Multigrid-convergence of digital curvature estimators

Jacques-Olivier Lachaud

Laboratoire de Mathématiques

CNRS / Université de Savoie

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Multigrid-convergence of digital curvature estimators

Objectives

A detour by digital straight segments

2D curvature by maximal digital circular arcs

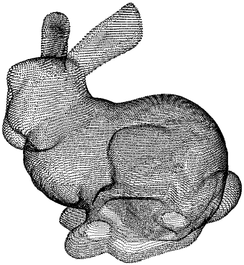
Curvature by constrained optimization

Curvature by digital integral invariants

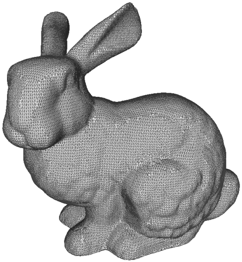
Parameter-free curvature estimation

Curvatures along discrete data

Discrete data may be . . .



cloud of points



triangulated mesh

digital sampling

- estimating curvatures (mean, Gaussian, curvature tensor) everywhere
- close to the curvatures of some underlying continuous object
- data may be subject to perturbations

Stability results for curvatures

Stability: estimated measure converge to the Euclidean one when sampling become denser [Amenta, Bern, Kamvysselis 1998]
Manifold $\mathcal{M}$, Points $\mathcal{V}$, Mesh $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, sampling density is ϵ,

1. Related works for meshes
   - many local estimators (1-ring or 2-ring)
     fitting, discrete methods, curvature tensor estimation
     (e.g. see [Surazhsky *et al.* 2003] [Gatzke, Grimm 2006])
   - general theory [Desbrun, Hirani, Leok, Marsden 2005] [Bobenko, Suris 2008]
   - Gaussian curvature by Gauss-Bonnet [Xu 2006]
     $O(\epsilon^{\min 3, a-2})$ stability of K if $\mathcal{V} \subset \mathcal{M} + O(\epsilon^a)$ and parallelogram and #6, $a > 2$.
   - normal cycle theory [Cohen-Steiner, Morvan, 2003 and 2006]
     $O(\delta)$ stability of $\int \kappa$ if $\mathcal{V} \subset \mathcal{M} + O(\epsilon^2)$, $d_H(\mathcal{T}, \mathcal{M}) \leq \delta \leq \epsilon$.
   - integral invariants [Pottman *et al.* 2007 and 2009]
     $O(\epsilon/r^2)$ stab. of rel. $\tilde{H}_r$ (param. r) if $\mathcal{T} \subset \mathcal{M} + O(\epsilon)$, $\epsilon \ll r$
   - discrete laplace operator [Belkin, Sun, Wang 2008]
     stability of h-laplace operator if $\mathcal{V} \subset \mathcal{M}$ for $h \geq \epsilon^{\frac{1}{2.5}}$

Stability results for curvatures

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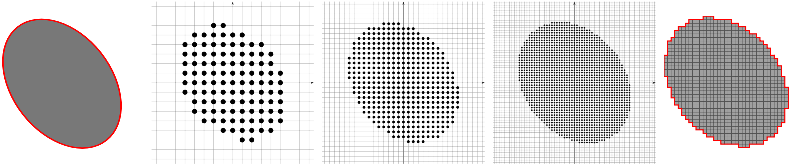
Manifold $\mathcal{M}$, Points $\mathcal{V}$, Mesh $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, sampling density is ϵ,

2. Related works for cloud of points
   - tangent space estimation: curvatures by polynomial fitting [Cazals, Pouget 2005]
     $O(\epsilon^2)$ stability (param N, degree) if $\mathcal{V} \subset \mathcal{M}$
   - curvature measures [Chazal, Cohen-Steiner, Lieutier, Thibert 2009] [Chazal, Cohen-Steiner, Mérigot 2010]
     $O(\sqrt{\delta})$ stability of $\int \kappa$ (param r) if $d_H(\mathcal{V}, \mathcal{M}) \leq \delta$.
   - orthogonal space estimation: covariance matrix on Voronoi diagram [Alliez *et al.* 2007, Mérigot *et al.* 2009 and 2011]
     $O(\sqrt{\delta})$ stability of χ_r-covariance (param r, R) if $\delta = d_H(\mathcal{V}, \mathcal{M})$
     $+$ same stability with outliers [Cuel *et al.*]

Digital data: stability = multigrid convergence

- digitization: shape $X \subset \mathbb{R}^d$, digitized as $\mathrm{Dig}_h(X) = (\frac{1}{h} \cdot X) \cap \mathbb{Z}^d$
- consider finer and finer digitizations [Serra 1982]



$X \in \mathbb{X}$, ∂X $\mathrm{Dig}_1(X)$ $\mathrm{Dig}_{1/2}(X)$ $\mathrm{Dig}_{1/4}(X)$ $\partial_h X$

Geom. estimator $\hat{E}$ multigrid convergent for $\mathbb{X}$ to a geom. quantity E
$\forall X \in \mathbb{X}, \quad |\hat{E}(\mathrm{Dig}_h(X), h) - E(X)| \leq \tau(h)$, with $\lim_{h \to 0} \tau(h) = 0$.

- area of convex set by counting: $O(h)$ conv. [Gauss, Dirichlet]
- digital moments by counting [Klette,Žunić 2000]
- perimeter [Kovalevsky, Fuchs 1992] [Sloboda,Zatko 1996] [Klette *et al.* 1998]

Multigrid convergence for curvature

Local geometric estimator $\hat{E}$ uniformly multigrid convergent for $\mathbb{X}$ to a local geom. quantity E iff

$$\forall X \in \mathbb{X}, \forall x \in \partial X, \forall y \in \partial_h X \text{ with } \|x - y\|_\infty \leq h,$$
$$|\hat{E}(\mathrm{Dig}_h(X), y, h) - E(X, x)| \leq \tau(h),$$

with $\lim_{h \to 0} \tau(h) = 0$.

- curvature of digital curves in 2D:
  - convolution by digital binomial kernel [Malgouyres *et al.* 2008] [Esbelin *et al.* 2011]
    convergence in $O(h^{\frac{4}{9}})$ (param binomial m) with $m = h^{\frac{4}{3}}$
  - polynomial fitting [Provot, Gérard 2011]
    convergence in $O(h^{\frac{1}{3}})$ (param roughness R) with R depending on 3rd derivatives

Multigrid convergence for curvature

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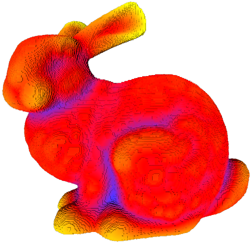
$$|\hat{E}(\mathrm{Dig}_h(X), y, h) - E(X, x)| \leq \tau(h),$$

with $\lim_{h \to 0} \tau(h) = 0$.

- curvature of digital curves in 2D:
- curvature of digital surfaces in 3D:
  - convolution along 3 slices Euler's theorem [Lenoir 1997] empirical approach (param variance σ^2)
  - local iterated convolutions [Fourey, Malgouyres 2008] empirical approach (param nb iteration k)

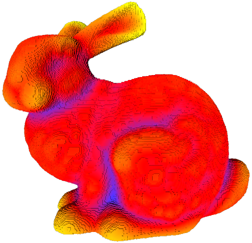
Requirements for a digital curvature estimator



A good digital curvature(s) estimator should be:

1. provably uniformly multigrid convergent
2. accurate in practice
3. computable in an exact manner
4. efficiently evaluable at one point or everywhere
5. robust to perturbations (i.e. bad digitization around the boundary, outliers)

Requirements for a digital curvature estimator



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1. provably uniformly multigrid convergent
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3. computable in an exact manner
4. efficiently evaluable at one point or everywhere
5. robust to perturbations (i.e. bad digitization around the boundary, outliers)
6. parameter free

Proposed estimators

Three different curvature estimators

1. (2D) Maximal Digital Circular Arcs (MDCA)
2. (2D/3D) Constrained minimization of squared curvature (Min-κ^2)
3. (2D/3D) digital Integral Invariants (II)

Proposed estimators

Three different curvature estimators

1. (2D) Maximal Digital Circular Arcs (MDCA)
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3. (2D/3D) digital Integral Invariants (II)

Is it possible to design a parameter free estimator ?

