Image restoration and segmentation using the Ambrosio-Tortorelli functional and discrete calculus

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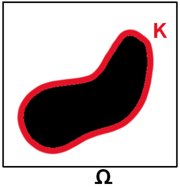
Mumford-Shah functional

[Mumford and Shah, 1989]

We minimize :

$$\mathcal{MS}(K,u) = \underbrace{\int_{\Omega\setminus K} |u-g|^2 \, \mathrm{dx}}_{\text{fidelity term}} + \alpha \underbrace{\int_{\Omega\setminus K} |\nabla u|^2 \, \mathrm{dx}}_{\text{smoothness term}} + \lambda \underbrace{\mathcal{H}^1(K\cap\Omega)}_{\text{discontinuities length}}$$

- Ω the image domain
- g the input image
- u a piecewise smooth approximation of g
- K the discontinuities set
- $\mathcal{H}^1$ the Hausdorff measure



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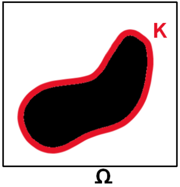

Ambrosio-Tortorelli functional

[Ambrosio and Tortorelli, 1992]

$$AT_\varepsilon(u,v) = \alpha \int_\Omega |u-g|^2 \, \mathrm{dx} + \int_\Omega v^2 |\nabla u|^2 \, \mathrm{dx} + \lambda \int_\Omega \varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} \frac{(1-v)^2}{4} \, \mathrm{dx}$$

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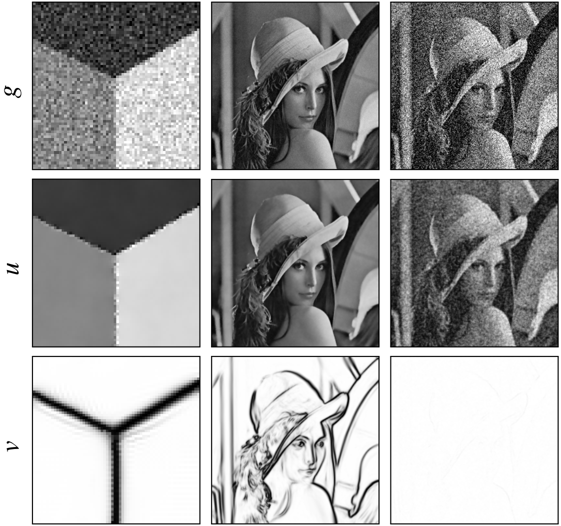
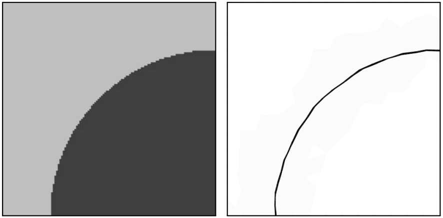
$$AT_\varepsilon(u,v) \xrightarrow{\Gamma} \mathcal{MS}$$

Finite differences implementation



Finite elements implementation

- Proposed in [Bourdin and Chambolle, 2000]



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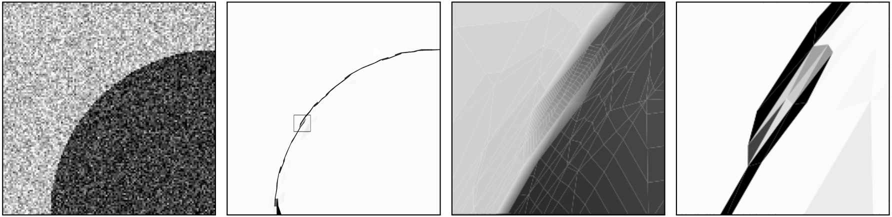

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- ▷ Proposed in [Bourdin and Chambolle, 2000]
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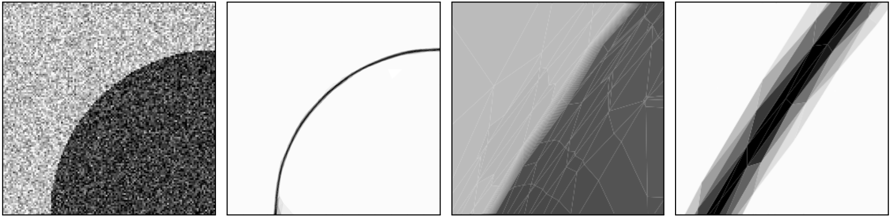

