Convergent geometric estimators with digital volume and surface integrals

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DGCI, Nantes

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Laboratoire de Mathématiques
Université de Savoie

UMR 5127

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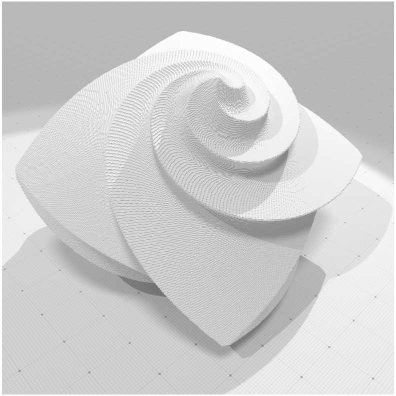
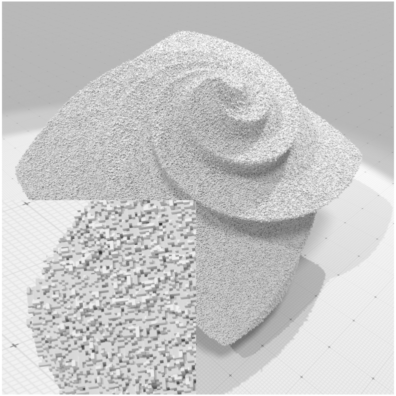
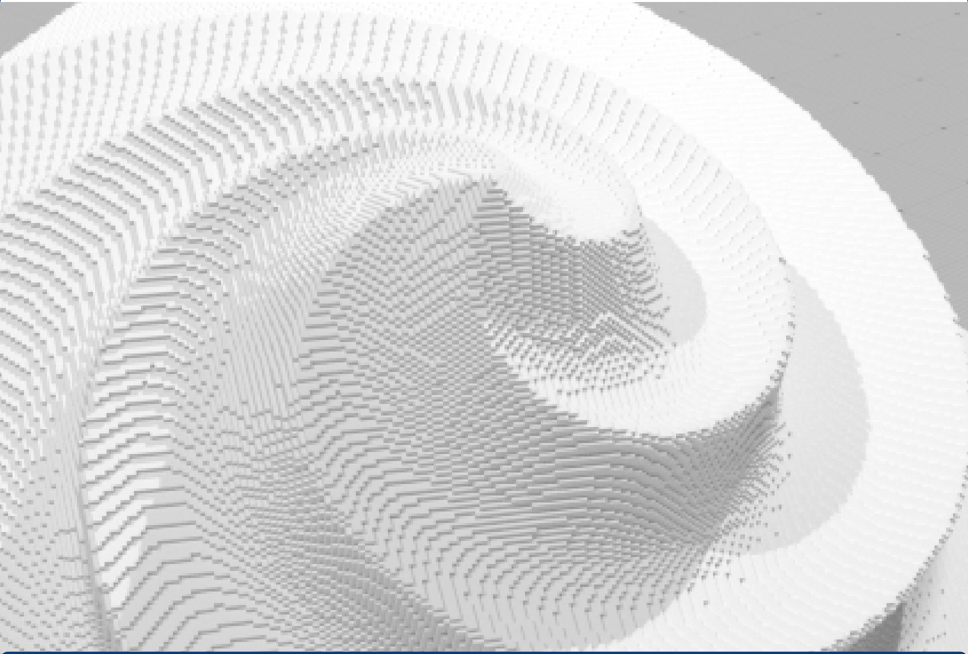
Geometry of digital shapes



set of voxels $\subset \mathbb{Z}^d$



possibly damaged data



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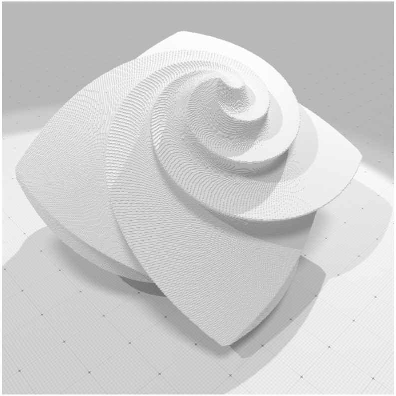

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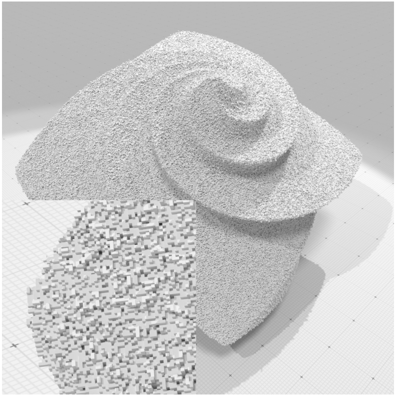

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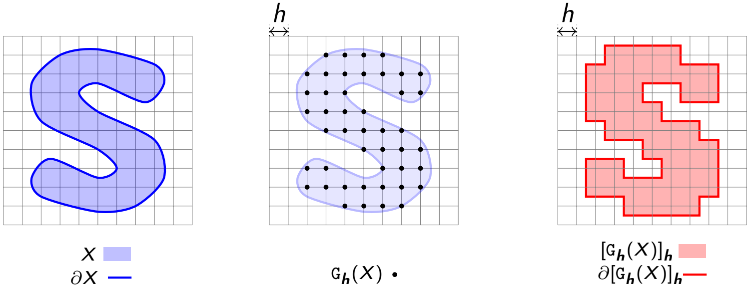
Underlying Euclidean shape $\subset \mathbb{R}^d \Rightarrow$ infer its differential geometry !

What are properties kept by digitization ?

Shape in $\mathbb{R}^d$ | Digitized shape in $h \cdot \mathbb{Z}^d$ | Digitized shape in $\mathbb{R}^d$



- digitization : any function that maps a subset $X \subset \mathbb{R}^d$ to a subset of $h \cdot \mathbb{Z}^d$, h is the sampling gridstep.
- Question: what are topological and geometric properties kept by digitization ?

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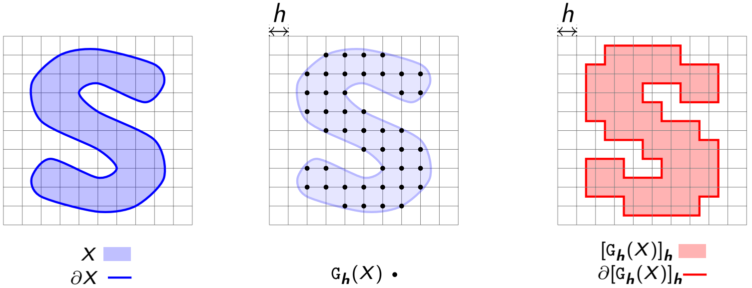

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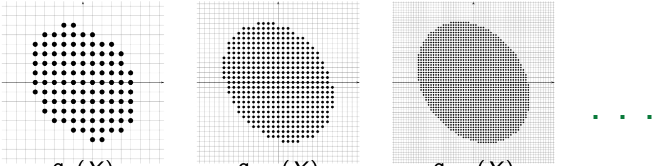
Almost nothing is "kept", a better word is "can be infered".

Multigrid convergence

For a fixed sampling grid h, **nothing** can be said !
A digital point may be anything... a disk, a cube, etc.