Multigrid-convergence of digital curvature estimators

Objectives

A detour by digital straight segments

(Joint work with F. de Vieilleville, F. Feschet, A. Vialard)

2D curvature by maximal digital circular arcs

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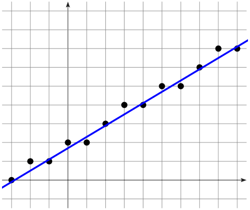
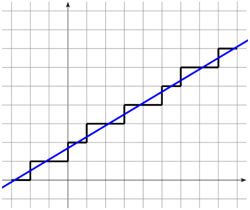
Parameter-free curvature estimation

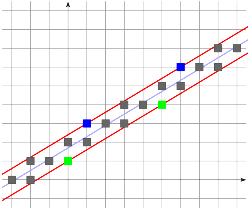
Standard digital straight line



[Bernouilli, 1772]
$y = \left\lfloor \frac{3}{5}x + \frac{3}{10} \right\rfloor$



digitization of half-plane



standard line
$-5 \leq 3x - 5y < 3$

(Arithmetic) standard line [Reveillès 91], [Kovalevsky 90]

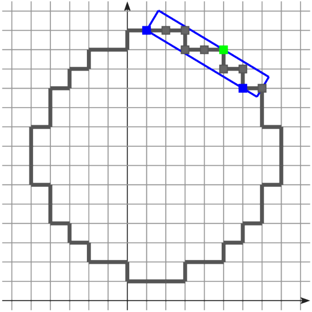
$\{(x, y) \in \mathbb{Z}^2, \mu \leq ax - by < \mu + |a| + |b|\}$, a, b, μ integers

- slope $\frac{a}{b}$, shift to origin μ
- simple 4-connected path in $\mathbb{Z}^2$

Maximal digital straight segments

Maximal segment on contour C

a piece of standard line $S \subset C$ such that $\forall P \in C \setminus S$, $S \cup P$ is not a piece of standard line.



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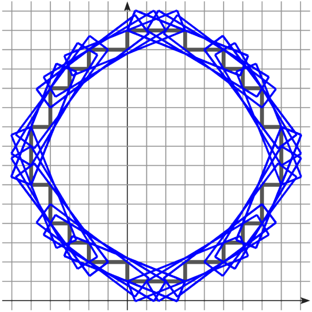


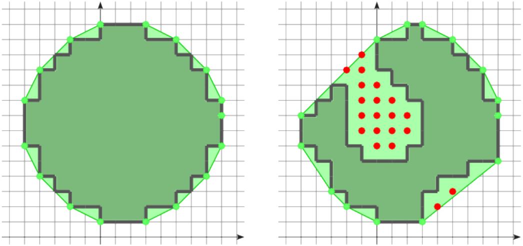
Definition ([Feschet, Tougne 1999], [Dorst, Smeulders 1991])

Tangential cover of C : sequence of all maximal segments of C

Theorem ([L., Vialard, de Vieilleville 07])

Tangential cover is computed in $O(n)$ time, where $n = \#C$

Digital convexity and maximal segments



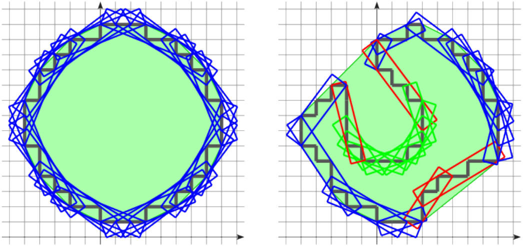
Definition (Convexity of digital shape $O \subset \mathbb{Z}^2$)

O convex iff $\mathrm{Conv}(O) \cap \mathbb{Z}^2 = O$ and O 4-connected.

[Kim, Rosenfeld 1983], [Minsky, Papert 1988] [Hübler, Eckhardt, Klette, Voss, ...],

..., [Brlek, L., Provençal, Reutenauer 2009]

Digital convexity and maximal segments

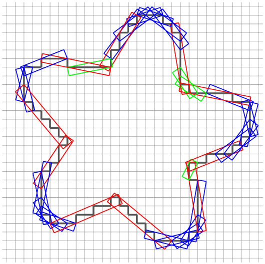


Theorem ([Debled-Rennesson, Reiter-Doerksen 2004])

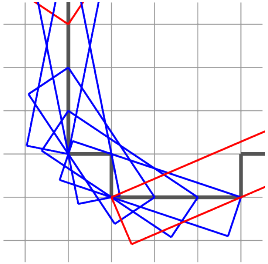
A 4-connected shape $O \subset \mathbb{Z}^2$ is digitally convex iff the directions of its maximal segments are monotonous.

Parameter-free tangent estimation

X



$\mathrm{Dig}_1(X)$

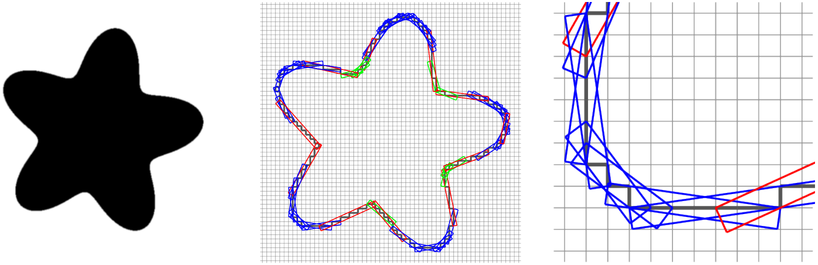


zoom

Definition (Tangent estimation at $y \in \partial_h X$)

$\hat{E}^{MS}(\mathrm{Dig}_h(X), y, h) =$ direction of any maximal segment covering y.

Parameter-free tangent estimation



X $\quad$ $\mathrm{Dig}_{\frac{1}{2}}(X)$ $\quad$ zoom

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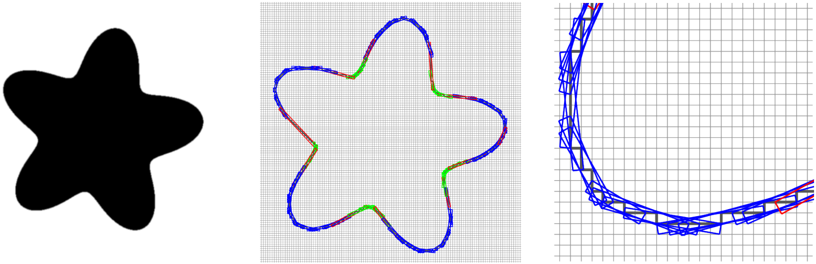

X | $\mathrm{Dig}_{\frac{1}{4}}(X)$ | zoom

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Theorem ([L. Vialard, de Vieilleville, 2007])

For piecewise C^3-convex shapes, $\hat{E}^{MS}$ is multigrid convergent toward tangent. Uniform speed is $O(h^{\frac{1}{3}})$

Parameter-free tangent estimation

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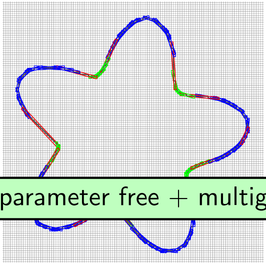

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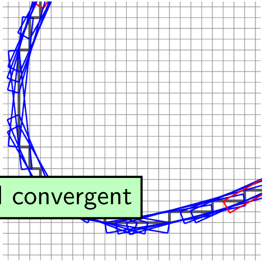

zoom

Tangent estimator: parameter free + multigrid convergent

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2D curvature by maximal digital circular arcs

(Joint work with T. Roussillon)

Curvature by constrained optimization

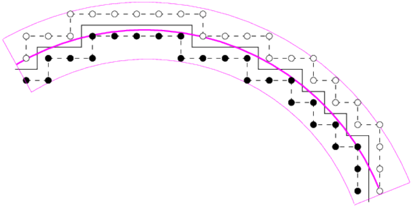
Curvature by digital integral invariants

Parameter-free curvature estimation

Digital circular arc

Digital circular arc

A connected part C' of a contour C is a Digital Circular Arc (DCA) iff its interior points and exterior points are circularly separable.



