# An alternative definition for digital convexity*

Jacques-Olivier Lachaud[1 [0000−0003−4236−2133]]

LAMA, Université Savoie Mont Blanc, France
jacques-olivier.lachaud@univ-smb.fr

**Abstract.** This paper proposes *full convexity* as an alternative definition of digital convexity, which is valid in arbitrary dimension. It solves many problems related to its usual definitions, like possible non connectedness or non simple connectedness, while encompassing its desirable features. Fully convex sets are digitally convex, but are connected and simply connected. They have a morphological characterisation, which induces a simple convexity test algorithm. Arithmetic planes are fully convex too. We obtain a natural definition of tangent subsets to a digital surface, which gives rise to the tangential cover in 2D, and to its extensions in arbitrary dimension. Finally it leads to a simple algorithm for building a polygonal mesh from a set of digital points.

**Keywords:** Digital geometry · Digital convexity · Simple connectedness · Arithmetic planes · Tangential cover · Digital surface reconstruction.

## 1 Introduction

A subset $X \subset \mathbb{Z}^d$ is generally said to be *digitally convex* whenever

$$X = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d, \tag{1}$$

where $\mathrm{cvxh}\,(\cdot)$ denotes the convex hull (so called $H$-convexity [Eck01]). In contrast with continuous convexity, this definition does not imply digital connectedness of $X$ starting from dimension $d \geq 2$ (see Figure 1abcd). Therefore, especially in 2D, many works add a connectedness constraint or propose a definition that implies it (e.g. [KR82b] or see overviews of [Ron89,Eck01]). As already foreseen in [KR82a], 2D definitions do not extend well to 3D. Their own 3D digital convexity definition relies on the triangle chordal property plus connectedness, and induces a quite burdensome convexity check algorithm. But for $d \geq 3$ a connectedness constraint is not enough to build meaningful digital convex sets. For instance, when cut by a slice, they may lose connectedness (see Figure 1e). Other convexity definitions rely on progressive intersections with half-planes [Soi04]. Connectedness is preserved in the first steps at the price of a coarse approximation of convexity, and at the limit this definition is equivalent to $H$-convexity.

We present here a more consistent definition of digital convexity, which naturally entails connectedness as well as simple connectedness, and that is valid

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in arbitrary dimension. This new definition, called *full convexity*, encompasses digital arithmetic planes or digitizations of thick enough convex shapes. We give a morphological characterisation of full convexity, which shares—but not originates from—the thickening idea present in [CdFG20] for connecting 2D digital convex sets. This induces a practical full convexity check algorithm. Finally full convexity nicely addresses classical digital geometry problems, like a tangential cover in arbitrary dimension or piecewise affine recontruction of digital shapes.

## 2 Full convexity

Let $\mathbb{Z}^d$ be the $d$-dimensional digital space, $d > 0$. Let $\mathscr{C}^d$ be the *(cubical) cell complex* induced by the lattice $\mathbb{Z}^d$: its 0-cells are the points of $\mathbb{Z}^d$, its 1-cells are the open unit segments joining two 0-cells at distance 1, its 2-cells are the open unit squares, etc, and its $d$-cells are the $d$-dimensional open unit hypercubes with vertices in $\mathbb{Z}^d$. We denote $\mathscr{C}^d_k$ the set of its $k$-cells. In the following, a *cell* will always designate an element of $\mathscr{C}^d$, and the term *subcomplex* always designates a subset of $\mathscr{C}^d$. A cell $\sigma$ is a *face* of another cell $\tau$ whenever $\sigma$ is a subset of the topological closure $\bar{\tau}$ of $\tau$, and we write $\sigma \preccurlyeq \tau$. Given any subcomplex $K$ of $\mathscr{C}^d$, the *closure* $\mathrm{Cl}\,(K)$ of $K$ is the complex $\{\tau \in \mathscr{C}^d,\ \text{s.t. } \exists \sigma \in K, \tau \preccurlyeq \sigma\}$ and the *star* $\mathrm{Star}\,(K)$ of $K$ is $\{\tau \in \mathscr{C}^d,\ \text{s.t. } \exists \sigma \in K, \sigma \preccurlyeq \tau\}$.

In combinatorial topology, a subcomplex $K$ with $\mathrm{Star}\,(K) = K$ is *open*, while being *closed* when $\mathrm{Cl}\,(K) = K$. The *body* of a subcomplex $K$, i.e. the union of its cells in $\mathbb{R}^d$, is written $\|K\|$. Finally, if $Y$ is any subset of the Euclidean space $\mathbb{R}^d$, we denote by $\bar{\mathscr{C}}^d_k[Y]$ the set of $k$-cells whose topological closure has a non-empty intersection with $Y$, i.e. $\bar{\mathscr{C}}^d_k[Y] := \{c \in \mathscr{C}^d_k, \bar{c} \cap Y \neq \emptyset\}$. The complex made of all $k$-cells having a non-empty intersection with $Y$, $0 \leqslant k \leqslant d$ is called the *intersection (cubical) complex of* $Y$ and denoted by $\bar{\mathscr{C}}^d[Y]$.

**Lemma 1** *The intersection complex of a set $Y$ is open and its body covers $Y$.*

*Proof.* If $Y$ is the empty set, $\bar{\mathscr{C}}^d[Y]$ is empty and is open. If $Y$ is not empty, let $\sigma$ be any cell of $\bar{\mathscr{C}}^d[Y]$. Let $\tau$ be any cell of $\mathscr{C}^d$ with $\sigma \preccurlyeq \tau$. Thus $\sigma \subset \bar{\tau} \Rightarrow \bar{\sigma} \subset \bar{\tau}$ (since topological closure is increasing and idempotent). It follows that $\sigma \in \bar{\mathscr{C}}^d[Y] \Leftrightarrow \bar{\sigma} \cap Y \neq \emptyset \Rightarrow \bar{\tau} \cap Y \neq \emptyset \Leftrightarrow \tau \in \bar{\mathscr{C}}^d[Y]$. We have just proved that $\mathrm{Star}\,(\sigma) \subset \bar{\mathscr{C}}^d[Y]$, hence $\mathrm{Star}\,\left(\bar{\mathscr{C}}^d[Y]\right) \subset \bar{\mathscr{C}}^d[Y]$. The converse inclusion being obvious, $\bar{\mathscr{C}}^d[Y]$ is open. The fact that $Y \subset \left\|\bar{\mathscr{C}}^d[Y]\right\|$ is straightforward. □