Multigrid convergence



$\mathtt{G}_h(X)$ $\mathtt{G}_{h/2}(X)$ $\mathtt{G}_{h/4}(X)$ $\ldots$

To get properties on digitized shapes, you need:

- specific families $\mathbb{X}$ of shapes in $\mathbb{R}^d$: compact, smooth, convex, etc
- specific digitization processes $\mathtt{D}$: Gauss, Jordan, etc
- a fine enough gridstep $h < h_0$

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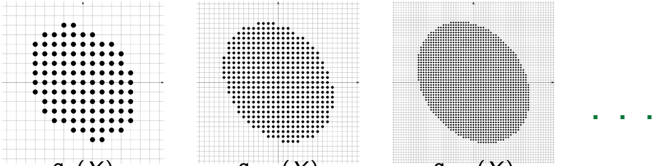

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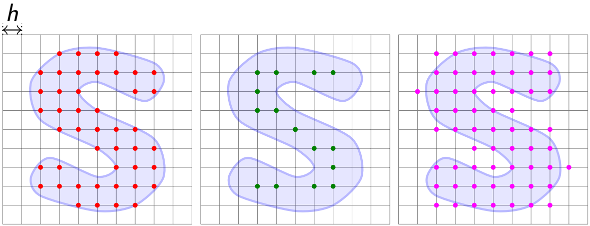
Definition (multigrid convergence [Pavlidis 1982, Serra 1982])

Let $X \in \mathbb{X}$. If E is some geometric quantity on X, then some discrete geometric estimator $\hat{E}$ converges towards E, iff

$$\exists h_0, \forall 0 < h < h_0, \|E(X) - \hat{E}(\mathtt{D}_h(X), h)\| \leq \tau(h), \quad \text{with} \lim_{h \to 0} \tau(h) = 0.$$

Digitizations

Gauss digitization $\mathtt{G}_h(X)$	Inner Jordan $\mathtt{J}_h^-(X)$	Outer Jordan $\mathtt{J}_h^+(X)$
$(h\mathbb{Z})^d \cap X$	$\{\mathbf{z} \in (h\mathbb{Z})^d, Q_h(\mathbf{z}) \subset X\}$	$\{\mathbf{z} \in \mathbb{Z}^d, Q_h(\mathbf{z}) \cap X \neq \emptyset\}$



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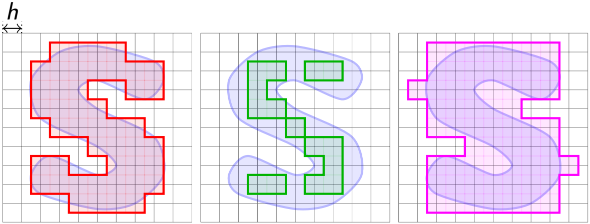

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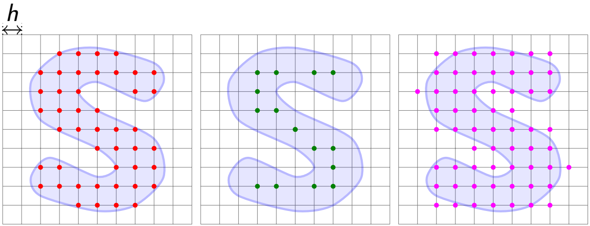

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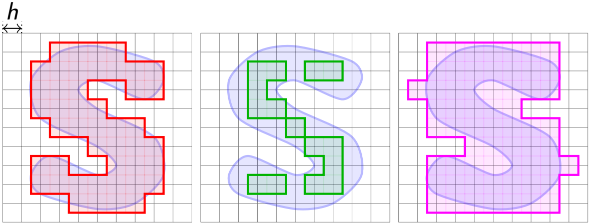

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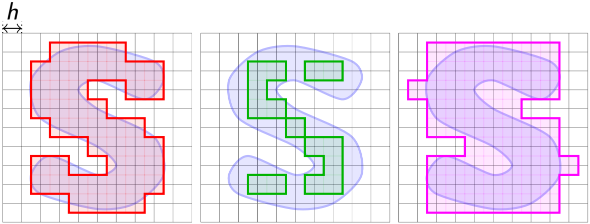

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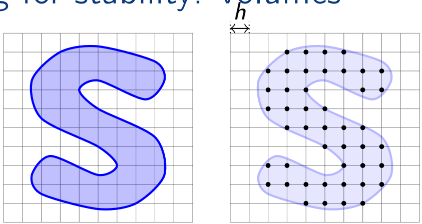
- $\partial X \subset [\mathtt{J}_h^+(X)]_h \setminus \mathrm{Int}[\mathtt{J}_h^-(X)]_h$
- *We know where lies* ∂X

Looking for stability: volumes



$X \subset \mathbb{R}^d$ $\mathtt{D}_h(X)$

- For $X \subset \mathbb{R}^d$, $\mathrm{Vol}(X) := \int \cdots \int_X 1 dx_1 \ldots dx_d$
- For $Z \subset (h\mathbb{Z})^d$, $\widehat{\mathrm{Vol}}(Z,h) := \sum_Z 1 h^d$

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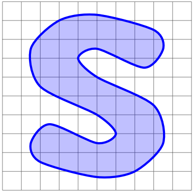

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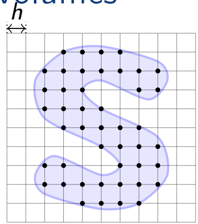
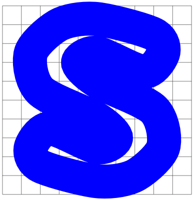


$\mathtt{D}_h(X)$



$\sqrt{d}h$-offset de ∂X

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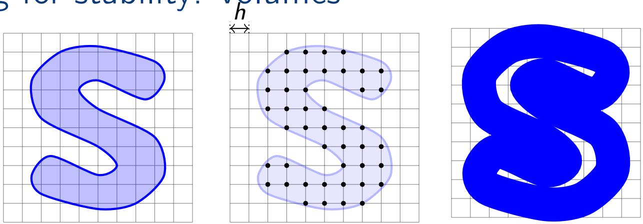
Theorem

Let X be a compact domain of $\mathbb{R}^d$. Let $\mathtt{D}$ be any digitization process such that $\mathtt{J}_h^-(X) \subset \mathtt{D}_h(X) \subset \mathtt{J}_h^+(X)$. Digital and continuous volumes are related as follows:

$$\left| \mathrm{Vol}(X) - \widehat{\mathrm{Vol}}(\mathtt{D}_h(X), h) \right| \leq \mathrm{Vol}(\partial X^{\sqrt{d}h}). \quad (1)$$

Looking for stability: volumes



$X \subset \mathbb{R}^d$ $\quad$ $\mathtt{D}_h(X)$ $\quad$ $\sqrt{d}h$-offset de ∂X

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Multigrid convergence only when ∂X is rectifiable !

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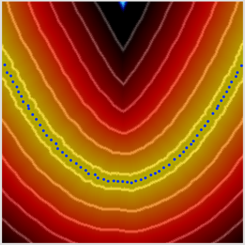
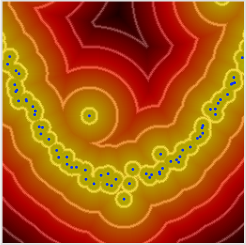
Looking for stability: distance



Hausdorff perturbation



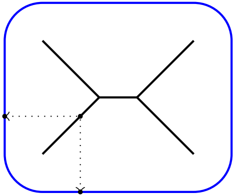
outliers

- distance d_K to a compact set K, projection ξ_K onto K
- distance d_K is Hausdorff stable

$\Rightarrow$ if $B_h := \partial[\mathtt{D}_h(X)]_h$ is close to ∂X, then d_{B_h} is close to $d_{\partial X}$.