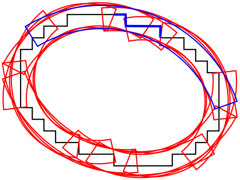
Curvature of a DCA A

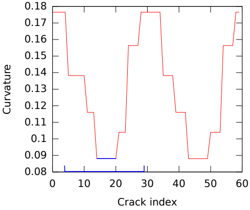
Curvature of A $\kappa(A) = \begin{cases} 0 \text{ if } A \text{ linearly separable,} \\ \text{inverse radius of any separating circle.} \end{cases}$

Curvature estimator based on circular arcs

Maximal Digital circular arc (MDCA)

A DCA A is a MDCA iff all the proper supsets C' of A in the contour C ($A \subset C' \subset C$) are not DCA.

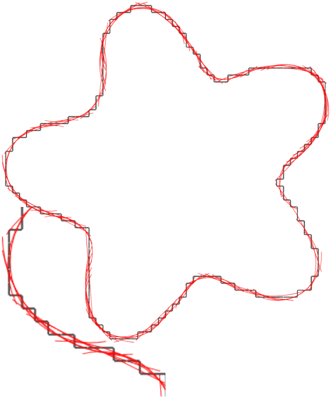


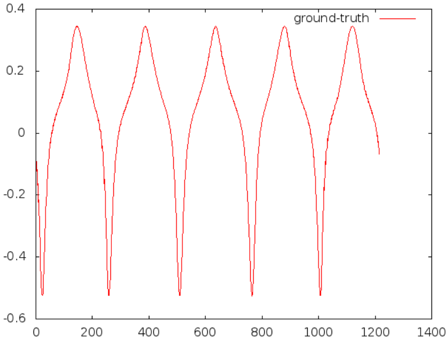


MDCA curvature estimator

Let $p \in C$, h the gridstep. Curvature $\hat{\kappa}^h(p) = \frac{1}{h}\kappa(A)$, where A is the most centered MDCA covering p.

Accuracy of MDCA





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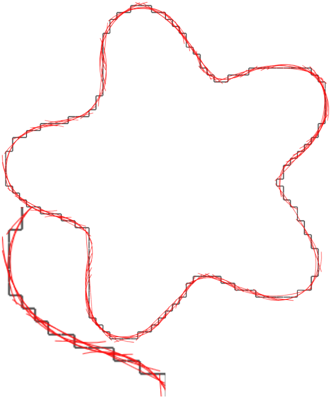

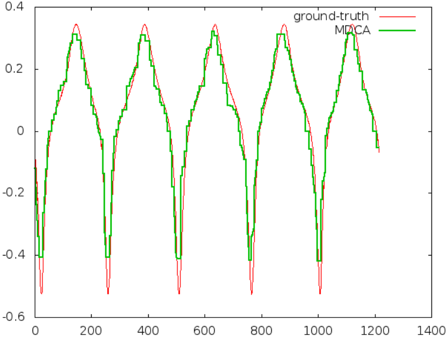

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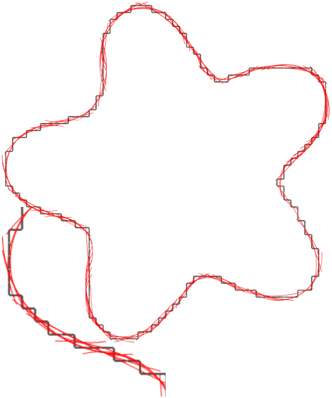

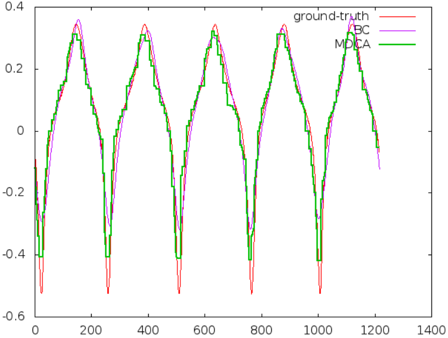

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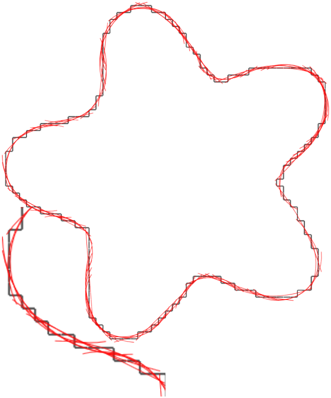

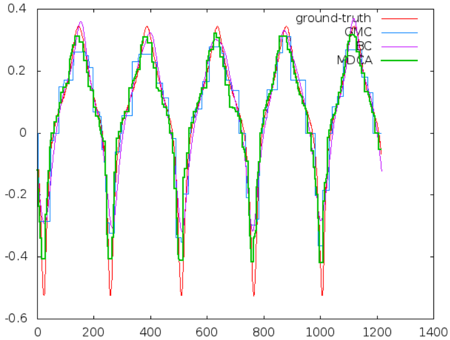

Other properties of MDCA estimator

- computable in an exact manner
- time complexity:
  - worst-case $O(n^2)$ for arbitrary digital curves
  - worst-case $O(n^{\frac{4}{3}})$ for digitization of C^3-convex shapes
  - quasi linear-time in practice.
- parameter free
- multigrid convergence
  - experimental convergence (between $O(h^{\frac{1}{3}})$ and $O(h^{\frac{1}{2}})$)
  - conditional theorem:
    if $\Omega(h^a) \leq$ length MDCA $\leq O(h^b)$, for $0 < b \leq a \leq \frac{1}{2}$,
    then $\hat{\kappa}$ is multigrid convergent in $O(h^{\min(1-2a,b)})$.
- not robust to perturbations
- limited to 2D curves

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(Joint work with E. Bretin, B. Kerautret, E. Oudet)

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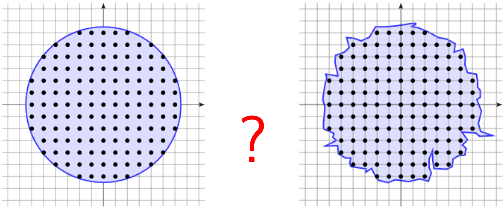
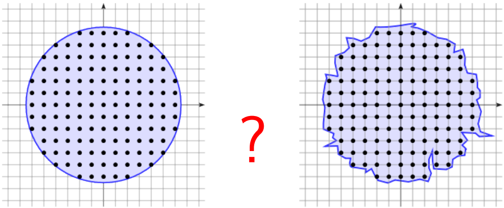
Parameter-free curvature estimation

Can we take into account all shapes with same digitization ?



$$\begin{array}{rcl} \mathrm{Dig}_h(X_1) & = & \mathrm{Dig}_h(X_2) \\ \text{but } \kappa(X_1, x) & \neq & \kappa(X_2, x) \end{array}$$

Can we take into account all shapes with same digitization ?



