Digital shape analysis with maximal segments

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Abstract We show in this paper how a digital shape can be efficiently analyzed through the maximal segments defined along its digital contour. They are efficiently computable. They can be used to prove the multigrid convergence of several geometric estimators. Their asymptotic properties can be used to estimate the local amount of noise along the shape, through a multiscale analysis.

Keywords discrete geometry; digital shape analysis; digital straight segments; geometric estimators; multigrid convergence; noise detection; digital convexity

1 Introduction

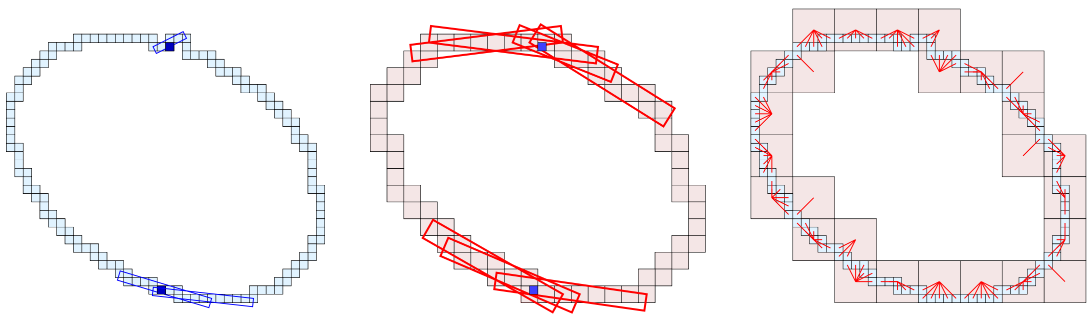
It is often interesting to study the geometry of digitization of Euclidean shapes in the plane, and to establish connections between the discrete geometry computed along the digital contour and the Euclidean geometry of the initial shape. This task is essential in image analysis, where the initial Euclidean shape has been lost through various acquisition and segmentation processes.

Maximal segments are the connected pieces of digital straight lines that are contained in the digital contour and that are not extensible [7, 6] (if they are extended on either side, the formed set is no more a digital straight segment). Maximal segments appear to hold many interesting properties for analyzing digital shapes. We will show here that they characterize the convex and concave parts of the shape [6, 5]. They induce discrete geometric estimators of length and tangent that are multigrid convergent, with a quantifiable error [11, 12]. These asymptotic properties of maximal segments [3] are also extremely useful to detect the local meaningful scales at which the shape should be analyzed: in this sense, they provide an unsupervised method to determine locally the level of noise that is damaging the shape [9].

2 Digital straightness, maximal segments and convexity

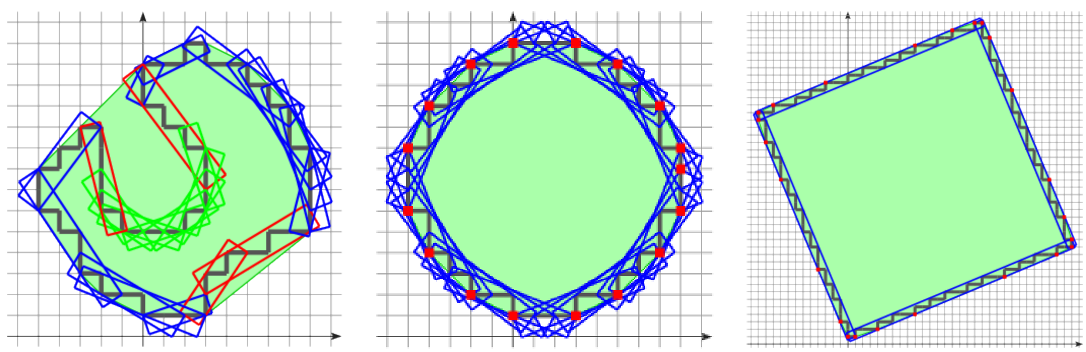
A *digital shape* is a subset of the digital plane $\mathbb{Z}^2$. To simplify the exposition, this shape is simply connected (i.e. a polyomino). Its interpixel boundary is therefore a 4-connected contour in the half-integer plane. By translating everything by vector $(\frac{1}{2}, \frac{1}{2})$, we get back that all pointels of the interpixel boundary have integer coordinates. Let us denote with C the digital contour of size N.