- Stability of distance used in computational topology and geometry
  - homotopy stability of offsets [Chazal, Lieutier 2008, Niyogi et al. 2008]
  - normal [Amenta et al. 1998] and covariance measure estimation [Mérigot et al. 2011]

Looking for stability: medial axis, reach, (par)-regularity



medial axis, reach

- medial axis $MA(\partial X)$ = points with more than one closest point on ∂X
- reach $\mathrm{reach}(\partial X)$ = infimum of distance ∂X to $MA(\partial X)$ [Federer 1959]

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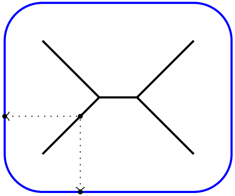

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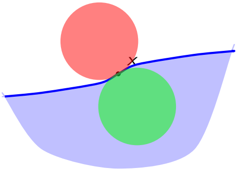

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- R-regularity [Pavlidis 1982, Serra 1982], par(R)-regularity [Latecki et al. 1998] = inside and outside osculating balls of radius R for each $\mathbf{x} \in \partial X$

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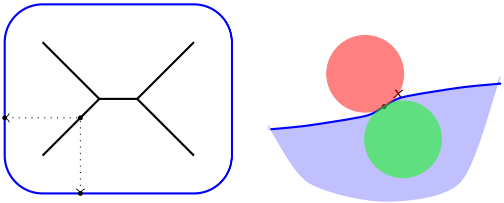

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Lemma

Let X be a d-dimensional compact domain of $\mathbb{R}^d$. Then

$$\mathrm{reach}(\partial X) \geq R \quad \Leftrightarrow \quad \forall R' < R,\ X \text{ is } par(R')\text{-regular}$$

Multigrid convergence of $\widehat{\mathrm{Vol}}$ toward Vol

- volume of convex sets by $\widehat{\mathrm{Vol}}$ [Gauss, Dirichlet]. $\tau(h) = O(h)$.
- better bounds for C^3-smooth strictly convex sets [Huxley 1990]
- volume under monotonic functions (see [Krätzle 1988, Krätzle, Nowak 1991]). $\tau(h) = O(h)$.

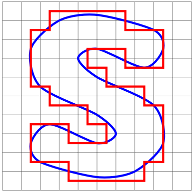
Theorem

Let X be a compact domain of $\mathbb{R}^d$, with $\mathrm{reach}(\partial X) \geq \rho$. Let $h < \frac{\rho}{\sqrt{d}}$. Let $\mathtt{D}$ be any digitization such that $\mathtt{J}_h^-(X) \subset \mathtt{D}_h(X) \subset \mathtt{J}_h^+(X)$. Digital and continuous volumes follows

$$\left|\mathrm{Vol}(X) - \widehat{\mathrm{Vol}}(\mathtt{D}_h(X), h)\right| \leq 2^{d+1}\sqrt{d}\,\mathrm{Area}(\partial X)h\,. \quad (2)$$

NB: the reach bounds the volume of an offset to ∂X.

Multigrid convergence of local geometric estimators



- slight difficulty to define it: must relate ∂X with $\partial_h X$

Definition (local multigrid convergence)

Let $X \in \mathbb{X}$. If E is some local geometric quantity on X, then some local discrete geometric estimator $\hat{E}$ converges towards E, iff

$$\exists h_0, \forall 0 < h < h_0, \forall \mathbf{x} \in \partial X, \forall \hat{\mathbf{x}} \in \partial_h X, \|\mathbf{x} - \hat{\mathbf{x}}\| \leq \Theta(h)$$

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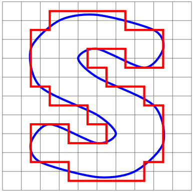

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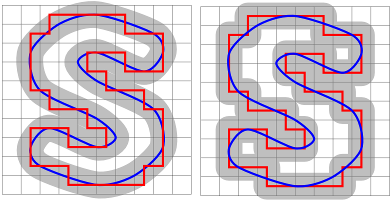
- or functional approach [Esbelin, Malgouyres, Provot, Gérard,. . .]
- or stability with measures [Chazal,Cohen-Steiner,Lieutier,Mérigot,Thibert,. . .]

Hausdorff dist. between continuous and digitized boundary



Theorem (boundary of Gauss digitized shape)

Let X be a compact domain of $\mathbb{R}^d$ with $\mathrm{reach}(\partial X) \geq R$. Then $\forall 0 < h < 2R/\sqrt{d}$, the Hausdorff distance between sets ∂X and $\partial_h X := \partial[\mathtt{G}_h(X)]_h$ is less than $\sqrt{d}h/2$. More precisely:

$$\forall \mathbf{x} \in \partial X, \exists \mathbf{y} \in \partial_h X, \|\mathbf{x} - \mathbf{y}\| \leq \frac{\sqrt{d}}{2} h \quad (\textit{with } \xi_{\partial X}(\mathbf{y}) = \mathbf{x}), \tag{3}$$

$$\forall \mathbf{y} \in \partial_h X, \|\mathbf{y} - \xi_{\partial X}(\mathbf{y})\| \leq \frac{\sqrt{d}}{2} h. \tag{4}$$

NB: Tight bound. Proof uses osculating balls and C^1-smoothness of ∂X.

Objectives

- multigrid convergence of curvature tensor with (digital) integral invariants

  (With D. Coeurjolly, J. Levallois)

- multigrid convergence of normals with (digital) Voronoi covariance measure even on noisy data

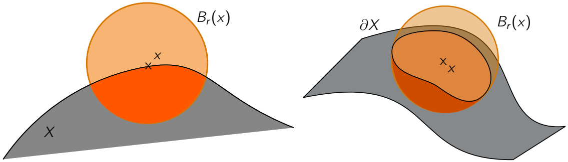
  (With L. Cuel, Q. Mérigot, B. Thibert)

- multigrid convergence of (digital) surface integrals

  (With B. Thibert)

Convergent geometric estimators with digital volume and surface integrals

Integral invariants



Let $X \subset \mathbb{R}^3$ smooth enough, $\mathbf{x} \in \partial X$. Let $r \in \mathbb{R}, r > 0$, radius of ball $B_r(\mathbf{x})$

$$V_r(X, \mathbf{x}) := \int_{B_r(\mathbf{x}) \cap X} dp.$$

volume

$$J_r(X, \mathbf{x}) := \int_{B_r(\mathbf{x}) \cap X} (p - \bar{p}) \otimes (p - \bar{p})^T dp.$$

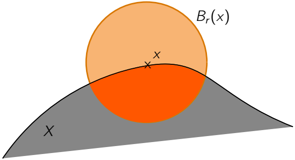
covariance matrix

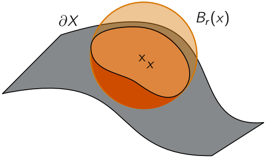
e.g. see [Pottmann et al 2007]

Mean curvature $H(X, \mathbf{x})$ follows:

$$H(X, \mathbf{x}) = \frac{8}{3r} - \frac{4 V_r(X, \mathbf{x})}{\pi r^4} + O(r)$$

Integral invariants





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covariance matrix

[Pottmann et al 2007], Theorem 2

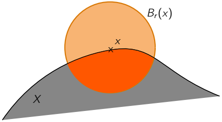
Principal curvatures $\kappa_1(X, \mathbf{x}), \kappa_2(X, \mathbf{x})$ follows

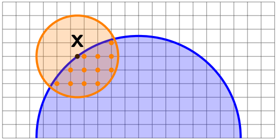
$$\kappa^1(X, \mathbf{x}) = \frac{6}{\pi r^6}(\lambda_2 - 3\lambda_1) + \frac{8}{5r} + O(r)$$

$$\kappa^2(X, \mathbf{x}) = \frac{6}{\pi r^6}(\lambda_1 - 3\lambda_2) + \frac{8}{5r} + O(r)$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$ eigenvalues of $J_r(X, \mathbf{x})$.

Digital integral invariants





princ. curv. $\hat{\kappa}^i_r$, curv. dir. $\hat{\mathbf{v}}^i_r$, normal $\hat{\mathbf{n}}_r$ estimators

Let $Z \subset (h\mathbb{Z})^3$ be a digital shape, $\mathbf{y}$ any point of $\mathbb{R}^3$.

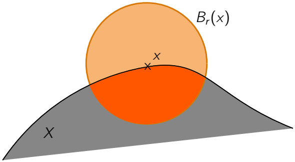
$$\hat{\kappa}^1_r(Z,\mathbf{y},h) = \frac{6}{\pi r^6}(\hat{\lambda}_2 - 3\hat{\lambda}_1) + \frac{8}{5r}, \quad \hat{\mathbf{v}}^1_r(Z,\mathbf{y},h) = \hat{\nu}_1, \quad \hat{\mathbf{n}}_r(Z,\mathbf{y},h) = \hat{\nu}_3,$$

$$\hat{\kappa}^2_r(Z,\mathbf{y},h) = \frac{6}{\pi r^6}(\hat{\lambda}_1 - 3\hat{\lambda}_2) + \frac{8}{5r}, \quad \hat{\mathbf{v}}^2_r(Z,\mathbf{y},h) = \hat{\nu}_2,$$

with $\hat{\lambda}_1$, $\hat{\lambda}_2$ two first eigenvalues of $\hat{J}_r(Z,\mathbf{y},h)$ (dig. cov. matrix of $Z \cap B_r(\mathbf{y})$), and $\hat{\nu}_1$, $\hat{\nu}_2$, $\hat{\nu}_3$ corresp. eigenvectors.