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1. Choose $\mathbb{X}$, s.t. $\mathrm{Dig}_h(X_1) = \mathrm{Dig}_h(X_2)$ implies $\kappa(X_1, x) \xrightarrow{h \to 0} \kappa(X_2, x)$.

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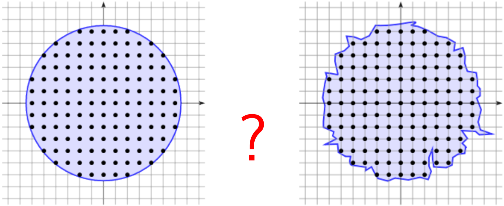

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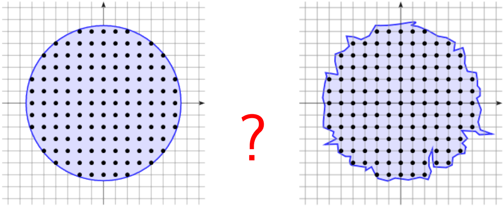

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2. Estimate integral quantities $\int_{A \cap \partial X} \kappa$ [Cohen-Steiner, Morvan 2003]
3. Favor one shape X^{ref} among all shapes with same digitization
   Estimator $\hat{\kappa} \stackrel{def}{=} \kappa(X^{ref})$

Shape that minimizes its squared curvature

Let $Z \subset \mathbb{Z}^2$ be some digital shape at step h.

- shape of reference = solution to a variational problem

$$X^{ref}(Z) = \underset{\{X, s.t. \mathrm{Dig}_h(\mathring{X}) = Z\}}{\arg\min} \int_{\partial X} P(s) ds$$

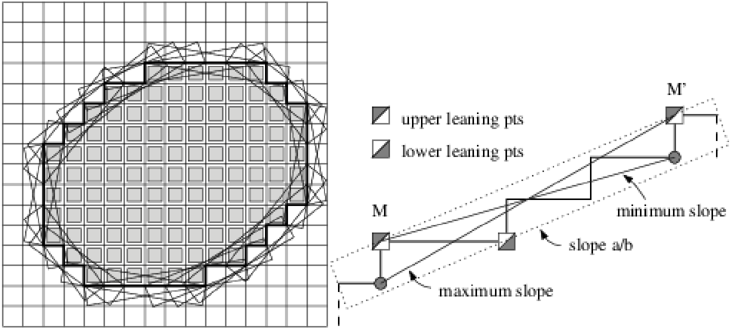
- if $P(s) = 1$, then $X^{ref}(Z)$ is the minimum perimeter polygon
$\Rightarrow$ multigrid convergent perimeter estimator [Sloboda *et al.* 1998]
- if $P(s) = \kappa^2(s)$, then $X^{ref}(Z)$ is smooth (Willmore energy)
Best curvature estimates minimize total squared curvature.

Three approaches to minimization

1. convex support functions
2. digital approach: minimization in tangent space
3. phase-field approximation

method	shapes	accuracy	complexity
(1. convex)	convex 2D	very good	iter $\times O(n)$
2. digital	any 2D	correct	iter $\times O(n^{\frac{2}{3}})$
3. phase-field	any 2D/3D	good	iter $\times O(n^2)$ / iter $\times O(n^{\frac{3}{2}})$

Digital approach: minimization in tangent space



- curvilinear abscissae are estimated (convergent estimator)
- each maximal segment define bounds for curve direction

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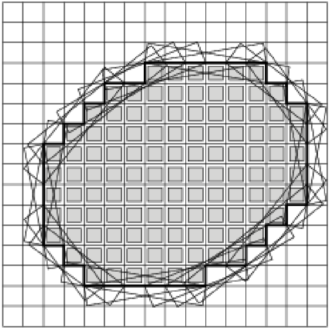

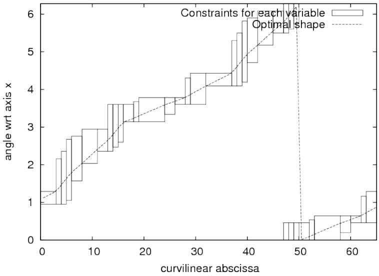

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- each maximal segment define bounds for curve direction
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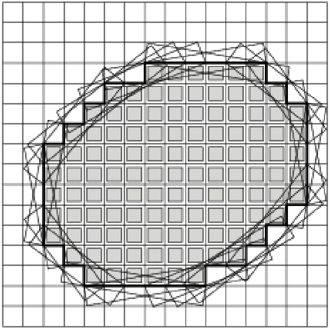
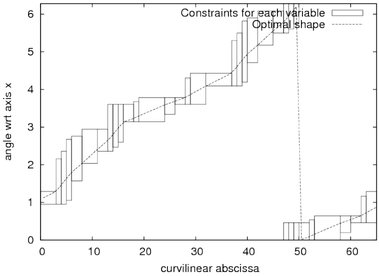
- shortest path gives solution: $\kappa(X^{ref}) \approx slope$

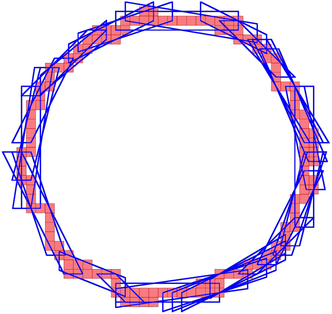
Digital approach: minimization in tangent space



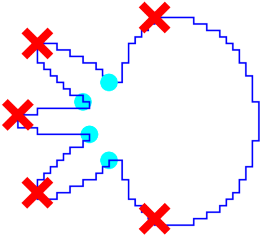


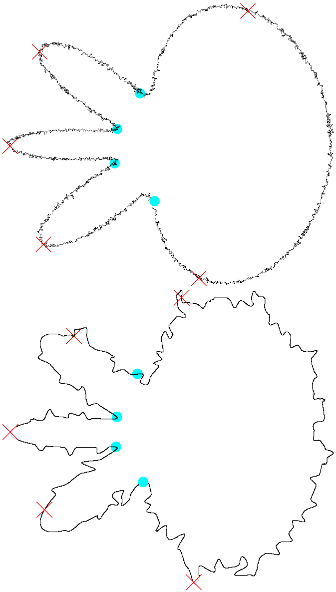
- curvilinear abscissae are estimated (convergent estimator)
- each maximal segment define bounds for curve direction
- cast bounds in tangent space
- shortest path gives solution: $\kappa(X^{ref}) \approx \mathit{slope}$
- solution: sequence of C^1-continuous circular arcs.

Adaptable to Hausdorff noise around curve

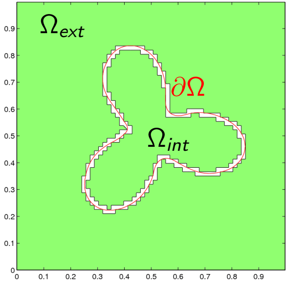


Thickness can be adapted





Phase field approach



$$\Omega^* = \underset{\Omega_{int} \subset \Omega \subset \Omega_{ext}}{\arg\min} \int_{\partial\Omega} \kappa^2 d\sigma.$$

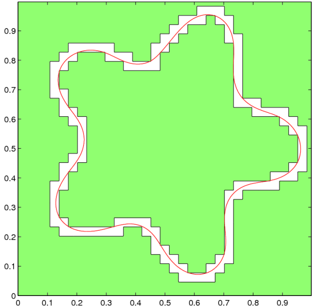
PF approx. [Du, Liu, Ryham, Wang 2005]

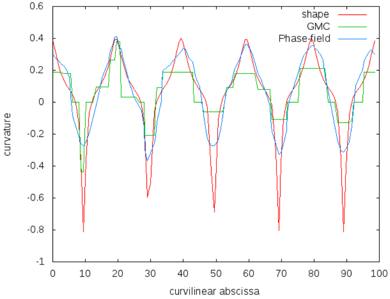
$$F_\epsilon(u) = \int_{\mathbb{R}^d} \frac{1}{\epsilon} \left(-\epsilon \triangle u + \frac{1}{\epsilon} \partial_u W(u,x) \right)^2 dx.$$

where $W(s,x) = \frac{1}{2} s^2 (1-s)^2$ if $x \in \Omega_{ext} \setminus \Omega_{int}$, and special values for x in Ω_{int} or Ω_{ext}.