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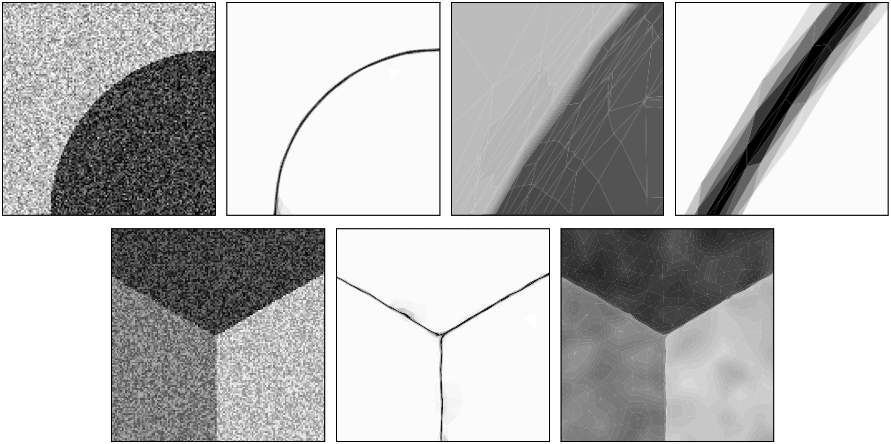
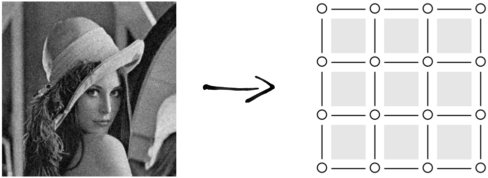


Discrete Calculus

Cell complex

- We decompose the image domain into a cell complexe
  - ▷ faces : pixels
  - ▷ edges : sides shared by 2 pixels
  - ▷ vertices : vertices shared by 4 pixels



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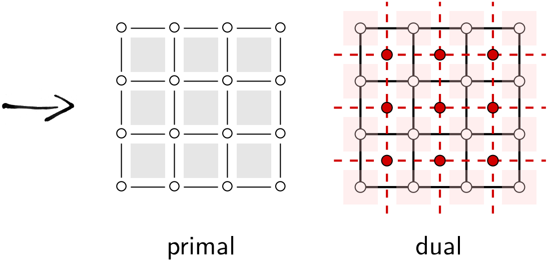
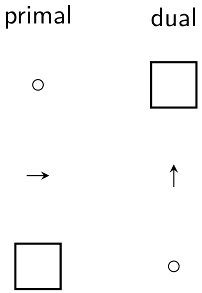
- Dual complex :
  - dual vertices : centers of primal faces
  - dual edges : edges orthogonal to primal edges
  - dual faces : delineated by dual vertices and edges

Discrete Calculus

Discrete operators

- **Discrete k-form** associates a scalar to a k-dimensional cell (represented by column vectors)



[Grady and Polimeni, 2010]

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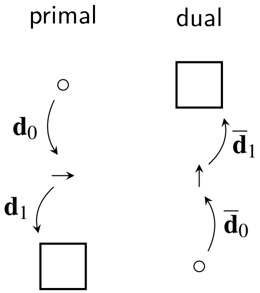

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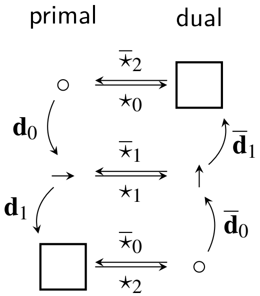

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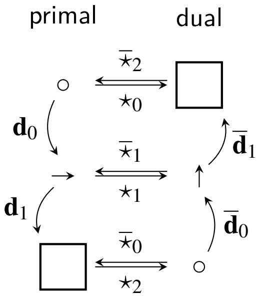
- $\mathbf{M}_{01} = \frac{1}{2}|\mathbf{d}_0|$ transforms a 0-form into a 1-form by averaging the values on the two edge extremities
- **"Edge Laplacian" :** $\bar{\star}_1 \bar{\mathbf{d}}_0 \star_2 \mathbf{d}_1 + \mathbf{d}_0 \bar{\star}_2 \bar{\mathbf{d}}_1 \star_1$



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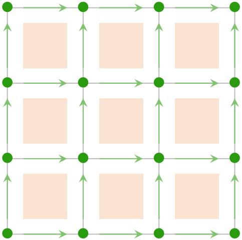
Discrete formulation of AT

On faces and vertices

$$AT_\varepsilon(u,v) = \alpha \int_\Omega |u-g|^2 \ \mathrm{dx} + \int_\Omega v^2|\nabla u|^2 \ \mathrm{dx} + \lambda \int_\Omega \varepsilon|\nabla v|^2 + \frac{1}{\varepsilon}\frac{(1-v)^2}{4} \ \mathrm{dx}$$



We choose :

- u, g to live on faces
  - ▷ u, g are 2-forms
- v to live on vertices •
  - ▷ v is a 0-form

Discrete formulation of AT

On faces and vertices

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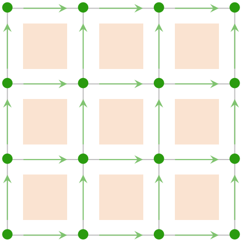