1. Convergence of covariance matrix: $X \cap B_r(\mathbf{x})$ has not positive reach,
2. Positionning error: $\hat{\mathbf{x}} \in \partial_h X$ is known, not $\mathbf{x} \in \partial X$
3. Stability of eigenvalues of covariance matrix,
4. Approximation error in previous equations: r must be small.

Digital covariance matrix from digital moments



(p,q,s)-moments of $Y \subset \mathbb{R}^3$

for non negative integers p, q and s

$$m_{p,q,s}(Y) := \iiint_Y x^p y^q z^s \, dxdydz$$

Covariance matrix of $A := B_r(\mathbf{x}) \cap X$

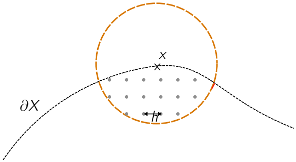
$$J_r(X,x) = \begin{bmatrix} m_{2,0,0}(A) & m_{1,1,0}(A) & m_{1,0,1}(A) \\ m_{1,1,0}(A) & m_{0,2,0}(A) & m_{0,1,1}(A) \\ m_{1,0,1}(A) & m_{0,1,1}(A) & m_{0,0,2}(A) \end{bmatrix} - \frac{1}{m_{0,0,0}(A)} \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix} \otimes \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix}^T$$

Digital covariance matrix from digital moments

digital (p,q,s)-moments of $Z \subset (h\mathbb{Z})^3$

for non negative integers p, q and s

$$\hat{m}_{p,q,s}(Z,h) := h^3 \sum_{(i,j,k)\in Z} i^p j^q k^s$$



digital covariance matrix of $A' := B_r(\mathbf{y}) \cap Z$

$$\hat{J}_r(Z,\mathbf{y},h) = \begin{bmatrix} \hat{m}_{2,0,0}(A',h) & \hat{m}_{1,1,0}(A',h) & \hat{m}_{1,0,1}(A',h) \\ \hat{m}_{1,1,0}(A',h) & \hat{m}_{0,2,0}(A',h) & \hat{m}_{0,1,1}(A',h) \\ \hat{m}_{1,0,1}(A',h) & \hat{m}_{0,1,1}(A',h) & \hat{m}_{0,0,2}(A',h) \end{bmatrix} - \frac{1}{\hat{m}_{0,0,0}(A',h)} \begin{bmatrix} \hat{m}_{1,0,0}(A',h) \\ \hat{m}_{0,1,0}(A',h) \\ \hat{m}_{0,0,1}(A',h) \end{bmatrix} \otimes \begin{bmatrix} \hat{m}_{1,0,0}(A',h) \\ \hat{m}_{0,1,0}(A',h) \\ \hat{m}_{0,0,1}(A',h) \end{bmatrix}^T$$

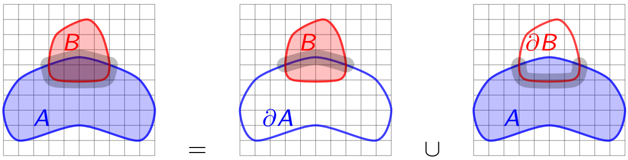
1. Convergence of covariance matrix

Theorem

Let X be a compact domain of $\mathbb{R}^d$, with $\mathrm{reach}(\partial X) \geq \rho$. Let $\mathtt{D}$ be any digitization process such that $\mathtt{J}_h^-(X) \subset \mathtt{D}_h(X) \subset \mathtt{J}_h^+(X)$. Let $\mathbf{x} \in \mathbb{R}^d$. Let radius r and gridstep h be such that $0 < h \leq \frac{r}{\sqrt{2d}}$ and $0 < 2r \leq \rho$. Then digital moments within a ball $B_r(\mathbf{x})$ are multigrid convergent toward continuous moments as follows

$$\begin{aligned} &|m_{p,q,s}(X \cap B_r(\mathbf{x})) - \hat{m}_{p,q,s}\left(\mathtt{D}_h(X \cap B_r(\mathbf{x})), h\right)| \\ &\qquad \leq K_1 r^2 (\|\mathbf{x}\|_\infty + 2r)^{p+q+s}\, h + \frac{\pi}{9} r^3\, h^4. \end{aligned} \quad (5)$$

NB: Proof uses convergence of digital volumes, and



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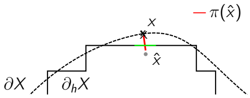
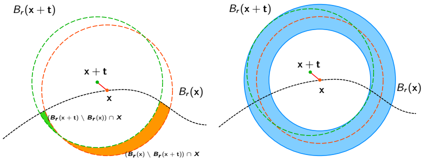
- Convergence of digital covariance matrix ($r \geq h$) for Gauss $\mathtt{G}$.

$$\forall \mathbf{x} \in \mathbb{R}^3, \| J_r(X \cap B_r(\mathbf{x}), \mathbf{x}) - \hat{J}_r(\mathtt{G}_h(X) \cap B_r(\mathbf{x}), \mathbf{x}, h) \| = O(r^4\, h).$$

  - invariance by translation of (dig. or cont.) covariance matrix
  - translation to origin by $-h[\frac{\mathbf{x}}{h}]$, $[\frac{\mathbf{x}}{h}]$ integer vector closer to $\frac{\mathbf{x}}{h}$.

2. Influence of position error ($\hat{x} \in \partial_h X$ known)





(a) (b)

Positioning error of moments with vector t

$$|m_{p,q,s}(B_r(\mathbf{x}+\mathbf{t}) \cap X) - m_{p,q,s}(B_r(\mathbf{x}) \cap X)| = \sum_{i=0}^{p+q+s} O(\|\mathbf{x}\|^i \|\mathbf{t}\| r^{2+p+q+s-i}).$$

Corollary, note that $\|\hat{\mathbf{x}} - \mathbf{x}\|_\infty \leq h$ thanks to shift to origin

$$\|\hat{J}_r(\mathtt{G}_h(X) \cap B_r(\mathbf{x}), \hat{\mathbf{x}}, h) - J_r(X \cap B_r(\mathbf{x}), \mathbf{x})\| = O(r^4\, h) + O(\|\mathbf{x} - \hat{\mathbf{x}}\| r^4).$$

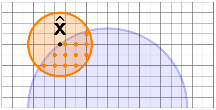
3. Stability of eigenvalues and eigenvectors

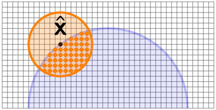
- if B and B' are two symmetric matrices, then errors on eigenvalues do not exceed errors on $\|B - B'\|$ (Lidskii-Weyl inequality)
- errors on eigenvectors do not exceed $\|B - B'\|$ divided by eigengap (Davis-Kahan $\sin\theta$ theorem)

Corollary

Eigenvalues of $\hat{J}_r$ and J_r are as close as matrix terms. Eigenvectors of $\hat{J}_r$ and J_r are as close as matrix terms, except around umbilic points.

Multigrid convergence of curvature tensor





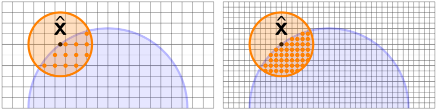
Theorem (Multigrid convergence of curvatures for Gauss digit.)