- Γ-convergence: as $\epsilon \to 0$, F_ϵ tends toward $cst(W) \int_{\partial\Omega} \kappa^2$
- gradient flow of F_ϵ approaches our optimization problem
- potential W forces $u \approx 1$ in Ω_{int} and $u \approx 0$ in Ω_{ext}.
- scheme use Fourier transform for bilaplacian then explicit integration.

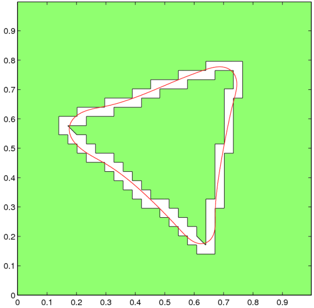
Experimental evaluation

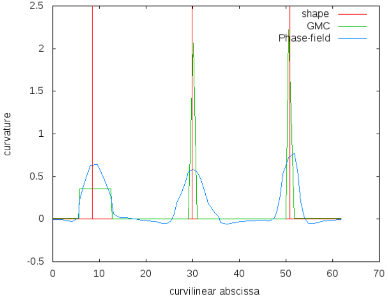




- good accuracy with PF, correct for digital approach (GMC)
- multigrid convergence is observed
- significant points are detected by both methods

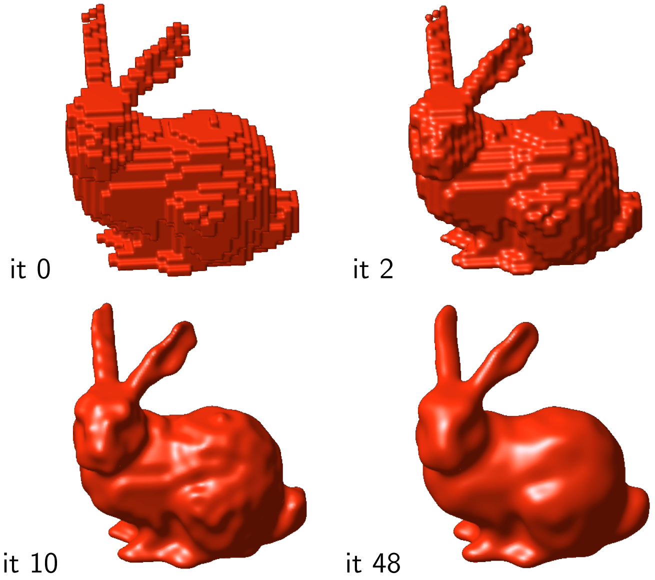
Experimental evaluation





- good accuracy with PF, correct for digital approach (GMC)
- multigrid convergence is observed
- significant points are detected by both methods
- GMC gives more natural results with linear parts.

3D extension of phase field method



Multigrid-convergence of digital curvature estimators

Objectives

A detour by digital straight segments

2D curvature by maximal digital circular arcs

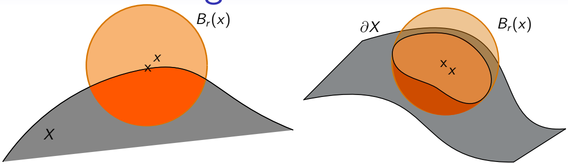
Curvature by constrained optimization

Curvature by digital integral invariants

(Joint work with D. Cœurjolly, J. Levallois)

Parameter-free curvature estimation

Integral invariants



Let $X \in \mathbb{X}$, $x \in \partial X$. Let $r \in \mathbb{R}$, $r > 0$, radius parameter of ball $B_r(x)$

$$V_r(X,x) \stackrel{def}{=} \int_{B_r(x)\cap X} dp.$$

volume

$$J_r(X,x) \stackrel{def}{=} \int_{B_r(x)\cap X} (p-\bar{p}) \otimes (p-\bar{p})^T dp.$$

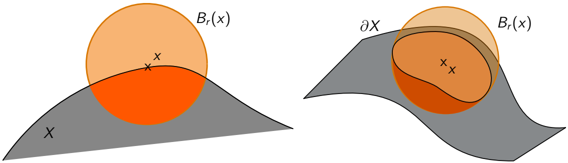
covariance matrix

e.g. see [Pottmann *et al.* 2007]

Mean curvature $H(X,x)$ follows:

$$H(X,x) = \frac{8}{3r} - \frac{4V_r(X,x)}{\pi r^4} + O(r)$$

Integral invariants



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covariance matrix

[Pottmann, Wallner, Yang, Lai, Yu 2007], Theorem 2

Principal curvatures $\kappa_1(X,x), \kappa_2(X,x)$ follows

$$\lambda_1 = \frac{2\pi}{15} r^5 - \frac{\pi}{48}(3\kappa^1(X,x) + \kappa^2(X,x))r^6 + O(r^7)$$

$$\lambda_2 = \frac{2\pi}{15} r^5 - \frac{\pi}{48}(\kappa^1(X,x) + 3\kappa^2(X,x))r^6 + O(r^7)$$

with λ_1, λ_2, λ_3 eigenvalues of $J_r(X,x)$.

How to cast Integral Invariants in the digital world ?

$$\lambda_1 = \frac{2\pi}{15} r^5 - \frac{\pi}{48}(3\kappa^1(X,x) + \kappa^2(X,x))r^6 + O(r^7)$$
$$\lambda_2 = \frac{2\pi}{15} r^5 - \frac{\pi}{48}(\kappa^1(X,x) + 3\kappa^2(X,x))r^6 + O(r^7)$$

Let $Z \subset \mathbb{Z}^3$ be a digital shape, y any point of $\mathbb{R}^3$.

$$\hat{\kappa}^1_r(Z,y,h) = \frac{6}{\pi r^6}(\hat{\lambda}_2 - 3\hat{\lambda}_1) + \frac{8}{5r},$$
$$\hat{\kappa}^2_r(Z,y,h) = \frac{6}{\pi r^6}(\hat{\lambda}_1 - 3\hat{\lambda}_2) + \frac{8}{5r},$$

with $\hat{\lambda}_1$, $\hat{\lambda}_2$ two first eigenvalues of $\hat{J}_r(Z,y,h)$

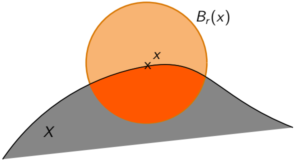
0. define a digital covariance matrix $\hat{J}_r$
1. approx. error of digital covariance matrix $\hat{J}_r \approx J_r$
2. influence of position error (x unknown, only $y \in \partial_h X$ known)
3. impact of these errors on eigenvalues, eigenvectors
4. valid only for $r \to 0$, but $r \gg h$. Best r ?

