}$ closest to $\mathbf{y}$ on ∂X, $\delta := \|\mathbf{y}-\mathbf{x}\|$
- $\mathbf{c} := \mathbf{x} - \rho\mathbf{n}$ if $\mathbf{n}$ normal at $\mathbf{x}$
- $B := B_\rho(\mathbf{c})$ is included in X (reach)
- $S_X := X \cap (2\mathbf{y}-X)$, $S_B := B \cap (2\mathbf{y}-B)$
- $S_B \subset S_X$ hence $\mathrm{Vol}^d(S_B) \leqslant \mathrm{Vol}^d(S_X)$
- S_B union of two spherical caps $\mathrm{Vol}^d(S_B) = \mathrm{cst}\cdot\int_0^\delta(\sqrt{2\rho t - t^2})^{d-1}dt = \mathrm{cst}\cdot\rho^{\frac{d-1}{2}}\delta^{\frac{d+1}{2}}$.
- Minkowski theorem (S_X symmetric): $\mathrm{Vol}^d(S_B) \leqslant \mathrm{Vol}^d(S_X) \leqslant (2h)^d$

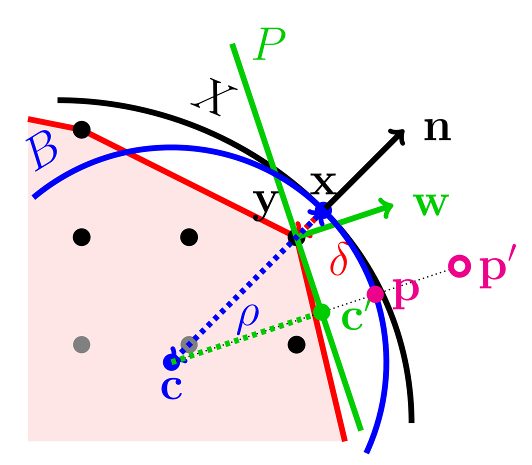
□

In 2D, $\delta \leqslant 1.041 h^{\frac{4}{3}}/\rho^{\frac{1}{3}}$. In 3D, $\delta \leqslant 1.129 h^{\frac{3}{2}}/\rho^{\frac{1}{2}}$.

Theorem 4 (Convergence of normals of Y_h to normals of X)

Let $\mathbf{y} \in \partial Y_h$ and $\mathbf{x}$ its closest point on ∂X. Let $\delta := \|\mathbf{x}-\mathbf{y}\|$. Let $\mathbf{n}$ normal to X at $\mathbf{x}$. Let $\mathbf{w} \in \mathrm{N}_{Y_h}(\mathbf{y})$ be any normal vector to Y_h at $\mathbf{y}$. For $0 < h < \frac{\rho}{\sqrt{d}}$, we have

$$\mathbf{n}\cdot\mathbf{w} \geqslant \frac{1-\sqrt{d}\frac{h}{\rho}}{1-\frac{\delta}{\rho}} \geqslant 1 - \sqrt{d}\frac{h}{\rho} > 0 \quad \text{i.e.} \quad \angle(\mathbf{n},\mathbf{w}) \leqslant O\left(\sqrt{\frac{h}{\rho}}\right).$$



Proof.
- Let $\mathbf{w} \in \mathrm{N}_{Y_h}(\mathbf{y})$ and $P \perp$ at $\mathbf{y}$
- $\mathbf{c} := \mathbf{x} - \rho\mathbf{n}$, $B := B_\rho(\mathbf{c}) \subset X$ (reach)
- $\mathbf{c}' := \pi_P(\mathbf{c})$, then $(\mathbf{c}'-\mathbf{c})\cdot\mathbf{w} = (\rho-\delta)\mathbf{n}\cdot\mathbf{w}$
- $\mathbf{p} := \mathbf{c} + \rho\mathbf{w}$ in B so in X, while $\mathbf{p}' := \mathbf{c}' + \sqrt{d}h\mathbf{w}$ outside X (Theorem 2)
- $\mathbf{p}\cdot\mathbf{w} \leqslant \phi_X(\mathbf{w}) < \mathbf{p}'\cdot\mathbf{w}$ (support fct)
  $\Rightarrow (\mathbf{c}+\rho\mathbf{w})\cdot\mathbf{w} < (\mathbf{c}'+\sqrt{d}h\mathbf{w})\cdot\mathbf{w}$
  $\Rightarrow \rho - \sqrt{d}h < (\rho-\delta)\mathbf{n}\cdot\mathbf{w}$

□

Tightness of normal convergence

The convergence $\sqrt{h}$ is tight. The constant $\sqrt{2\sqrt{d}/\rho}$ is almost reached (20%).

Conclusion & Perspectives

The convex hull of X_h is a good geometric inference of X in any dimension

- It is Hausdorff close to the input X, with distance $O(h)$
- Its vertices are even much closer to X, with distance $O(h^{\frac{2d}{d+1}})$
- Its normal vectors are multigrid convergent to the normals of X, with speed $O(\sqrt{h})$

Future works

- Experimental evaluation of this normal estimator wrt state-of-the-art.
- How to exploit proximity of vertices in designing new normal estimators ?

Acknowledgements

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