**Definition 1 (Full convexity)** *A non empty subset $X \subset \mathbb{Z}^d$ is* digitally $k$-convex *for $0 \leqslant k \leqslant d$ whenever*

$$\bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)]. \tag{2}$$

*Subset $X$ is* fully (digitally) convex *if it is digitally $k$-convex for all $k, 0 \leqslant k \leqslant d$.*

Equivalently, the intersection complex of a fully convex set $Z$ covers the convex hull of $Z$. We can already make the following observation:

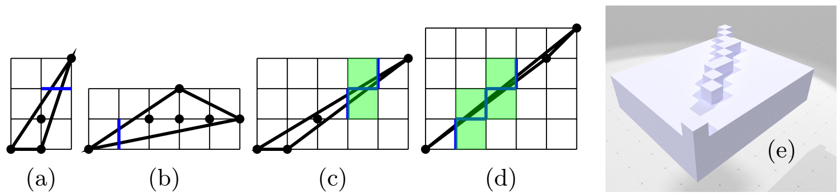



**Fig. 1.** (abcd) Digital triangles that are not fully convex: digital points are depicted as black disks, missing 1-cells for digital 1-convexity as blue lines, missing 2-cells for digital 2-convexity as green squares. (e) Usual digital convexity (1) plus 3D connectivity does not imply connectedness on the upper slice; it is also not fully convex.

**Lemma 2** *Common digital convexity is the digital* 0*-convexity.*

*Proof.* Remark (1): for any $Y \subset \mathbb{R}^d$, we have $\bar{\mathscr{C}}_0^d[Y] = \{c \in \mathscr{C}_0^d, c \cap Y \neq \emptyset\} = \{c \in \mathbb{Z}^d, c \cap Y \neq \emptyset\} = Y \cap \mathbb{Z}^d$. Now, from (1), $X \subset \mathbb{Z}^d$ is digitally convex iff $X = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$, otherwise said $X \cap \mathbb{Z}^d = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$. With remark (1), it is equivalent to $\bar{\mathscr{C}}_0^d[X] = \bar{\mathscr{C}}_0^d[\mathrm{cvxh}\,(X)]$, which is exactly (2) for $k = 0$. $\square$

Figure 1 shows several digitally 0-convex sets, but which are not full convex. Clearly full convexity forbids too thin convex sets, which are typically the ones that are not connected or simply connected in the digital sense. Denoting by $\#\,(X)$ the cardinal of a finite set $X$, the straightforward lemma below shows that is suffices to count intersected cells to check for full convexity.

**Lemma 3** *A finite non-empty subset $X \subset \mathbb{Z}^d$ is* digitally $k$-convex *for $0 \leqslant k \leqslant d$ iff $\#\left(\bar{\mathscr{C}}_k^d[X]\right) \geqslant \#\left(\bar{\mathscr{C}}_k^d[\mathrm{cvxh}\,(X)]\right)$.*

For example, the tetrahedra $T(l) = \{(0,0,0), (1,0,0), (0,1,0), (1,1,l)\}$, for positive integer $l$, is digitally 0-convex. However $\mathrm{cvxh}\,(T(l))$ intersects as many 2-cells and 3-cells as wanted above the unit square with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,0)$, just by increasing $l$. Meanwhile, $T(l)$ only intersects the same finite number of 2-cells and 3-cells. Hence, for $l \geqslant 2$, $T(l)$ is not fully convex.

It is not necessary to check digital $d$-convexity to verify if a digital set is fully convex, and this property is useful to speed up algorithms to check for full convexity. You can observe the contraposition of this lemma on Figure 1, left, where non digitally 2-convex sets in 2D are not digitally 1-convex too.

**Lemma 4** *If $Z \subset \mathbb{Z}^d$ is digitally $k$-convex for $0 \leqslant k < d$, it is also digitally $d$-convex, hence fully convex.*

*Proof.* Let $Z$ be such set, so $Z$ not empty, and let $\sigma \in \bar{\mathscr{C}}_d^d[\mathrm{cvxh}\,(Z)]$. Let $B = \partial\bar{\sigma}$ be the topological boundary of $\sigma$. By hypothesis, we have $\bar{\sigma} \cap \mathrm{cvxh}\,(Z) \neq \emptyset$, hence $(B \cup \sigma) \cap \mathrm{cvxh}\,(Z) \neq \emptyset$.

The surface $B$ separates $\mathbb{R}^d$ into two components, one finite equal to $\sigma$, the other infinite. Assume $B \cap \mathrm{cvxh}\,(Z) = \emptyset$. Since $\mathrm{cvxh}\,(Z)$ is arc-connected, then cvxh $(Z)$ lies entirely in one component. The relation $(B \cup \sigma) \cap \mathrm{cvxh}\,(Z) \neq \emptyset$ then implies $\mathrm{cvxh}\,(Z) \subset \sigma$. This is impossible since $\mathrm{cvxh}\,(Z) \cap \mathbb{Z}^d = Z$ while $\sigma \cap \mathbb{Z}^d = \emptyset$.

It follows that $B \cap \mathrm{cvxh}\,(Z) \neq \emptyset$. But $B$ is a union of $k$-cells $(b_i)_{i=0\ldots m}$ of $\mathscr{C}^d$, with $0 \leqslant k < d$. There exists at least one $k$-cell $b_j$ with $b_j \cap \mathrm{cvxh}\,(Z) \neq \emptyset$. Thus $b_j \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(Z)]$. But $Z$ is digitally $k$-convex for $0 \leqslant k < d$, so $b_j \in \bar{\mathscr{C}}^d_k[Z]$. To conclude, $\sigma \in \mathrm{Star}\,(b_j)$ and $\bar{\mathscr{C}}^d[Z]$ is open, so $\sigma$ belongs also to $\bar{\mathscr{C}}^d[Z]$. We have just shown that every $d$-cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Z)]$ are in $\bar{\mathscr{C}}^d[Z]$, so $Z$ is digitally $d$-convex and hence fully convex. □

Other implications of digital $k$-convexities over digital $l$-convexities are unlikely. For instance in 3D, some digital sets are digitally 0-convex, 1-convex, 3-convex but are not 2-convex, like $\{(0,0,0), (1,1,2), (1,1,3), (1,2,3), (2,1,3)\}$.