Let X be a compact domain of $\mathbb{R}^3$ with $\mathrm{reach}(\partial X) \geq \rho$ and C^3-continuity.

$$\exists h_0, \forall 0 < h < h_0, \forall \mathbf{x} \in \partial X, \forall \hat{\mathbf{x}} \in \partial[\mathtt{G}_h(X)]_h \text{ with } \|\hat{\mathbf{x}} - \mathbf{x}\|_\infty$$

$$|\hat{\kappa}^i_r(\mathtt{G}_h(X), \hat{\mathbf{x}}, h) - \kappa^i(X, \mathbf{x})| \leq \underbrace{O(r)}_{\text{Taylor expansion}} + \underbrace{O(h/r^2)}_{\text{dig. cov. mat.}} + \underbrace{O(h/r^2)}_{\text{positioning}}$$

Multigrid convergence of curvature tensor



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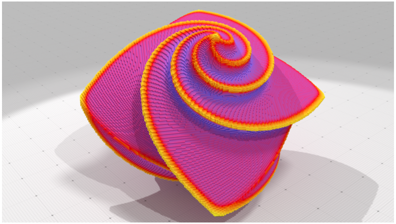
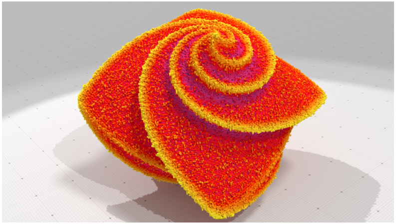
$$|\hat{\kappa}^i_r(\mathtt{G}_h(X), \hat{\mathbf{x}}, h) - \kappa^i(X, \mathbf{x})| \leq \underbrace{O(r)}_{\text{Taylor expansion}} + \underbrace{O(h/r^2)}_{\text{dig. cov. mat.}} + \underbrace{O(h/r^2)}_{\text{positioning}}$$

- balancing error terms give $r = kh^{\frac{1}{3}}$ with k some constant
- convergence of $\hat{\kappa}^i_r$ toward princ. curv. κ^i at speed $O(h^{\frac{1}{3}})$
- convergence of $\hat{\mathbf{v}}^i_r$ toward princ. dir. $\mathbf{v}^i$ at speed $\frac{1}{|\kappa^1 - \kappa^2|} O(h^{\frac{1}{3}})$
- convergence of $\hat{\mathbf{n}}_r$ toward normal $\mathbf{n}$ at speed $O(h^{\frac{2}{3}})$

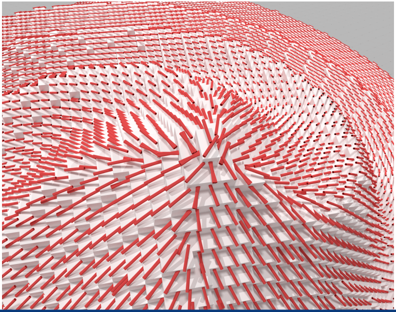
Experimental results

mean curvature $(\hat{\kappa}_r^1 + \hat{\kappa}_r^2)/2$

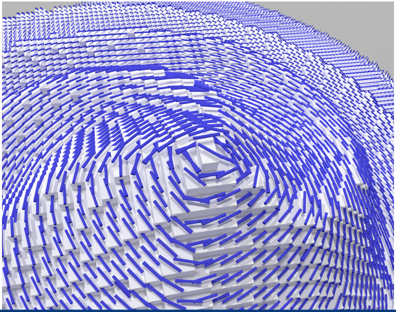




first princ. dir. $\hat{\mathbf{v}}_r^1$



second princ. dir. $\hat{\mathbf{v}}_r^2$



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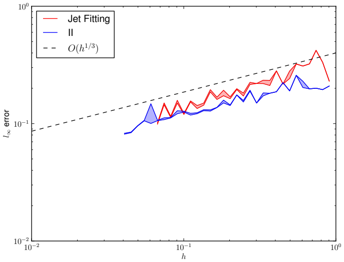
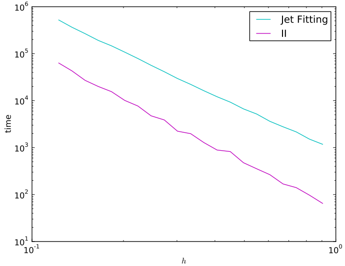


mean curvature l_∞-error



timings

- comprehensive experimental evaluation wrt existing estimators
- expected accuracy
- computationally efficient (in $O(N^{\frac{10}{3}})$ for digital image of size N^3)
- robust to noise **in practice**

Convergent geometric estimators with digital volume and surface integrals

Shapes versus digitized shapes

Curvatures with Digital Integral Invariants

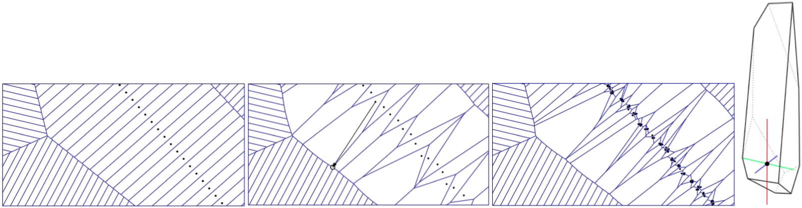
Digital Voronoi Covariance Measure

(With L. Cuel, Q. Mérigot, B. Thibert)

Digital surface integration

Conclusion

Origin of Voronoi Covariance Measure



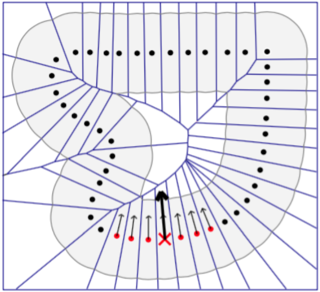
input: arbitrary cloud of points K

idea: detect normal vector using geometry of Voronoi cells

origin: poles [Amenta, Bern 1999], PCA per Voronoi cells [Alliez, Cohen-Steiner, Desbrun, Tong 2007]

vcm: integrate this information as a measure [Mérigot et al. 2011]

Voronoi Covariance Measure



χ is included in some ball

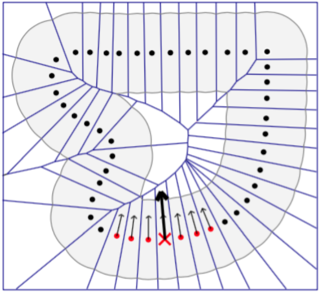
Definition (Voronoi Covariance Measure)

Let K a compact. Given any non-negative *probe function* χ, i.e. an integrable function on $\mathbb{R}^d$, we associate a positive semi-definite matrix defined by

$$\mathcal{V}_K^R(\chi) := \int_{K^R} \underbrace{\mathbf{N}_K(\mathbf{x}) \otimes \mathbf{N}_K(\mathbf{x})}_{\text{PCA of each Vor. cell}} \cdot \chi\left(\mathbf{x} - \mathbf{N}_K(\mathbf{x})\right) dx$$

where $\mathbf{N}_K(\mathbf{x}) := \mathbf{x} - \xi_K(\mathbf{x})$

Voronoi Covariance Measure



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where $\mathbf{N}_K(\mathbf{x}) := \mathbf{x} - \xi_K(\mathbf{x})$

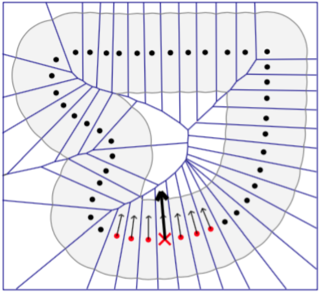
Theorem (Stability of VCM [Mérigot et al. 2011])

Let K, K' be two compacts, χ a probe function of support $\subset B_r(\mathbf{p})$

$$\|\mathcal{V}_K^R(\chi) - \mathcal{V}_{K'}^R(\chi)\| \leq O(d_H(K, K')),$$

where constant O depends on χ and r.