0. Digital covariance matrix from digital moments

(p,q,s)-moments of $Y \subset \mathbb{R}^3$

for non negative integers p, q and s

$$m_{p,q,s}(Y) \stackrel{def}{=} \iiint_Y x^p y^q z^s \, dxdydz$$



Covariance matrix of $A \stackrel{def}{=} B_r(x) \cap X$

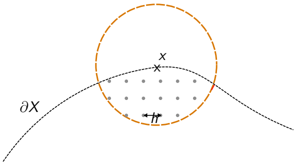
$$J_r(X,x) = \begin{bmatrix} m_{2,0,0}(A) & m_{1,1,0}(A) & m_{1,0,1}(A) \\ m_{1,1,0}(A) & m_{0,2,0}(A) & m_{0,1,1}(A) \\ m_{1,0,1}(A) & m_{0,1,1}(A) & m_{0,0,2}(A) \end{bmatrix} - \frac{1}{m_{0,0,0}(A)} \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix} \otimes \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix}^T$$

0. Digital covariance matrix from digital moments

digital (p,q,s)-moments of $Z \subset \mathbb{Z}^3$

for non negative integers p, q and s

$$\hat{m}_{p,q,s}(Z,h) \stackrel{def}{=} h^{3+p+q+s} \sum_{i,j,k \in Z} i^p j^q k^s$$



digital covariance matrix of $A' \stackrel{def}{=} B_{r/h}(\frac{1}{h} \cdot y) \cap Z$

$$\hat{J}_r(Z,y,h) = \begin{bmatrix} \hat{m}_{2,0,0}(A',h) & \hat{m}_{1,1,0}(A',h) & \hat{m}_{1,0,1}(A',h) \\ \hat{m}_{1,1,0}(A',h) & \hat{m}_{0,2,0}(A',h) & \hat{m}_{0,1,1}(A',h) \\ \hat{m}_{1,0,1}(A',h) & \hat{m}_{0,1,1}(A',h) & \hat{m}_{0,0,2}(A',h) \end{bmatrix} - \frac{1}{\hat{m}_{0,0,0}(A',h)} \begin{bmatrix} \hat{m}_{1,0,0}(A',h) \\ \hat{m}_{0,1,0}(A',h) \\ \hat{m}_{0,0,1}(A',h) \end{bmatrix} \otimes \begin{bmatrix} \hat{m}_{1,0,0}(A',h) \\ \hat{m}_{0,1,0}(A',h) \\ \hat{m}_{0,0,1}(A',h) \end{bmatrix}^T$$

1. Approximation error of digital covariance matrix

Multigrid convergence of digital moments ($\sigma \stackrel{def}{=} p + q + s$)

$$\forall x \in \mathbb{R}^3, |\hat{m}_{p,q,s}(\mathrm{Dig}_h(X), x, h) - m_{p,q,s}(X, x)| = O(h^{\mu_\sigma})$$

- $\mu_0 \geq 1$ (monotonic [Krätzel 1988]), $\mu_0 \approx \frac{66}{43}$ (C^∞-convex [Müller 1999])
- $\mu_1, \mu_2 \geq 1$ (monotonic [Krätzel 1988], [Klette, Žunić 2000])

Get rid of hidden scale constant in O

$$|\hat{m}_{p,q,s}(A', x, h) - m_{p,q,s}(A, x)| \leq O(r^{3+\sigma-\mu_\sigma} h^{\mu_\sigma}).$$

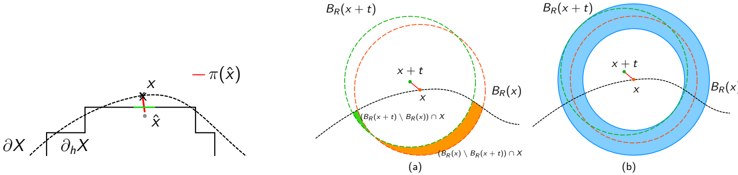
- Trick: scale $A \stackrel{def}{=} B_r \cap X$ to $B_1 \cap \frac{1}{r} \cdot X$ and digitized by h/r.
- Limit object is half ball of radius 1.

Multigrid convergence of digital covariance matrix ($r \geq h$)

$$\forall x \in \mathbb{R}^3, \|\hat{J}_r(A', x, h) - J_r(A, x)\| = \sum_{i=0}^{2} O(r^{5-\mu_i} h^{\mu_i}).$$

- invariance by translation of (dig. or cont.) covariance matrix
- shift to origin by $t = x - h[\frac{x}{h}]$, $[\frac{x}{h}]$ integer vector closer to $\frac{x}{h}$.

2. Influence of position error ($\hat{x} \in \partial_h X$ known)



Positioning error of moments with vector $\mathbf{t}$

$$|m_{p,q,s}(B_r(x+\mathbf{t}) \cap X) - m_{p,q,s}(B_r(x) \cap X)| = \sum_{i=0}^{\sigma} O(\|x\|^i \|t\| r^{2+\sigma-i}).$$

Corollary, with $\mathbf{t} = \hat{x} - x$, $\|\mathbf{t}\|_\infty \leq h$

$$\|\hat{J}_r(A', \hat{x}, h) - J_r(A, x)\| = \sum_{i=0}^{2} O(r^{5-\mu_i} h^{\mu_i}) + O(\|x - \hat{x}\| \|r^4).$$

3. Impact of these errors on eigenvalues, eigenvectors

Theorem (Lidskii-Weyl inequality), e.g. see [Stewart 1990] [Bhatia 1997]

If $\lambda_i(B)$ denotes the ordered eigenvalues of some symmetric matrix B and $\lambda_i(B+E)$ the ordered eigenvalues of some symmetric matrix $B+E$, then $\max_i |\lambda_i(B) - \lambda_i(B+E)| \leq \|E\|$.

Hence, eigenvalues of $\hat{J}_r$ and J_r are as close as matrix terms.

$$\hat{\lambda}_i = \lambda_i + \sum_{i=0}^{2} O(r^{5-\mu_i} h^{\mu_i}) + O(\|x - \hat{x}\| r^4).$$

4. Theorem. What is the best r ?

Theorem (Multigrid convergence of curvature estimators $\hat{\kappa}^i_r$)
Let $X \in \mathbb{X}$. Then, $\exists h_X \in \mathbb{R}^+$, for any $h \leq h_X$, we have:
$\forall x \in \partial X, \forall \hat{x} \in \partial_h X, \|\hat{x} - x\|_\infty \leq h,$ *then*

$$|\hat{\kappa}^i_r(\mathrm{Dig}_h(X), \hat{x}, h) - \kappa^i(X, x)| \leq O(r) + \sum_{i=0}^{2} O(h^{\mu_i}/r^{1+\mu_i}) + O(h/r^2)$$

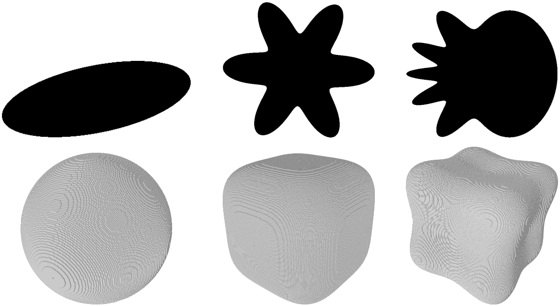
Reminder: O(r) comes from [Pottmann *et al.* 2007].

- What is the best r ?
- Without more hypothesis, $\mu = 1$, then setting $r = kh^\alpha$, we got $\alpha_m = \frac{1}{3} \Rightarrow |\hat{\kappa}^i_r - \kappa^i(X, x)| \leq (cst)h^{\frac{1}{3}}$
- With more hyp. ($C^3 - convex$, $K > 0$) and better positioning of $\hat{x}$, we could have $\alpha_m \approx 0.434$ and error in $O(h^{0.434})$.