## 3 Topological properties of fully convex digital sets

We give below the main topological properties of fully convex digital sets.

**Theorem 1** *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection cubical complex is connected.*

*Proof.* Let $x, x'$ be two points of $\left\|\bar{\mathscr{C}}^d[X]\right\|$. Since $\mathscr{C}^d$ is a partition of $\mathbb{R}^d$, there are two cells $c, c'$ of $\bar{\mathscr{C}}^d[X]$ such that $x \in c$, $x' \in c'$. Since $Z$ is fully convex, then $\bar{\mathscr{C}}^d[X] = \bar{\mathscr{C}}^d[\mathrm{cvxh}\,(X)]$. Hence there exist $y \in \bar{c} \cap \mathrm{cvxh}\,(X)$ and $y' \in \bar{c} \cap \mathrm{cvxh}\,(X)$. By convexity of cells, the segment $[x, y[$ lies entirely in $c$ hence in $\left\|\bar{\mathscr{C}}^d[X]\right\|$. Similarly, the segment $]y', x']$ lies entirely in $c'$ hence in $\left\|\bar{\mathscr{C}}^d[X]\right\|$.

Now by definition of convexity, the segment $[y, y']$ lies in $\mathrm{cvxh}\,(X)$. But $\mathrm{cvxh}\,(X) \subset \left\|\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(X)]\right\| = \left\|\bar{\mathscr{C}}^d[X]\right\|$ by cell convexity. We have just built an arc from $x$ to $x'$ which lies entirely in $\left\|\bar{\mathscr{C}}^d[X]\right\|$. We conclude since arc-connectedness implies connectedness. □

Two elements $x, y$ of $\mathbb{Z}^d$ are *$d$-adjacent* if $\|x - y\|_\infty \leqslant 1$. The transitive closure of this relation defines the *$d$-connectedness* relation. Historically, it was called 8-connectivity in 2D, and 26-connectivity in 3D.

**Theorem 2** *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then $Z$ is $d$-connected.*

*Proof.* We show first that 0-cells of $\bar{\mathscr{C}}^d[Z]$ are face-connected, i.e. for any points $z, z' \in Z = \bar{\mathscr{C}}^d_0[Z]$, there is a path of cells $(c_i)_{i=0..m}$ of $\bar{\mathscr{C}}^d[Z]$, such that $c_0 = \sigma$, $c_m = \tau$, and for all $i \in \mathbb{Z}, 0 \leqslant i < m$, either $c_i \preccurlyeq c_{i+1}$ or $c_{i+1} \preccurlyeq c_i$.

The straight segment $[z, z']$ is included in $\mathrm{cvxh}\,(Z)$, hence any one of its point belongs to a cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Z)]$ so a cell of $\bar{\mathscr{C}}^d[Z]$ by full convexity.

Let $p(t) = (1-t)z + tz'$ for $0 \leqslant t \leqslant 1$ be a parameterization of segment $[z, z']$. The above remark implies that, for any $t \in [0,1]$, the point $p(t)$ belongs to a cell $c(t)$ of $\bar{\mathscr{C}}^d[Z]$. The sequence of intersected cells from $t = 0$ to $t = 1$ is obviously finite, and we denote it by $(c_0, c_1, \ldots, c_m)$ with $c_0 = p(0) = z$ and $c_m = p(1) = z'$.

Since it corresponds to an infinitesimal change of $t$, two consecutive cells of this sequence are necessary in the closure of one of them, hence $c_i \preccurlyeq c_{i+1}$ or $c_{i+1} \preccurlyeq c_i$.

We use Lemma 5 given just after. We associate to each cell $c_i$ one of its $Z$-corner, denoted $z_i$. We obtain a sequence of digital points $z = z_0, z_1, \ldots, z_m = z'$. Now any two incident faces (like $c_i$ and $c_{i+1}$) belong to the closure of a $d$-cell $\sigma$. It follows that both corner $z_i$ and $z_{i+1}$ are vertices of $\bar{\sigma}$, a unit hypercube. Obviously $\|z_i - z_{i+1}\|_\infty \leqslant 1$ and these two points are $d$-adjacent. We have just built a sequence of $d$-adjacent points in $Z$, which concludes. □

**Lemma 5** *Let $Z \subset \mathbb{Z}^d$. If $\sigma$ is a cell of $\bar{\mathscr{C}}^d[Z]$, there there exists $z \in \bar{\mathscr{C}}^d_0[Z] = Z$ such that $z \preccurlyeq \sigma$. We call such digital point a $Z$-*corner *for $\sigma$. If $\sigma$ is a 0-cell, its only $Z$-corner is itself.*

*Proof.* By definition of $\bar{\mathscr{C}}^d[Z]$ we have $\bar{\sigma} \cap Z \neq \emptyset$. It follows that $\exists z \in Z$ such that $z \in \bar{\sigma}$. So $z \preccurlyeq \sigma$ and also $z \in Z = \bar{\mathscr{C}}^d_0[Z]$.

We can show an even stronger result on fully convex sets: they present no topological holes. Indeed, we have:

**Theorem 3** *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection cubical complex is* simply *connected.*

*Proof.* Let $\mathscr{A} := \{x(t), t \in [0,1]\}$ be a closed curve in $\left\|\bar{\mathscr{C}}^d[X]\right\|$, i.e. $x(0) = x(1)$ and $x(t) \in \left\|\bar{\mathscr{C}}^d[X]\right\|$. We must show that there is a homotopy from $\mathscr{A}$ to a point $a \in \left\|\bar{\mathscr{C}}^d[X]\right\|$.

The curve $x(t)$ visits cells of $\bar{\mathscr{C}}^d[X]$. Let $c(t)$ be these cells. By finiteness of $\mathscr{A}$, $c(t)$ defines a finite sequence of cells $c_0, c_1, \ldots, c_m$ from $t = 0$ to $t = 1$, with $c_m = c_0$. We can also associate a sequence of parameters $t_0, t_1, \ldots, t_m$, such that $x(t_i) \in c_i = c(t_i)$. As in the proof of Theorem 2, two consecutive cells of this sequence are necessary in the closure of one of them. Let us set $d_i$ to $c_i$ or $c_{i+1}$ such that both are in $\bar{d}_i$.