Voronoi Covariance Measure



χ is included in some ball

Definition (Voronoi Covariance Measure)

Let K a compact. Given any non-negative *probe function* χ, i.e. an integrable function on $\mathbb{R}^d$, we associate a positive semi-definite matrix defined by

$$\mathcal{V}_\delta^R(\chi) := \int_{\delta^R} \underbrace{\mathbf{N}_\delta(\mathbf{x}) \otimes \mathbf{N}_\delta(\mathbf{x})}_{\text{PCA of each Vor. cell}} \cdot \chi\left(\mathbf{x} - \mathbf{N}_\delta(\mathbf{x})\right) dx$$

where δ is distance to K and $\mathbf{N}_\delta(\mathbf{x}) := \frac{1}{2}\nabla\delta^2$

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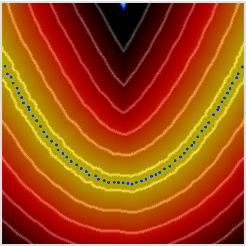
where constant O depends on χ and r.

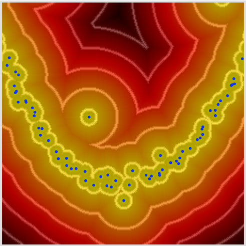
Generalized Voronoi Covariance Measure

Hausdorff pert.



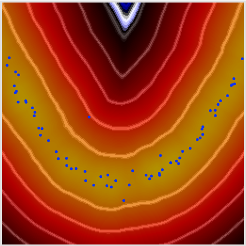
δ is distance to K

H + outliers



δ is distance to K

H + outliers



δ is k-distance to K

Definition (Generalized Voronoi Covariance Measure)

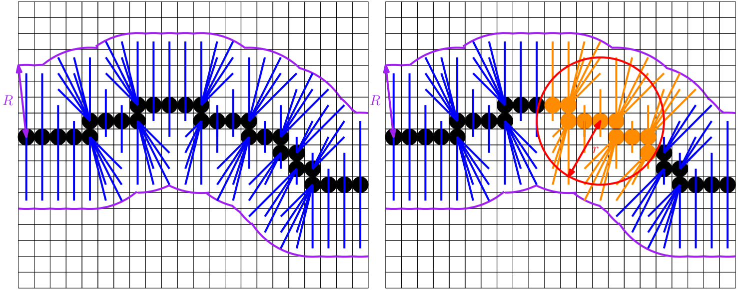
Let K a compact and χ a probe function

$$\mathcal{V}_{\delta}^{R}(\chi) := \int_{\delta^{R}} \mathbf{N}_{\delta}(\mathbf{x}) \otimes \mathbf{N}_{\delta}(\mathbf{x}) \cdot \chi\left(\mathbf{x} - \mathbf{N}_{\delta}(\mathbf{x})\right) dx$$

where δ is *distance-like*, $\mathbf{N}_{\delta}(\mathbf{x}) := \frac{1}{2}\nabla\delta^2$, $\delta^R := \delta^{-1}(]-\infty, R])$.

NB: robust to Hausdorff perturbations + outliers [Cuel, L., Mérigot, Thibert 2015].

Digital Voronoi Covariance Measure



Definition

Let $Z \subset (h\mathbb{Z})^d$ and $h > 0$. The *digital Voronoi Covariance Measure* of Z at step h and radius R associates to a probe function χ the matrix:

$$\hat{\mathcal{V}}^R_{Z,h}(\chi) := \sum_{\mathbf{z} \in Z^R} h^d \mathbf{N}_{d_Z}(\mathbf{z}) \otimes \mathbf{N}_{d_Z}(\mathbf{z}) \chi(\mathbf{z} - \mathbf{N}_{d_Z}(\mathbf{z})), \tag{6}$$

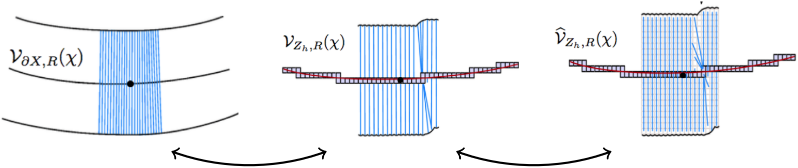
where d_Z is the distance to Z function, $\mathbf{N}_{d_Z} = \frac{1}{2}\nabla d_Z^2$.

Stability of Digital Voronoi Covariance Measure

Theorem

Let X compact domain of $\mathbb{Z}^3$ with C^2-smooth boundary and reach $\geq \rho$. Let $R < \rho/2$ and probe function χ with finite support diameter r. Let $Z = \partial[\mathtt{G}_h(X)]_h \cap h(\mathbb{Z} + \frac{1}{2})^3$. For h small enough, we have:

$$\|\mathcal{V}^R_{\partial X}(\chi) - \hat{\mathcal{V}}^R_{Z,h}(\chi)\|_{\mathrm{op}} \leq O\big(\mathrm{Lip}\chi(r^3 R^{\frac{5}{2}} + r^2 R^3 + r R^{\frac{9}{2}}) h^{\frac{1}{2}} + \|\chi\|_\infty [(r^3 R^{\frac{3}{2}} + r^2 R^2 + r R^{\frac{7}{2}}) h^{\frac{1}{2}} + r^2 R h]\big).$$



localization error (stability) — digitization error (digital volumes)

Stability of Digital Voronoi Covariance Measure

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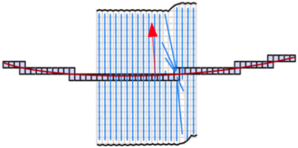
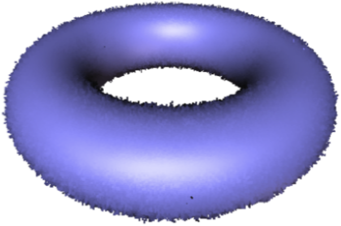
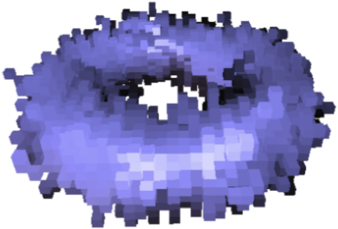
Corollary

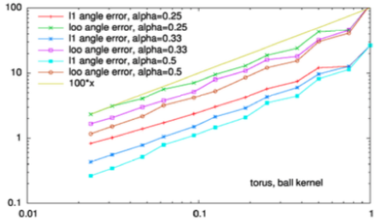
Let $\hat{\mathbf{n}}_{R,r}(Z, \mathbf{y}, h)$ be first eigenvector of $\hat{\mathcal{V}}^R_{Z,h}(\chi)$ with χ hat function of radius r centered on $\mathbf{y}$. For $R = \Theta(h^{\frac{1}{4}})$ and $r = \Theta(h^{\frac{1}{4}})$, then $\hat{\mathbf{n}}_{R,r}$ is multigrid convergent toward the true normal at speed $O(h^{\frac{1}{8}})$.

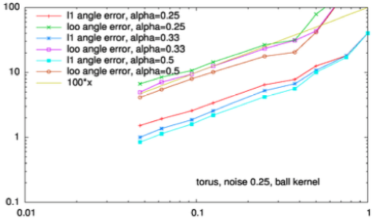


Experimental evaluation of DVCM: normals



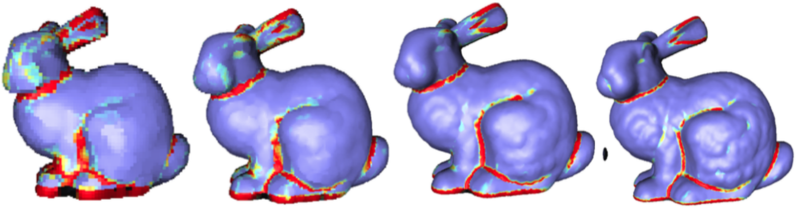






Much better in practice: speed in $O(h)$

Experimental evaluation of DVCM: features



$$R = r = 3h^{\frac{1}{2}}$$

Definition (Feature selection [Mérigot et al. 2011])

Let $Z \subset (h\mathbb{Z})^3$. Let T some angle threshold (here 0.1). Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ the three eigenvalues of $\hat{\mathcal{V}}^R_{Z,h}(\chi)$ with χ hat function of radius r centered on $\mathbf{y}$.

$$\mathbf{y} \text{ is a feature} \Leftrightarrow \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \geq T.$$

Convergent geometric estimators with digital volume and surface integrals

Shapes versus digitized shapes

Curvatures with Digital Integral Invariants

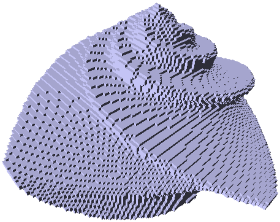
Digital Voronoi Covariance Measure

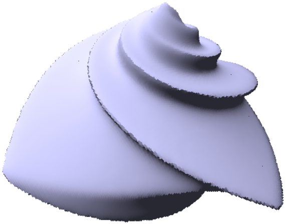
Digital surface integration

(With B. Thibert)

Conclusion

What about surface integrals ?