Experimentation

Evaluation framework

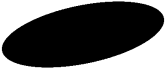
- Family of Euclidean shapes with *exact* curvature information
- Digitization process at resolution h
- Error metrics l_∞ error (worst case), l_2 error

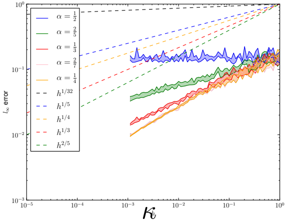


Validation of α parameter

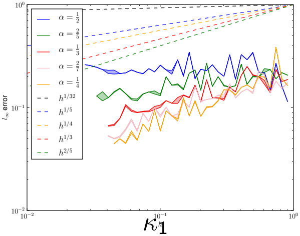
Convolution kernel radius

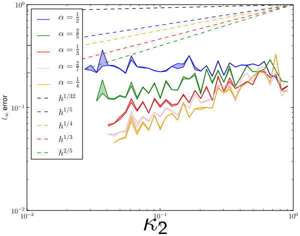
Asymptotic l_∞ error when $h \to 0$, for varying α, $r = 5h^\alpha$





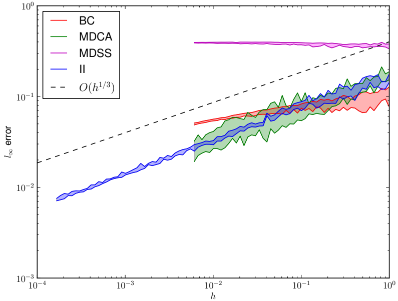
ellipse

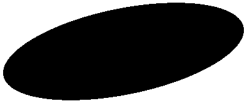


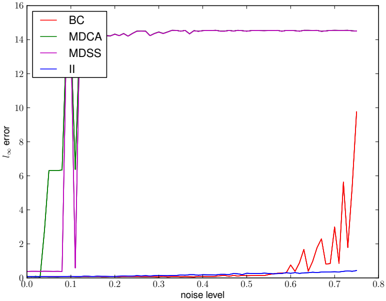


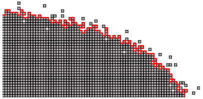
smooth cube

Comparison on 2D shapes



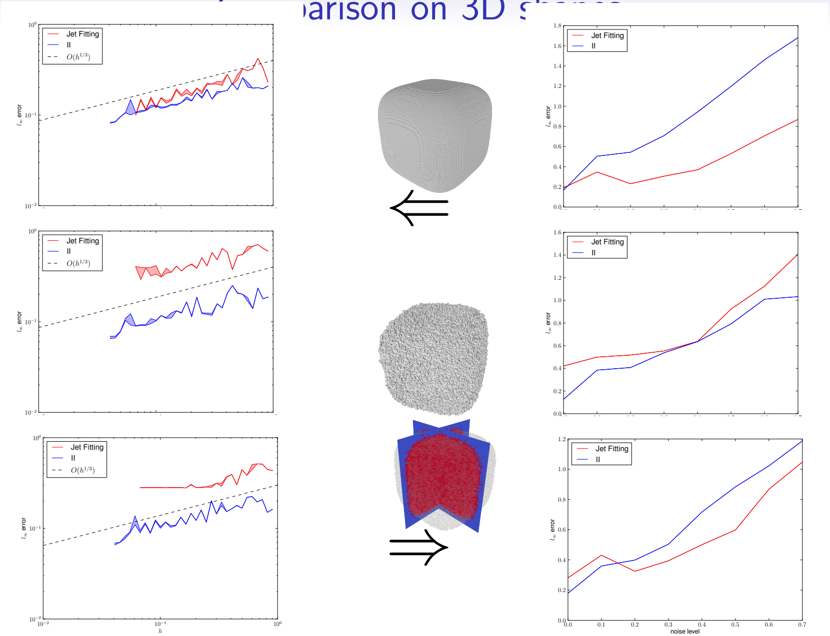




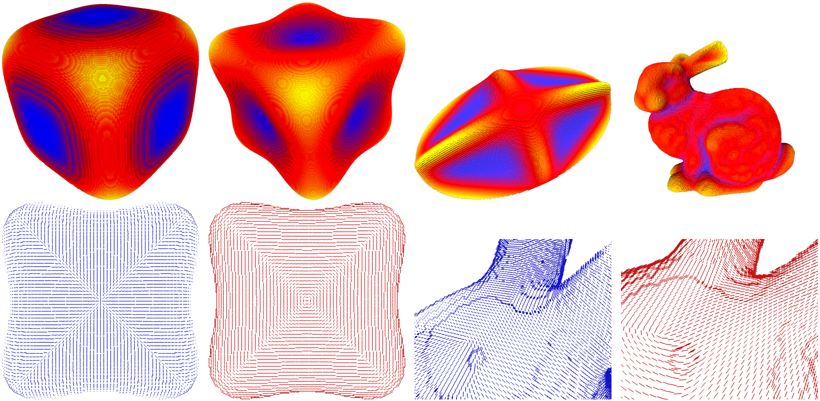


- BC: binomial covolution
- MDCA: circular arcs
- II: integral invariant

Comparison on 3D shapes



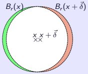
Mean curvature and principal directions



Some words about implementation

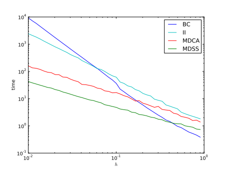
- Open-source C++ library
- Geometry structures, algorithm & tools for digital data
- `http://libdgtal.org`

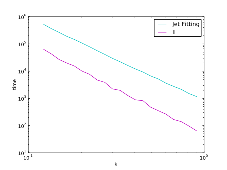
Optimization with displacement masks + depth-first traversal



Complexity:

- without optimization: $O(n(r/h)^{d})$
- with optimization: $O(n(r/h)^{d-1})$

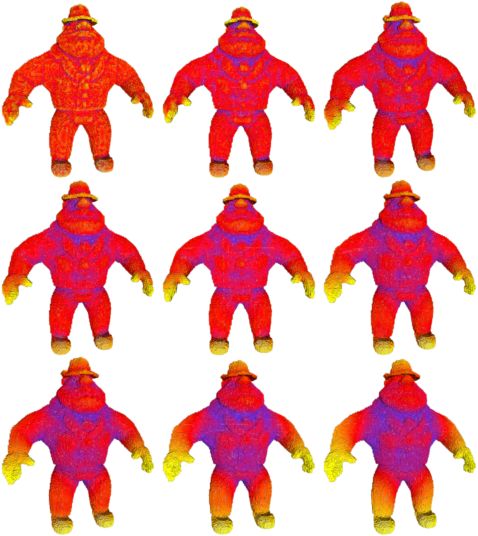




Summary of digital curvature estimators

	convergence	accuracy	exact comp.	efficient	robust	param. free
(2D estimator)						
MDCA	?	$O(h^{\frac{1}{3}})$	yes	$O(n^{\frac{4}{3}})$	no	scale
Min κ^2 (GMC)	?	$\approx O(h^{\frac{1}{3}})$	Opt.	iter $\times O(n^{\frac{2}{3}})$	Hausdorff	$\approx$ scale
Min κ^2 (PF)	?	$O(h^{\frac{1}{3}})$	Opt.	iter $\times O(n^2)$	no	$\approx$ scale
II	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h
(3D estimator)						
Min κ^2 (PF)	?	?	Opt.	iter $\times O(n^{\frac{3}{2}})$	no	$\approx$ scale
II	$O(h^{\frac{1}{3}})$	$O(h^{\frac{1}{3}})$	Yes	$O(n^{\frac{5}{3}})$	yes	need h

Is there an automatic way to set parameter $r = kh^{\frac{1}{3}}$?



Multigrid-convergence of digital curvature estimators

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A detour by digital straight segments

2D curvature by maximal digital circular arcs

Curvature by constrained optimization

Curvature by digital integral invariants

Parameter-free curvature estimation

How to estimate consistently the radius $r = kh^{\frac{1}{3}}$?