The path $x([t_i, t_{i+1}])$ lies in $c_i \cup c_{i+1}$. For each cell $c_i$ we pick one of its $Z$-corner $z_i$. Clearly $z_i$ and $z_{i+1}$ belong to $\bar{d}_i$. By convexity of $\bar{d}_i$, it is in particular simply-connected and there is a homotopy in $\bar{d}_i$ between $x([t_i, t_{i+1}])$ and the segment $[z_i, z_{i+1}]$. Since $\bar{\mathscr{C}}^d[Z]$ is open and both points are in $\left\|\bar{\mathscr{C}}^d[Z]\right\|$, $[z_i, z_{i+1}] \subset \left\|\bar{\mathscr{C}}^d[Z]\right\|$ as well as the whole homotopy. Gathering all these local homotopies for every $i$, $0 \leqslant i < m$, we have defined a homotopy between $\mathscr{A}$ and the polyline $[z_i]_{i=0\ldots m}$.

By full convexity, every $z_i \in Z$ is also in $\mathrm{cvxh}\,(Z)$. It follows that the vertices of the polyline $[z_i]_{i=0\ldots m}$ belong to $\mathrm{cvxh}\,(Z)$. By convexity of $\mathrm{cvxh}\,(Z)$, the whole polyline is a subset of $\mathrm{cvxh}\,(Z)$. Being a closed curve in a simply connected set, the polyline $[z_i]_{i=0\ldots m}$ is continuously deformable to a point of this set, say $z_0$, by some homotopy. Composing the two homotopies finishes the argument. □

Finally we can determine a relation between the numbers of $k$-cells of the intersection complex of a fully convex set. For $K$ a subcomplex, let $\#_k(K)$ be its number of $k$-cells. The *Euler characteristic* of a subcomplex $K$ is $\chi(K) := \sum_{k=0}^{d}(-1)^k \#_k(K)$. Its proof is omitted for space reasons.

**Theorem 4** *The Euler characteristic of the intersection cubical complex of a fully convex set is* $(-1)^d$.

## 4 Morphological properties and recognition algorithm

We provide first a morphological characterization of full convexity that will help us to design a practical algorithm for checking this property.

*Morphological characterisation.* Let $I^d := \{1, \ldots, d\}$. The set of subsets of cardinal $k$ of $I^d$ is denoted by $I^d_k$, for $0 < k \leqslant d$. For $i \in I^d$, let $\mathcal{S}_i := \{t\mathbf{e}_i, t \in [0,1]\}$ be the unit segments aligned with axis vectors $\mathbf{e}_i$. For any point $x$ of $\mathbb{R}^d$, we write its $d$ coordinates with superscripts: $x^1, \ldots, x^d$. Let us also denote the Minkowski sum of two sets $A$ and $B$ by $A \oplus B$. We further build axis-aligned unit squares, cubes, etc, by suming up the unit segments: for any $\alpha \in I^d_k$, $\mathcal{S}_\alpha := \bigoplus_{i\in\alpha} \mathcal{S}_i$. For instance, in 3D, the three unit segments are $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ (or equiv. $\mathcal{S}_{\{1\}}, \mathcal{S}_{\{2\}}, \mathcal{S}_{\{3\}}$), the three unit squares are $\mathcal{S}_{\{1,2\}}, \mathcal{S}_{\{1,3\}}, \mathcal{S}_{\{2,3\}}$, the unit cube is $\mathcal{S}_{\{1,2,3\}}$. To treat the 0-dimensional case uniformly, we set $I^d_0 = \{0\}$ and $\mathcal{S}_{\{0\}} = \{\vec{0}\}$.

We can partition the $k$-cells of $\mathscr{C}^d_k$ into $\#\left(I^d_k\right)$ subsets such that, for any $\alpha \in I^d_k$, each subset denoted by $\mathscr{C}^d_\alpha$ contains all the $k$-cells parallel to $\mathcal{S}_\alpha$. For instance, $\mathscr{C}^d_{\{1\}}$ and $\mathscr{C}^d_{\{2\}}$ partition the set $\mathscr{C}^d_1$ in dimension $d = 2$. Now let us define the mapping $\mathcal{Z} : \mathscr{C}^d \rightarrow \mathbb{Z}^d$ which associates to any cell $\sigma$, the digital vertex of $\bar{\sigma}$ with highest coordinates. Its restriction to $\mathscr{C}^d_\alpha$ is denoted by $\mathcal{Z}_\alpha$.

**Lemma 6** *For any* $\alpha \in I^d_k$, *the mapping* $\mathcal{Z}_\alpha$ *is a bijection.*

*Proof.* Clearly every digital point of $\mathbb{Z}^d$ forms the highest vertex of all possible kind of cells, so $\mathcal{Z}_\alpha$ is a surjection. Now no two cells of $\mathscr{C}^d_\alpha$ can have the same highest vertex, since all cells of $\mathscr{C}^d_\alpha$ are distinct translations of the same set. □

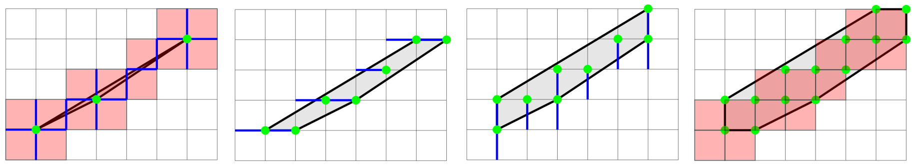
The intersection subcomplex of some set $Y$ restricted to cells of $\mathscr{C}^d_\alpha$ is naturally denoted by $\bar{\mathscr{C}}^d_\alpha[Y]$. We relate $k$-cells intersected by set $Y$ to digital points included in the set $Y$ dilated in some directions, as illustrated in Figure 2.

**Lemma 7** *For any* $Y \subset \mathbb{R}^d$, *for any* $\alpha \in I^d_k$, $\mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[Y]) = \bar{\mathscr{C}}^d_0[Y \oplus \mathcal{S}_\alpha]$.