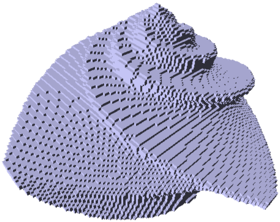


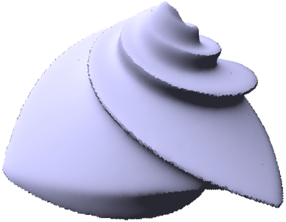


$\sum_{\partial[\mathtt{G}_{(}(h)X)]_h} f \, d\sigma \quad \xrightarrow{h \to 0} \quad \int_{\partial X} f \, ds$

- perimeter, area estimation, measures, local area estimation
- calculus over surface: geodesics, diffusion, PDE, etc.
- transform volume integral into surface integral for speed up

What about surface integrals ?



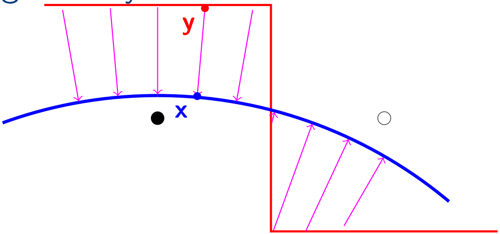


$\sum_{\partial[\mathtt{G}_(h)X)]_h} f\, {\color{red}d\sigma}$ $\xrightarrow{h\to 0}$ $\int_{\partial X} f\, ds$

- perimeter, area estimation, measures, local area estimation
- calculus over surface: geodesics, diffusion, PDE, etc.
- transform volume integral into surface integral for speed up

Naive approach with ${\color{red}d\sigma} = h^{d-1}$ does not work !

Differential geometry

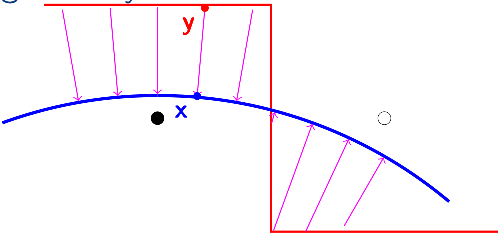


- a mapping g from $\partial[\mathtt{G}_h(X)]_h$ to ∂X, Jg Jacobian determinant

$$\int_{\partial X} f(\mathbf{x})d\mathbf{x} = \int_{\partial[\mathtt{G}_h(X)]_h} f(g(\mathbf{y}))Jg\,d\mathbf{y} \qquad \text{(substitution rule)}$$

- g should be bijective, differentiable a.e.

Differential geometry



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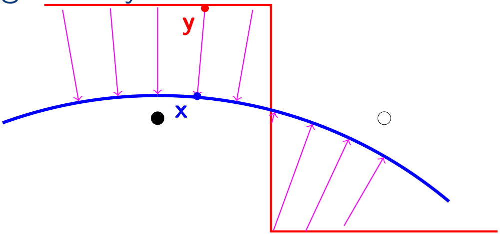
Differential geometry



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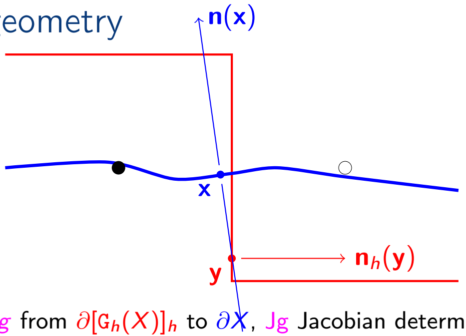
Differential geometry



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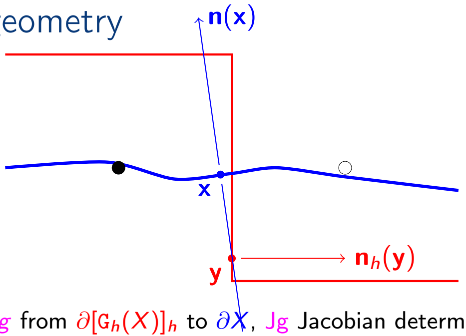
Differential geometry



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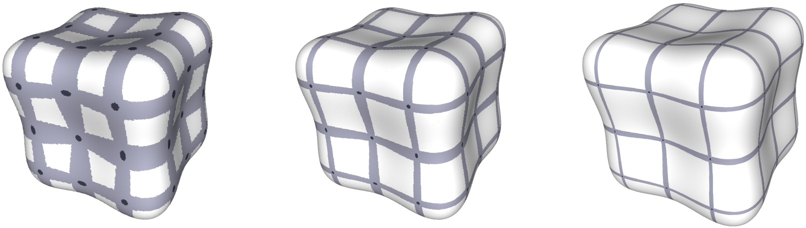
- g should be bijective, differentiable a.e.
- but such g is unknown (since ∂X also)
- the projection is a natural choice: $g = \xi_{\partial X}$
- But $\xi_{\partial X}$ is generally not injective !
- In 3D $\partial[\mathtt{G}_h(X)]_h$ and ∂X may not be homeomorphic [Stelldinger et al. 2007]

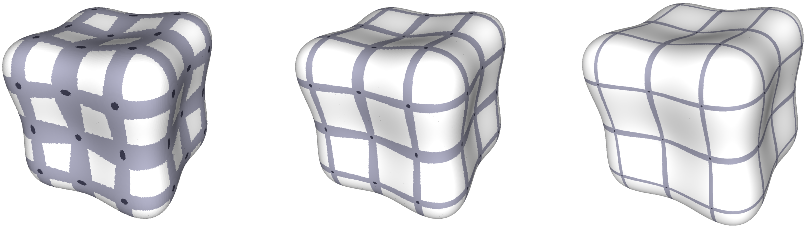
Non-manifold places and non-injective places



Theorem (Localization of non-manifold places)

*Let $X \subset \mathbb{R}^3$ compact domain with positive reach ρ. Non-manifoldness of $\partial[\mathtt{G}_h(X)]_h$ only occurs at places of ∂X where $\xi_{\partial X}$ is **aligned** with an axis (angle $< 1.260 h/\rho$).*

Non-manifold places and non-injective places



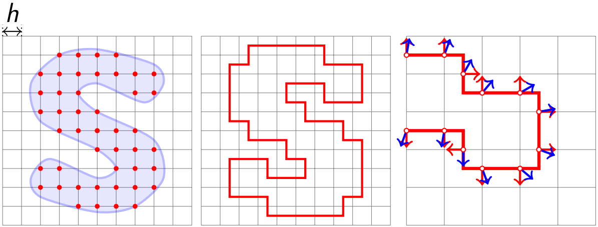
Theorem (Localization and size of non-injective parts)

*$X \subset \mathbb{R}^d$ compact domain with positive reach ρ. Places where $\xi_{\partial X}$ is not injective from $\partial[\mathtt{G}_h(X)]_h$ to ∂X correspond to places where $\xi_{\partial X}$ is **orthogonal** to some axis. If $h \leq \rho/\sqrt{d}$, then one has*

$$\mathrm{Area}(\mathrm{mult}(\partial X)) \leq K_1(h)\ \mathrm{Area}(\partial X)\ h,$$

where $K_1(h) = \frac{8d^2}{\rho} + O(h) \leq \frac{d^2\ 4^{d+1}}{\rho}$.