- suppose h is not given/known ?
- we cannot estimate the scale of some X, since $\mathrm{Dig}_h(X) = \mathrm{Dig}_{sh}(s \cdot X)$
- however we can estimate the appropriate radius $r = kh^{\frac{1}{3}}$ without more knowledge
- hence we will get the same relative error on curvature estimates whatever the scale or gridstep.
- we use asymptotic properties of maximal segments

Asymptotic results on max. segments

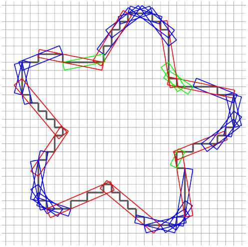
Thm [L. 2006],[de Vieilleville, L., Feschet 2007]

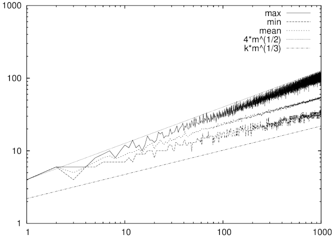
Along digitizations of C^3-convex shapes (curvature $\kappa > 0$), digital length L_D and Euclidean length L of maximal segments follow:

	shortest	average	longest
$L_D(MS)$	$\Omega(h^{-\frac{1}{3}})$	$\Theta(h^{-\frac{1}{3}}) \leq \cdot \leq \Theta(h^{-\frac{1}{3}} \log \frac{1}{h})$	$O(h^{-\frac{1}{2}})$
$L(MS)$	$\Omega(h^{\frac{2}{3}})$	$\Theta(h^{\frac{2}{3}}) \leq \cdot \leq \Theta(h^{\frac{2}{3}} \log \frac{1}{h})$	$O(h^{\frac{1}{2}})$

- average length:
  - lattice polytope theory (Thm by [Balog, Bárány 91])
    $c_1(X)h^{-\frac{2}{3}} \leq n_{vtx}(\mathrm{Conv}(\mathrm{Dig}_h(X)) \leq c_2(X)h^{-\frac{2}{3}}$
  - relating maximal segments to edges of convex hull
  - log term linked to depth of continued fractions
- longest max. segment $= O(h^{-\frac{1}{2}})$ (geometry)
- shortest max. segment $= \Omega(h^{-\frac{1}{3}})$ [L. 06] (separating circles)

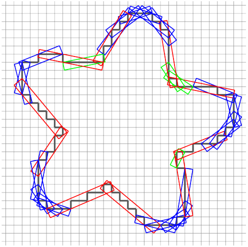
Estimating $r = kh^{\frac{1}{3}}$ with max. segments

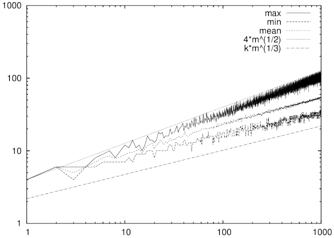




- Let $\mathcal{L} = \text{Average}(L_D(MS))$
- Hence $\Omega(h^{-\frac{1}{3}}) \leq \mathcal{L} \leq O(h^{-\frac{1}{3}} \log(1/h))$
- thus $\mathcal{L}^2 \approx \Theta(h^{-\frac{2}{3}}) \propto r/h$
- $\mathcal{L}^2$ is proportional to the digital radius r/h

Estimating $r = kh^{\frac{1}{3}}$ with max. segments





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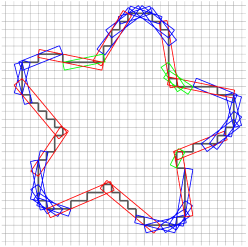
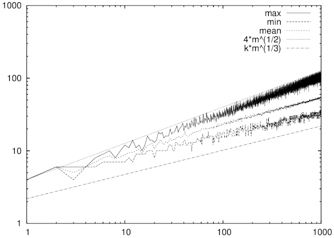
- Everything can be done in the digital domain !

Estimating $r = kh^{\frac{1}{3}}$ with max. segments





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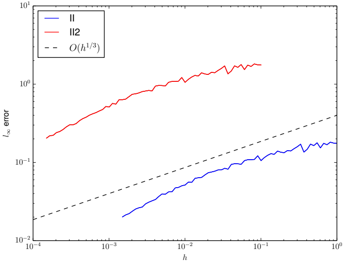
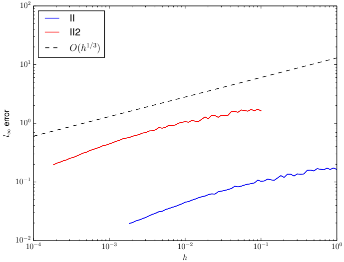
- $\mathcal{L}^2$ is proportional to the digital radius r/h
- Everything can be done in the digital domain !
- For constant k, we choose $\frac{1}{5}$ since $L_D(MS) = 5$ for the sphere of radius 1

(Completely) parameter-free integral invariant

l_∞ error



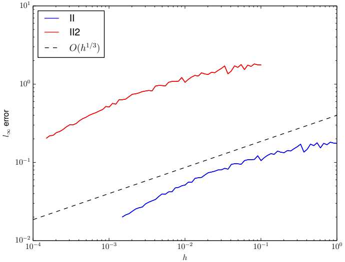
ellipse at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}$·ellipse at $h = \frac{1}{10}, \frac{1}{20}, \ldots$



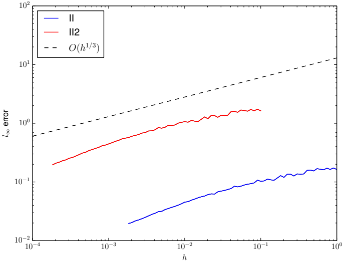
flower at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}$·flower at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

(Completely) parameter-free integral invariant

l_∞ error



ellipse at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}\cdot$ellipse at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

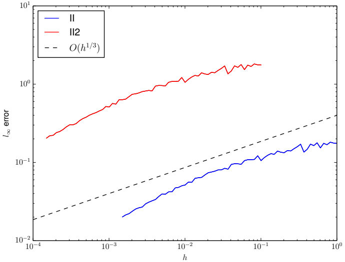


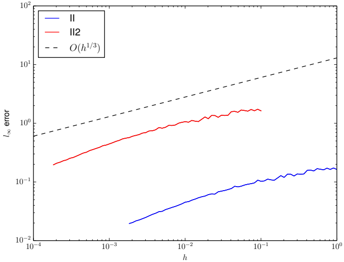
flower at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}\cdot$flower at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

- curvature error $\times 10$!

(Completely) parameter-free integral invariant

l_∞ error





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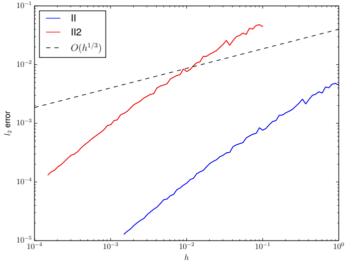
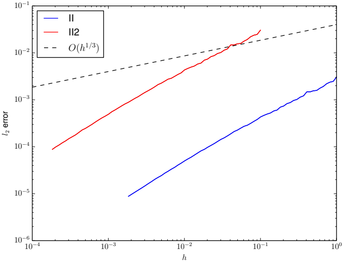
flower at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}$·flower at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

- curvature error $\times 10$!
- but curvature $\times 10$, hence relative curvature error is constant

(Completely) parameter-free integral invariant

l_2 error





ellipse at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}\cdot$ellipse at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

flower at $h = 1, \frac{1}{2}, \ldots$
$\frac{1}{10}\cdot$flower at $h = \frac{1}{10}, \frac{1}{20}, \ldots$

- curvature error $\times 10$!
- but curvature $\times 10$, hence relative curvature error is constant

Bibliography

Thank you for your attention !

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