*Proof.* We proceed by equivalences (the logical "and", symbol $\wedge$, has more priority than "if and only if", symbol $\Leftrightarrow$, but less than any other operations):

$$
\begin{aligned}
z \in \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[Y]) &\Leftrightarrow \sigma \in \bar{\mathscr{C}}^d_\alpha[Y] \wedge \sigma = \mathcal{Z}_\alpha^{-1}(z) \qquad (\mathcal{Z}_\alpha \text{ is a bijection, Lemma 6})\\
&\Leftrightarrow \exists y \in Y, y \in \bar{\sigma} \wedge \sigma = \mathcal{Z}_\alpha^{-1}(z)\\
&\Leftrightarrow \exists y \in Y, (\forall i \in \alpha, z^i - 1 \leqslant y^i \leqslant z^i \wedge \forall j \in I^d \setminus \alpha, z^j = y^i)\\
&\Leftrightarrow \exists y \in Y, (\forall i \in \alpha, 0 \leqslant x^i \leqslant 1 \wedge \forall j \in I^d \setminus \alpha, x^j = 0) \wedge x = z - y\\
&\Leftrightarrow \exists y \in Y, x \in \mathcal{S}_\alpha \wedge z = x + y \in \mathbb{Z}^d\\
&\Leftrightarrow z \in (Y \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d.
\end{aligned}
$$

We conclude since $(Y \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = \bar{\mathscr{C}}^d_0[Y \oplus \mathcal{S}_\alpha]$. □



(a) $\mathcal{P}$ (b) $\mathcal{P} \oplus \mathcal{S}_{\{1\}}$ (c) $\mathcal{P} \oplus \mathcal{S}_{\{2\}}$ (d) $\mathcal{P} \oplus \mathcal{S}_{\{1,2\}}$

**Fig. 2.** Let $\mathcal{P} = \mathrm{cvxh}\left(\{(0,0),(2,1),(5,3)\}\right)$. (a) $\bar{\mathscr{C}}^d[\mathcal{P}]$. (bcd) We can see that $\#\left(\bar{\mathscr{C}}^d_{\{1\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{1\}}]\right) = 7$, $\#\left(\bar{\mathscr{C}}^d_{\{2\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{2\}}]\right) = 9$, and $\#\left(\bar{\mathscr{C}}^d_{\{1,2\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{1,2\}}]\right) = 14$. The bijections $\mathcal{Z}_{\{1\}}$, $\mathcal{Z}_{\{2\}}$, $\mathcal{Z}_{\{1,2\}}$ are made clear in (b), (c), (d) respectively.

We arrive to our morphological characterization of full convexity: full convexity can thus be checked with common algorithms for checking digital convexity. We denote by $\mathbf{x}(Z)$ the set $Z$ translated by some lattice vector $\mathbf{x}$. Let $U_\emptyset(Z) := Z$, and, for $\alpha \subset I^d$ and $i \in \alpha$, we define recursively $U_\alpha(Z) := U_{\alpha\setminus i}(Z) \cup \mathbf{e}_i(U_{\alpha\setminus i}(Z))$. The previous definition is consistent since it does not depend on the order of the sequence $i \in \alpha$.

**Theorem 5** *A non empty subset $X \subset \mathbb{Z}^d$ is digitally $k$-convex for $0 \leqslant k \leqslant d$ iff*

$$\forall \alpha \in I^d_k, (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d, \tag{3}$$

$$\textit{or } (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X \oplus \mathcal{S}_\alpha)) \cap \mathbb{Z}^d, \tag{4}$$

$$\textit{or } U_\alpha(X) = \mathrm{cvxh}\,(U_\alpha(X)) \cap \mathbb{Z}^d. \tag{5}$$

*It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.*

*Proof.* We proceed by equivalence for (3):

$$\begin{aligned}
& \bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)] \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \bar{\mathscr{C}}^d_\alpha[X] = \bar{\mathscr{C}}^d_\alpha[\mathrm{cvxh}\,(X)] \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[X]) = \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[\mathrm{cvxh}\,(X)]) && (\mathcal{Z}_\alpha \text{ is a bijection}) \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \bar{\mathscr{C}}^d_0[X \oplus \mathcal{S}_\alpha] = \bar{\mathscr{C}}^d_0[\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha] && (\text{Lemma 7}) \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d.
\end{aligned}$$

(4) follows since convex hull operation commutes with Minkowski sum. Using Lemma 9 on left handside and Lemma 10 on right handside implies (5). □

Finally we can remark that, if $X \subset \mathbb{Z}^d$ is $d$-connected, then necessarily all $U_\alpha(X)$ are by construction $d$-connected.

*Recognition algorithm.* Algorithm 1 checks the full convexity of a digital set $Z \subset \mathbb{Z}^d$. Due to the bijections $\mathcal{Z}_\alpha$, all the processed sets are subsets of $\mathbb{Z}^d$.

**Algorithm 1:** Given the dimension $d$ of the space and a subset $Z$ of the digital space $\mathbb{Z}^d$, IsFullyConvex returns true iff $Z$ is fully convex.

**Function** IsConvex( **In** $S$: subset of $\mathbb{Z}^d$ ) : boolean;
**begin**
1 Polytope $\mathcal{P} \leftarrow$ ConvexHull($S$);
2 **return** Cardinal($S$) = CountLatticePoint($\mathcal{P}$);

**Function** IsFullyConvex( **In** d : integer, **In** $Z$: subset of $\mathbb{Z}^d$ ) : boolean;
**Var** $C$ : array[0 . . . d-1] of lists of subsets of $I^d$;
**Var** $X$ : map associating subsets of $I^d \rightarrow$ sets of digital points ;
**begin**
3 **if** Cardinal($Z$) = 0 ***or*** $\neg$IsDConnected($Z, d$) **then return** *false*;
$C[0] \leftarrow (\{0\})$ ; $X[0] \leftarrow Z$ ;
4 **if** $\neg$IsConvex($Z$) **then return** *false*;
5 **for** $k \leftarrow 1$ **to** $d-1$ **do**
$C[k] \leftarrow \emptyset$;
**foreach** $\beta \in C[k-1]$ **do**
**for** $j \leftarrow 1$ **to** $d$ **do**
$\alpha \leftarrow$ Append($\beta, j$) ;
6 **if** IsStrictlyIncreasing($\alpha$) **then**
$C[k] \leftarrow$ Append($C[k], \alpha$);
7 $X[\alpha] \leftarrow$ Union($X[\beta], \mathbf{e}_j(X[\beta])$) ;
8 **if** $\neg$IsConvex($X[\alpha]$) **then return** *false*;
**return** *true*

**Theorem 6** *Algorithm 1 correctly checks if a digital set $Z$ is fully convex.*

*Proof.* First of all, IsConvex checks the classical digital convexity of any digital set $S$ by counting lattice points within $\mathrm{cvxh}(S)$ (Lemma 3).