Digital surface integral



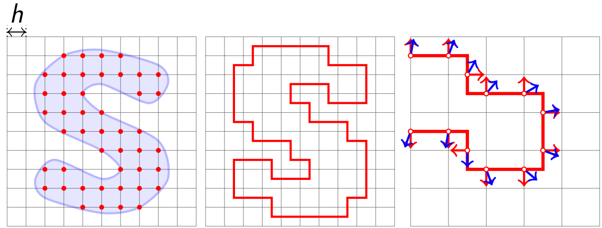
Definition

Let $Z \subset (h\mathbb{Z})^d$ be a digital set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and $\hat{\mathbf{n}}$ be a digital normal estimator. We define the **digital surface integral** by

$$\mathrm{DI}_h(f, Z, \hat{\mathbf{n}}) := \sum_{d-1\text{-cell}\, c \in \partial[Z]_h} h^{d-1} f(\dot{c}) |\hat{\mathbf{n}}(\dot{c}) \cdot \mathbf{n}(\dot{c})|,$$

where $\dot{c}$ is the centroid o of the $(d-1)$-cell c and $\mathbf{n}(\dot{c})$ is its trivial normal as a point on the digitized boundary.

Digital surface integral



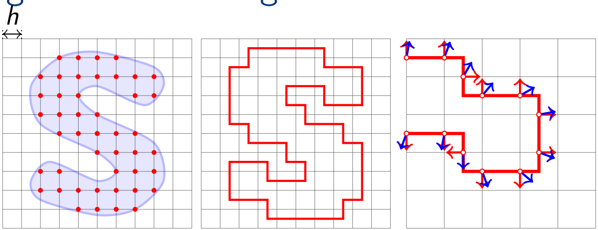
Theorem

Let X be a compact domain where $\mathrm{reach}(\partial X) \geq \rho$. For $h \leq \frac{\rho}{\sqrt{d}}$, the digital surface integral is multigrid convergent toward the integral over ∂X.

$$\left| \int_{\partial X} f(x)dx - \mathrm{DI}_h(f, \mathtt{G}_h(X), \hat{\mathbf{n}}) \right| \leq \mathrm{Area}(\partial X) \; \|f\|_{\mathrm{BL}} \; \Big(O(h) + O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\mathrm{est}}) \Big).$$

Constants in O only depends on dimension d.

Digital surface integral



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Constants in O only depends on dimension d.

Taking $f = 1$ and a conv. normal estimator gives a convergent area estimator.

Steps of the proof

1. First $\int_{\partial X} f(x) dx = \int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x) dx + K_1(h) \mathrm{Area}(\partial X) \|f\|_\infty h$. (size of non injective part).

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2. Then, $\int_{\partial X \setminus \mathrm{mult}(\partial X)} f(x)dx = \int_{\partial_h X \setminus \mathrm{mult}(\partial_h X)} f(\xi(y))J\xi(y)dy$. (diffeomorphism of ξ + change of variable formula)

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$\mathrm{Area}(\partial X)\ \mu\ \|f\|_\infty O(h)$
(multiplicity ae bounded by $\mu := d\lfloor\sqrt{d}+1\rfloor$ and coarea formula)

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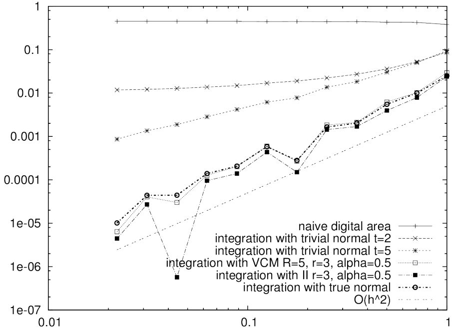
5. $\left| \int_{\partial_h X} f(\xi(y))|\mathbf{n}(\xi(y)) \cdot \mathbf{n}_h(y)|dy - \mathrm{DI}_h(f, \mathtt{G}_h(X), \hat{\mathbf{n}}) \right| \leq$
   $\mathrm{Area}(\partial X)\Big(\mathrm{Lip}(f)O(h) + \|f\|_\infty O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\mathrm{est}})\Big)$.
   (sum cell by cell plus error between $\mathbf{n}(\xi(\mathbf{y}))$ and $\hat{\mathbf{n}}(c)$)

Experimental evaluation



Area estimation error of the digital surface integral with several digital normal estimators. The shape of interest is 3D ellipsoid of half-axes 10, 10 and 5, for which the area has an analytical formula giving $A \approx 867.188270334505$. The abscissa is the gridstep h at which the ellipsoid is sampled by Gauss digitization. For each normal estimator, the digital surface integral $\hat{A}$ is computed with $f = 1$, and the relative area estimation error $\frac{|\hat{A}-A|}{A}$ is displayed in logscale.

Convergent geometric estimators with digital volume and surface integrals

Shapes versus digitized shapes

Curvatures with Digital Integral Invariants

Digital Voronoi Covariance Measure

Digital surface integration

Conclusion

(With)

Conclusion

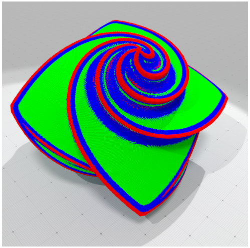
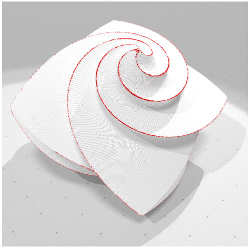
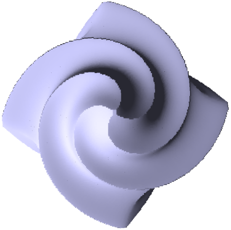
At the term of this journey, we have multigrid convergence of:

estimator	shapes	convergence speed	noise robustness
volume counting	reach > 0	$O(h)$	in practice
dig. moments in B_r	reach > 0	$O(r^{p+q+s}\,h)$	in practice
normal DII	C^3 + reach > 0	$O(h^{\frac{2}{3}})$	in practice
princ. dir. DII	C^3 + reach > 0	$O(h^{\frac{1}{3}})$	in practice
princ. curv. DII	C^3 + reach > 0	$O(h^{\frac{1}{3}})$	in practice
DVCM	C^2 + reach > 0	function(R, r, h)	yes
normal DVCM	C^2 + reach > 0	$O(h^{\frac{1}{8}})$ (obs. $O(h)$)	yes
dig. surf. integral	reach > 0	$O(h) + O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\text{est}})$	?
area	reach > 0	$O(h) + O(\|\hat{\mathbf{n}} - \mathbf{n}\|_{\text{est}})$	?

- Everything is implemented in **DGtal** library: `dgtal.org`
- Digital Integral Invariants are computable in real-time on GPU (see H. Perrier's talk on Wednesday).

Perspectives

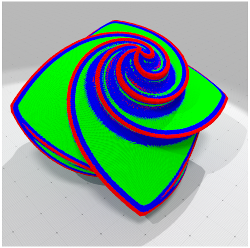
- Good geometry has numerous applications: feature detection, reconstruction

DII [Levallois et al. 2015]   normals + discrete calculus = pw-smooth reconst.

Perspectives

- Good geometry has numerous applications: feature detection, reconstruction



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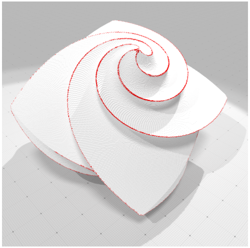

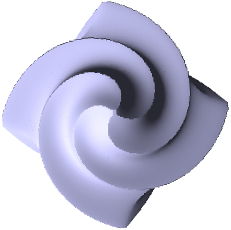

normals + discrete calculus = pw-smooth reconst.

- ANR project CoMeDiC: convergent metrics for digital calculus

Convergent geometric estimators in $\mathbb{Z}^d$ $\overset{\text{(metrics)}}{+}$ Discrete calculus

$=$ Convergent digital calculus in $\mathbb{Z}^d$

Perspectives

Thank you for your attention !

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