A *standard digital straight line (DSL)* is a 4-connected digital set $\{(x, y) \in \mathbb{Z}^2, \mu \leq ax - by < \mu + |a| + |b|\}$, all parameters being integers [14]. Geometrically, the fraction a/b represents the slope of the line while parameter μ quantifies its shift at the origin. A *digital straight segment (DSS)* is a 4-connected piece of DSL. If we consider the 4-connected path C, a *maximal segment* M is a subset of C that is a DSS and which is no more a DSS when adding any other points of $C \backslash M$. Fig. 1(a,b) displays the set of all the maximal segments covering the dark pixels. The sequence of all maximal segments along a digital contour is called the *tangential cover* [6]. It is worthy to note that the whole tangential cover of C can be computed in $O(N)$ time complexity [12]. Indeed, online recognition of DSS takes $O(1)$ time complexity when adding a point [4], while updating the DSS characteristics when removing a point takes also $O(1)$ [12].



(a) (b) (c)

Figure 1: Maximal segments on (a) an initial contour C and (b) on its subsampled contour $\phi_3^{0,0}(C)$. (c) Function $f_5^{0,0}$ (represented by lines) associating each pixel of C to its pixel of $\phi_5^{0,0}(C)$.



(a) (b) $n_{MS} = 24, n_e = 16$ (c) $n_{MS} = 4, n_e = 24$

Figure 2: (a) Maximal segments and convexity. (b) and (c) number of maximal segments wrt number of edges of convex hull.

Maximal segments are characteristics of the global convexity, but also give insights to the local convexity or concavity of the contour (illustrated on Fig. 2):

Theorem 1 ([5]) *A polyomino (simply connected subset of $\mathbb{Z}^2$) is digitally convex if and only if the directions of its maximal segments are monotonous.*

Inflexion maximal segments, where slope directions are increasing on one side and decreasing on the other, therefore cut the shape into its convex and concave parts.

3 Multigrid convergence and asymptotic properties

Multigrid convergence is an interesting way of relating digital and Euclidean geometries. The idea is to ask for discrete geometric estimations to converge toward the corresponding Euclidean quantity when considering finer and finer shape digitizations (here, Gauss digitization).

Definition 1 (Definition 2.10 of [10]) *A discrete geometric estimator $\hat{\mathcal{Q}}$ is* multigrid convergent *for a family of shapes $\mathcal{F}$ and a digitization process* Dig. *iff for all shape $X \in \mathcal{F}$, there exists a grid step $h_X > 0$ such that the estimate $\hat{\mathcal{Q}}(\mathrm{Dig}_h(X))$ is defined for all $0 < h < h_X$ and*

$$|\hat{\mathcal{Q}}(\mathrm{Dig}_h(X)) - \mathcal{Q}(X)| \leq \tau(h),$$

where $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with null limit at 0. This function is the speed of convergence *of the estimator.*

For instance, when $\mathcal{Q}$ is the area A of the shape, the estimator $\hat{A}(O) = h^2 \mathrm{Card}(O)$ is multigrid convergent for most family of shapes (Gauss, Dirichlet, [8]). Multigrid convergence has also been established for several length estimators (reported in [10]). The *minimum perimeter polygon* of a digital shape is multigrid convergent with speed $O(h)$ [15]. The minimum perimeter polygon of a digital contour C can be computed in optimal time $O(N)$ from its maximal segments [13]. Therefore, maximal segments are useful to estimate global geometric quantities.

Turning ourselves to evaluate the multigrid convergence for local geometric quantities such as tangent or curvature, Definition 1 must be adapted. This is formally done in Definition 4.15 of [11] but not detailed here for space reasons.

As observed in [2] and stated in [12, 3], the slope of maximal segments tend to approximate the slope of the tangent of the underlying points. This result is achieved by establishing some asymptotic properties of maximal segments along a digitized shape as the digitization step tends to 0. Although the number n_{MS} of maximal segments is not obviously related to the number n_e of edges of its convex hull (e.g., see Fig. 2(b,c)), we have:

Theorem 2 ([3]) *For a convex shape X with $\mathcal{C}^3$ boundary,*

$$\frac{n_e(\mathrm{Dig}_h(X))}{\Theta(\log \frac{1}{h})} \leq n_{MS}(\mathrm{Dig}_h(X)) \leq 3 n_e(\mathrm{Dig}_h(X)).$$

By relating this result to Theorem 2 of Balog et Bárány [1], we get results on the *digital length* of maximal segments, for shapes X as above and strictly positive curvature:

Theorem 3 (Theorem 5.1 of [3] and Theorem 5.26 of [11])

$$\text{average length of maximal segments } \overline{L}_{MS}\text{:} \quad \Theta(h^{-\frac{1}{3}}) \leq \overline{L}_{MS}(\mathrm{Dig}_h(X)) \leq \Theta(h^{-\frac{1}{3}} \log \frac{1}{h}) \quad (1)$$

$$\text{shortest maximal segment } L_{MS}^{\min}\text{:} \quad \Theta(h^{-\frac{1}{3}}) \leq L_{MS}^{\min}(\mathrm{Dig}_h(X)) \quad (2)$$

$$\text{longest maximal segment } L_{MS}^{\max}\text{:} \quad L_{MS}^{\max}(\mathrm{Dig}_h(X)) \leq \Theta(h^{-\frac{1}{2}}) \quad (3)$$

As one can see, the digital length of maximal segments grows as the resolution gets finer. Therefore, estimating the tangent direction at some point as the direction of any maximal segment covering it leads to a discrete tangent estimator that is uniformly convergent in $O(h^{\frac{1}{3}})$ (from (2) and Taylor expansion [11, 12]). The convergence speed is experimentally $O(h^{\frac{2}{3}})$ nearly everywhere.

Furthermore, the length of any digital path can be estimated by integrating at each linel the scalar product of its tangent estimation and the linel direction. The preceding result induces a multigrid convergent length estimator with speed $O(h^{\frac{1}{3}})$. It is also interesting to notice that (1) refutes the hypothesis used in the proof of the multigrid convergence of the curvature estimator by circumscribed circle (Theorem B.4, [2]). This estimator is also not convergent experimentally.

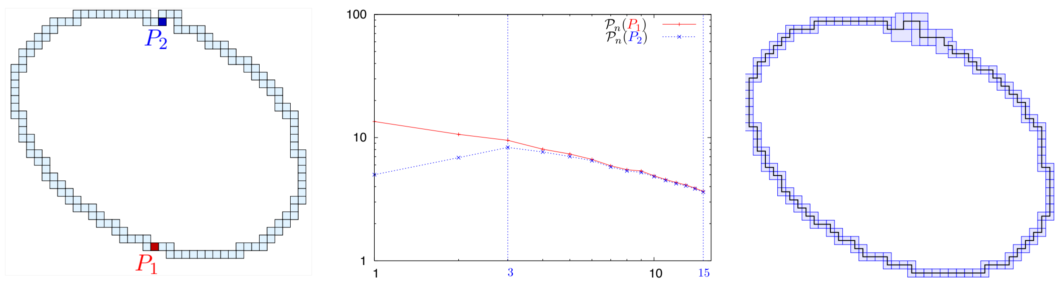
4 Meaningful scales and noise detection

The preceding asymptotic properties can be used to detect the meaningful scales at which a shape should be locally considered [9]. Indeed, let P be some point on ∂X. We denote by (L_j^h) the discrete lengths of the maximal segments, defined along $\partial \mathrm{Dig}_h(X)$, and which cover P. If U is an open connected neighborhood of P on X, Theorem 3 induces

$$\text{if } U \text{ is strictly convex or concave, then } \Omega(1/h^{1/3}) \leq L_j^h \leq O(1/h^{1/2}) \quad (4)$$

$$\text{if } U \text{ has null curvature everywhere, then } \Omega(1/h) \leq L_j^h \leq O(1/h). \quad (5)$$

Since in practice, it is not possible to obtain the asymptotic digitizations of the initial shape O with finer and finer grid steps h, a solution is to consider the subsampling $\phi_i^{x_0,y_0}(O)$ with increasing covering pixel sizes $i \times i$ for $i = 2, .., n$ and with shift x_0, y_0. Several subsampling processes can be



(a) (b) (c)

Figure 3: Illustration of multiscale profile (b) on several points of the contour (a). (c) shows the resulting noise level estimation represented by a centered box of size $\nu(P) + 1$.

considered at this stage, but it is necessary to maintain a surjective map $f_i^{x_0,y_0}$ which associates any point P of C to its image point in the subsampled contour $\phi_i^{x_0,y_0}(C)$. Such a function is illustrated on Fig. 1(c). Then, we can consider the discrete lengths $(L_j^{h_i,x_0,y_0})$ of the maximal segments on the subsampled shapes $\phi_i^{x_0,y_0}(C)$ containing $f_i^{x_0,y_0}(P)$ with the increasing sequence of digitization grid steps $h_i = ih$ (see Fig. 1(a,b)). For a given subsampling size i, the average discrete length of all the maximal segments containing the subsampled pixel is denoted as $\overline{L}^{h_i}$.