Looking now at IsFullyConvex, line 3 checks the $d$-connectedness of $Z$ and outputs false if $Z$ is not connected (valid since Lemma 2).

Line 4 checks for digital 0-convexity (i.e. usual digital convexity). The loop starting at line 5 builds, for each dimension $k$ from 1 to $d-1$ the possible $\alpha \in I^d_k$, and stores them in $C[k]$. Line 6 guarantees that each possible subsets of $I^d_k$ is used exactly once.

Line 7 builds $X[\alpha] = U_\alpha(Z)$. Indeed, by induction assume $X[\beta] = U_\beta(Z)$. Then $X[\alpha] = X[\beta] \cup \mathbf{e}_j(X[\beta])$ which is exactly the definition of $U_\alpha(Z)$.

Finally line 8 verifies $U_\alpha(Z) = \mathrm{cvxh}(U_\alpha(Z)) \cap \mathbb{Z}^d$. Since it does this check for every $\alpha \in I^d_k$, it checks digital $k$-convexity according to Theorem 5, (5). Now Lemma 4 tells that it is not necessary to check digital $d$-convexity if all the other digital $k$-convexities are satisfied. This establishes the correctness. $\square$

First, letting $n = \#(Z)$, function IsDConnected takes $O(n)$ operations by depth first algorithm and bounded number of adjacent neighbors. There are less than $d2^d$ calls to IsStrictlyIncreasing, which takes $O(d)$ time complexity.

However the total number of calls to IsConvex is exactly $2^d - 1$, and its time complexity dominates (by far) the previous $d^2 2^d$ in practical uses. The overall complexity $T(n)$ of this algorithm is thus governed by (1) the complexity $T_1(n)$ for computing the convex hull and (2) the complexity $T_2(n)$ for counting the lattice points within. For $T_1(n)$, it is: $O(n)$ for $d = 2$ (since we know $S$ is $d$-connected, it can be done in linear time [HKV81] or [BLPR09] for a fast practical algorithm), $O(n \log n)$ in 3D [Cha96] and $n^{\lfloor d/2 \rfloor} / \lfloor d/2 \rfloor!$ in $d$D [Cha93,BDH96]. For $T_2(n)$, it is $O(n)$ in 2D using Pick's formula, and in general dimension it is related to Ehrhart's theory [Ehr62], where best algorithms run in $O(n^{O(d)})$ [Bar94].

In practice, when the number of facets of the convex hulls is low, counting lattice points within the hull is done efficiently by visiting all lattice points within the bounding box, which is not too big since all digital sets are connected.

## 5 Digital planarity, tangency and linear reconstruction

We sketch here a few nice geometric applications of full convexity.

*Thick enough arithmetic planes are fully convex.* An arithmetic plane of intercept $\mu \in \mathbb{Z}$, positive thickness $\omega \in \mathbb{Z}, \omega > 0$, and irreducible normal vector $N \in \mathbb{Z}^d$ is defined as the digital set $P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$.

**Theorem 7** *Arithmetic planes are digitally 0-convex for arbitrary thickness, and fully convex for thickness $\omega \geqslant \|N\|_\infty$.*

*Proof.* Let $Q = P(\mu, N, \omega)$ be some arithmetic plane. Let $Y^- := \{x \in \mathbb{R}^d, \mu \leqslant x \cdot N\}$, $Y^+ := \{x \in \mathbb{R}^d, x \cdot N < \mu + \omega\}$, and $Y = Y^- \cap Y^+$. We have $Q = Y \cap \mathbb{Z}^d$, with $Y$ convex, so $Q$ is digitally 0-convex (non emptyness comes from $\omega > 0$).

Now let $c$ be any $k$-cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Q)]$, $1 \leqslant k \leqslant d$. Let us show that at least one vertex of $c$ is in $\bar{\mathscr{C}}^d[Q]$. There exists $x \in c$ with $x \in \mathrm{cvxh}\,(Q) = \mathrm{cvxh}\left(Y \cap \mathbb{Z}^d\right) \subset \mathrm{cvxh}\,(Y) = Y$. It follows that $\mu \leqslant x \cdot N < \mu + \omega$.

Let $(z_i)_{i=1\ldots 2^k}$ be the vertices of $\bar{c}$ ordered from lowest to highest scalar product with $N$. There exists $j \in \{1, 2^k\}$ such that $\forall i \in \{1, j\}, z_i \cdot N \leqslant x \cdot N$ and $\forall i \in \{j+1, 2^k\}, x \cdot N < z_i \cdot N$. Should no $z_i$ belong to $Y$, since $x$ belongs to $Y$, we have :

$$\forall i \in \{1, j\}, z_i \cdot N < \mu, \quad \text{and} \quad \forall i \in \{j+1, 2^k\}, \mu + \omega \leqslant z_i.$$

It follows that $(z_{j+1} - z_j) \cdot N > \omega$. Now the $(z_i)$ are vertices of a hypercube of side one, and ordered according to their projection along vector $N$. It is easy to see that any $(z_{i+1} - z_i) \cdot N \leqslant \max_{i=1\ldots d}(|N_i|)$ (for instance by constructing a subsequence moving along axes in order, it achieves the bound, and then the actual sequence $(z_i)$ is much finer than this one), so it holds in particular for $i = j$. To sum up, should no $z_i$ belongs to $Y$, then $\omega < \max_{i=1\ldots d}(|N_i|)$.

Otherwise, for $\omega \geqslant \max_{i=1\ldots d}(|N_i|)$, either $z_j$ or $z_{j+1}$ or both belong to $Y$, and thus to $Q$. It follows that $c \in \bar{\mathscr{C}}^d_k[Q]$, which concludes. □

The classic 2D and 3D machinery of digital straight lines and planes, very rich in results and applications, thus belongs to the fully convex framework.

*Tangent subsets.* Tangency as defined below induces strong geometric properties. In $\mathbb{Z}^d$, two cotangent points of a digital surface $X$ delineates a straight segment that stays "in" the surface, i.e. a tangent vector. A simplex made of $d$ cotangent points to $X$ lies "in" the surface, so defines a (local) tangent plane.

**Definition 2** *The digital set* $A \subset X \subset \mathbb{Z}^d$ *is said to be* $k$-tangent to $X$ *for* $0 \leqslant k \leqslant d$ *whenever* $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathscr{C}}^d_k[X]$. *It is* tangent to $X$ *if the relation holds for all such* $k$. *Elements of* $A$ *are called* cotangent.

As always in digital geometry, objects that are too local are not precise enough. We are thus more interested in "big" tangent subsets: a set $A$, tangent to $X$, that is not included in any other tangent set to $X$ is said *maximal in* $X$. In 2D, they give rise to the classical tangential cover of a contour [FT99]:

**Theorem 8** *When* $d = 2$, *if* $C$ *is a simple 2-connected digital contour (i.e. 8-connected in Rosenfeld's terminology), then the fully convex subsets of* $C$ *that are maximal and tangent are the classical maximal naive digital straight segments.*

*Proof.* Let $M$ be a fully convex subset of $C$, both maximal and tangent. Full convexity implies that $M$ is 2-connected (Theorem 2). Full convexity implies $H$-convexity, and connected convex subsets of simple 2-connected contours are digital straight segments. Maximality implies that they are inextensible. The converse is obvious from Theorem 7. □

Maximal fully convex tangent subsets to $X$ seem a good candidate for a sound definition of maximal digital plane segments. Our definition avoids the classical problem of 3D planes that are not tangent to the surface (as noted in [CL11]) as well as the many heuristics to cope with this issue [SDC04,PDR09].

*An elementary linear reconstruction algorithm for digital sets.* Let $X$ be a finite subset of $\mathbb{Z}^d$ and let $\mathrm{Del}\,(X)$ be its Delaunay complex.

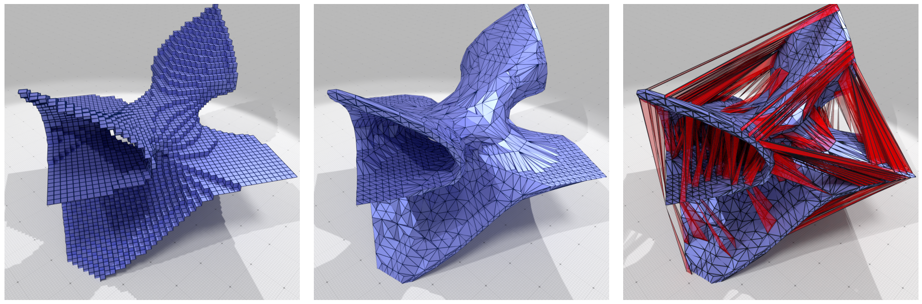
**Definition 3** *The* tangent Delaunay complex $\mathrm{Del}_{\mathrm{T}}\,(X)$ *to* $X$ *is the complex made of the cells* $\tau$ *of* $\mathrm{Del}\,(X)$ *such that the vertices of* $\tau$ *are tangent to* $X$.

The boundary of tangent Delaunay complexes is a linear reconstruction of voxel shapes, and is the boundary of the convex hull for fully convex shapes:

**Lemma 8** *If* $X$ *fully convex,* $\mathrm{Del}_{\mathrm{T}}\,(X) = \mathrm{Del}\,(X)$. *So* $\partial \left\|\mathrm{Del}_{\mathrm{T}}\,(X)\right\| = \partial \mathrm{cvxh}\,(X)$.

*Proof.* Let $\tau \in \mathrm{Del}\,(X)$. Let $A \subset X$ be the vertices of cell $\tau$ (i.e. $\tau = \mathrm{cvxh}\,(A)$). For all $k$, $0 \leqslant k \leqslant d$, $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)] = \bar{\mathscr{C}}^d_k[X]$ (by full convexity). This shows that $A$ is tangent to $X$. □

From Theorem 7, the tangent Delaunay complex of an arithmetic plane is the boundary of its convex hull, hence its facets have exactly the same normal as the arithmetic plane. Furthermore, since the tangent Delaunay complex is built with local geometric considerations, it is able to capture the geometry of local pieces of planes on digital objects, and nicely reconstructs convex and concave parts (see Figure 3). It is also a tight and reversible reconstruction of $X$:



Input digital shape $X$ Our reconstruction $\mathrm{Del}_{\mathrm{T}}(X)$ Bad simplices of $\mathrm{Del}(X)$

**Fig. 3.** The tangent Delaunay complex $\mathrm{Del}_{\mathrm{T}}(X)$ (middle) is a piecewise linear reconstruction of the input digital surface $X$ (left). On (right), we display in red simplices of $\mathrm{Del}(X)$ which avoid lattice points of $\mathbb{Z}^3 \setminus X$ but are not tangent to $X$. Tangency thus eliminates the "sliver" simplices of $\mathrm{Del}(X)$ that are not geometrically informative. ($\#(X) = 6013$, $\mathrm{Del}(X)$ has 36361 tetrahedra, $\mathrm{Del}_{\mathrm{T}}(X)$ has 25745 tetrahedra, computing $\mathrm{Del}(X)$ takes 70ms, computing $\mathrm{Del}_{\mathrm{T}}(X)$ takes 773ms.)

**Theorem 9** *The body of* $\mathrm{Del}_{\mathrm{T}}(X)$ *is at Hausdorff* $L_\infty$*-distance* 1 *to* $X$. $\mathrm{Del}_{\mathrm{T}}(X)$ *is a reversible polyhedrization, i.e.* $\|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d = X$.

*Proof.* First the distance of any point of $X$ to $\|\mathrm{Del}_{\mathrm{T}}(X)\|$ is zero, since any point of $X$ is tangent to $X$ and is also a 0-cell of $\mathrm{Del}(X)$. Second, any point $y$ of $\|\mathrm{Del}_{\mathrm{T}}(X)\|$ belongs to a simplex $\tau \in \mathrm{Del}_{\mathrm{T}}(X)$. Let $A$ be the vertices of $\tau$. By tangency, for any $k$, $0 \leqslant k \leqslant d$, $\bar{\mathscr{C}}^d_k[\tau] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}(A)] \subset \bar{\mathscr{C}}^d_k[X]$. Hence the point $y$ belongs to some cell $\sigma$ of $\bar{\mathscr{C}}^d_k[X]$ and is at most at $L_\infty$-distance 1 of any one of its $X$-corner. Finally by tangency, $\|\mathrm{Del}_{\mathrm{T}}(X)\| \subset \bar{\mathscr{C}}^d[X]$, so $\|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d \subset \bar{\mathscr{C}}^d[X] \cap \mathbb{Z}^d = X$. $X \subset \|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d$ is obvious. □