The *multiscale profile* $\mathcal{P}_n(P)$ at point P is defined as the sequence of samples $(X_i, Y_i) = (log(i), log(\overline{L}^{h_i}))_{i=1..n}$ (see Fig. 3(a,b)). According to (4) (resp. (5)), if P is located on a curved (resp. flat) part, the slope of an affine approximation of the multiscale profile should be in $[-\frac{1}{2}, -\frac{1}{3}]$ (resp. $[-1, -\frac{1}{2}]$). Since for noisy contour parts the preceding properties are not valid, an invalid slope detects them directly. A threshold t_m is given to determine the *meaningful scale* defined as a pair (i_1, i_2), $1 \leq i_1 < i_2 \leq n$, such that for all i, $i_1 \leq i < i_2$, $\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} \leq t_m$. For the example of Fig. 3, the meaningful scales of the points P_1 and P_2 are respectively equal to $(1, 15)$ and $(3, 15)$.

The *noise level* $\nu(P)$ of a point P is the integer $i_1 - 1$, where (i_1, i_2) is the first meaningful scale at P. Experimentally the threshold value $t_m = 0$ gives best results both on curved or flat noisy parts. Fig. 4 shows some results obtained on various shapes. The noise detection appears to be well linked to the amount of noise, an is accurate and fast to compute.

Further details on maximal segments and their applications can be found in [11, 3, 12, 13, 9].

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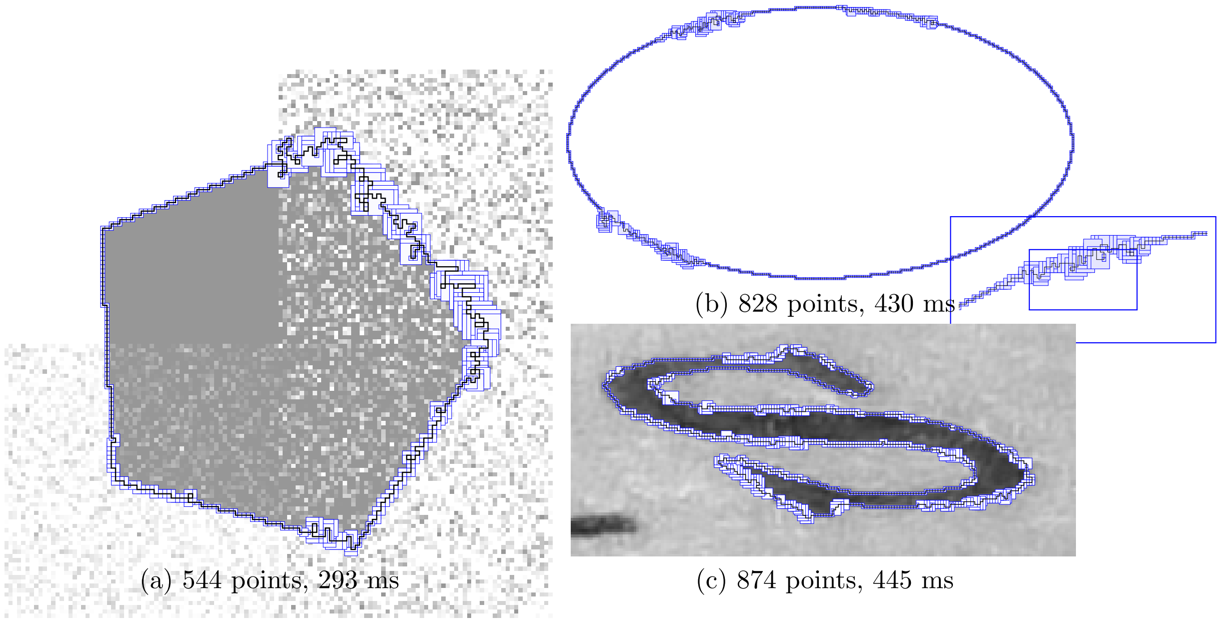

(a) 544 points, 293 ms

(b) 828 points, 430 ms

(c) 874 points, 445 ms

Figure 4: Noise detection obtained on various shapes (noise level locally represented by a centered box of size $\nu(P) + 1$). The contour in (a) is a thresholding of the background image (Gaussian noise of variances $\sigma = 0, 50, 100, 150$ added by quadrant). (b) Noise detection on a synthetic object with noise added locally to the curve. (c) Experiments on a photography of a letter. Timings obtained on an *Intel Pentium 4, 3GHz, 1Go* with a maximal scale n equal to 15.

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