## 6 Conclusion and perspectives

We have proposed an original definition for digital convexity in arbitrary dimension, called full convexity, which possesses topological and geometric properties that are more akin to continuous convexity. We exhibited an algorithm to check full convexity, which relies on standard algorithms. We illustrated the potential of full convexity for addressing classic discrete geometry problems like building a tangential cover or reconstructing a reversible first-order polygonal surface approximation. We believe that full convexity opens the path to $d$-dimensional digital shape geometry analysis. This work opens many perspectives. On a fundamental level, we work on a variant of full convexity that keeps the intersection property of continuous convexity. We also wish to improve the convexity check algorithm, especially in 3D, for instance when $\mathrm{cvxh}(X)$ has few facets.[1] Finally we wish to explore the properties of a tangential cover made of the maximal fully convex tangent subsets that are included in some arithmetic plane.

[1] See for instance the full convexity implementation in DGtal.

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## A Proofs of some properties

**Lemma 9** *For any $X \subset \mathbb{Z}^d$, for any $\alpha \subset I^d$, $(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = U_\alpha(X)$.*

*Proof.* Let $\alpha = \{i_1, \ldots, i_k\} \subset I^d$, non empty. Note that $\mathcal{S}_\alpha = \bigoplus_{j=1}^k \mathcal{S}_j$ and that it does not depend on the chosen order. For conciseness, we write $X^\gamma := X \oplus \mathcal{S}_\gamma$ for any subset $\gamma$ of $I^d$. Let $\beta = \{i_1, \ldots, i_{k-1}\}$ and let us first show that:

$$(X^\alpha) \cap \mathbb{Z}^d = (X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d). \tag{6}$$

$\boxed{\supset}$ $(X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d) = (X^\beta \cup \mathbf{e}_{i_k}(X^\beta)) \cap \mathbb{Z}^d \subset (X^\beta \oplus \mathcal{S}_{i_k}) \cap \mathbb{Z}^d = (X^\alpha) \cap \mathbb{Z}^d$.
$\boxed{\subset}$ Let $z \in X^\alpha \cap \mathbb{Z}^d$. We can write $z$ as $z = x + t_{i_1}\mathbf{e}_{i_1} + \cdots + t_{i_k}\mathbf{e}_{i_k}$, with $x \in X$ and every $t_{i_j} \in [0, 1]$. More precisely, since $z \in \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ and $e_{i_j}$ is a unit vector, every $t_{i_j} \in \{0, 1\}$. Clearly $z' = z + \sum_{j=1}^{k-1} t_{i_j}\mathbf{e}_{i_j}$ belongs to $X^\beta \cap \mathbb{Z}^d$. If $t_{i_k} = 0$ then $z' = z$ and we are done. Otherwise, $t_{i_k} = 1$ then $z' = z + \mathbf{e}_{i_k}$, which belongs to $\mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d)$.

We prove the lemma by induction on the cardinal of $\alpha$. For $\alpha = \emptyset$, $(X \oplus \mathcal{S}_\emptyset) \cap \mathbb{Z}^d = X = U_\emptyset(X)$. Assume the lemma is true for any $\beta$ of cardinal $k - 1 \geq 0$, and let us show it for $\alpha = \beta \cup \{i\}$.

$$\begin{aligned}
(X^\alpha) \cap \mathbb{Z}^d &= (X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_i(X^\beta \cap \mathbb{Z}^d) && \text{(Using (6))} \\
&= U_\beta(X) \cup \mathbf{e}_i(U_\beta(X)) && \text{(Induction)} \\
&= U_\alpha(X) && \text{(Definition).}
\end{aligned}$$

□

**Lemma 10** *For any $X \subset \mathbb{Z}^d$, for any $\alpha \subset I^d$,* $\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha = \mathrm{cvxh}\,(U_\alpha(X))$.

*Proof.* We prove it by induction on the cardinal $k$ of $\alpha$. It holds obviously for $k = 0$. Otherwise let $\alpha = \beta \cup \{i\}$ of cardinal $k$.
$\boxed{\supset}$ We have $U_\alpha(X) = U_\beta(X) \cup \mathbf{e}_i(U_\beta(X)) \subset U_\beta(X) \oplus \mathcal{S}_i$. Since convex hull is increasing, $\mathrm{cvxh}\,(U_\alpha(X)) \subset \mathrm{cvxh}\,(U_\beta(X) \oplus \mathcal{S}_i)$ holds. But convex hull commutes with Minkowski sum, so $\mathrm{cvxh}\,(U_\beta(X) \oplus \mathcal{S}_i) = \mathrm{cvxh}\,(U_\beta(X)) \oplus \mathcal{S}_i = \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\beta \oplus \mathcal{S}_i$ by induction hypothesis. We conclude with $\mathcal{S}_\beta \oplus \mathcal{S}_i = \mathcal{S}_\alpha$.
$\boxed{\subset}$ Let $y \in \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha = \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\beta \oplus \mathcal{S}_i = \mathrm{cvxh}\,(U_\beta(X)) \oplus \mathcal{S}_i$. Denoting by $z_j$, $j \in B$ the points of $U_\beta(X)$, it follows that $y$ can be written as a convex linear combination of these points plus a point of $\mathcal{S}_i$, i.e. $y = (\sum_{j \in B} \mu_j z_j) + t\mathbf{e}_i$, $\sum_{j \in B} \mu_j = 1$, $\forall j \in B, \mu_j \geq 0$ and $t \in [0, 1]$. All following sums are taken over $j \in B$. Since $\sum \mu_j = 1$, we rewrite $y$ as

$$\begin{aligned}
y &= \left(\sum \mu_j z_j\right) + t\left(\sum \mu_j\right)\mathbf{e}_i \\
&= \sum (1-t)\mu_j z_j + t\mu_j z_j + t\mu_j \mathbf{e}_i \\
&= \left(\sum (1-t)\mu_j z_j\right) + \left(\sum t\mu_j(z_j + \mathbf{e}_i)\right) \\
&\in \mathrm{cvxh}\,(U_\beta(X) \cup \mathbf{e}_i(U_\beta(X))),
\end{aligned}$$

which shows that $y \in \mathrm{cvxh}\,(U_\alpha(X))$. □