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$$\mathrm{AT}^{2,0}_\varepsilon(u,v) = \alpha \langle u-g, u-g\rangle_2 + \left\langle \mathbf{M}_{01} v \wedge \bar{\star}_1 \bar{\mathbf{d}}_0 \star_2 u, \mathbf{M}_{01} v \wedge \bar{\star}_1 \bar{\mathbf{d}}_0 \star_2 u \right\rangle_1 + \lambda \varepsilon \langle \mathbf{d}_0 v, \mathbf{d}_0 v\rangle_1 + \frac{\lambda}{4\varepsilon} \langle 1-v, 1-v\rangle_0$$

Discrete formulation of AT

On vertices and edges

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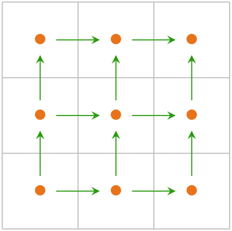

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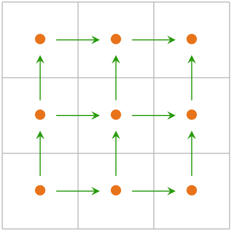

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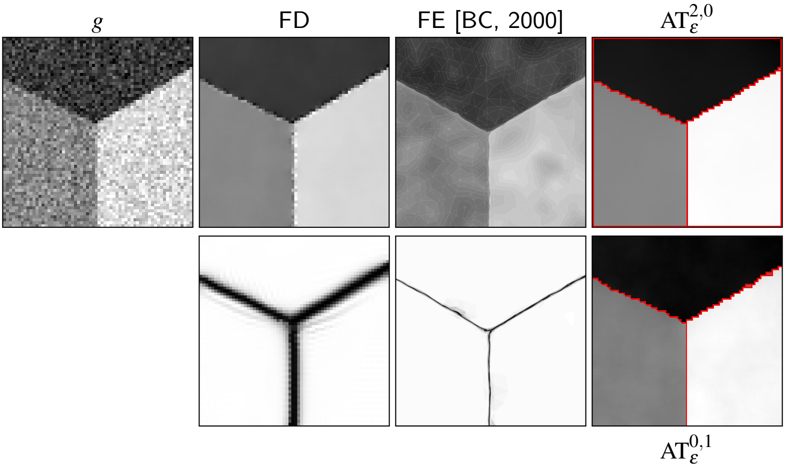
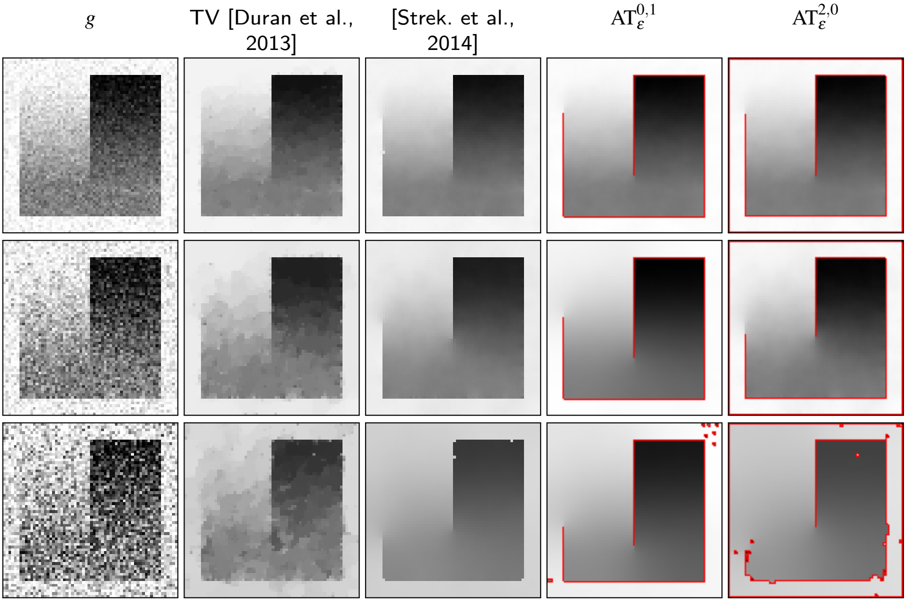
$$\mathrm{AT}_\varepsilon^{0,1}(u,v) = \alpha \langle u-g, u-g \rangle_0 + \langle v \wedge \mathbf{d}_0 u, v \wedge \mathbf{d}_0 u \rangle_1$$
$$+ \lambda \varepsilon \left\langle (\mathbf{d}_1 + \bar{\star}_2 \bar{\mathbf{d}}_1 \star_1) v, (\mathbf{d}_1 + \bar{\star}_2 \bar{\mathbf{d}}_1 \star_1) v \right\rangle_2 + \frac{\lambda}{4\varepsilon} \langle 1-v, 1-v \rangle_1$$


g
FD
FE [BC, 2000]
$AT_\varepsilon^{2,0}$
$AT_\varepsilon^{0,1}$


g
TV [Duran et al., 2013]
[Strek. et al., 2014]
$\mathrm{AT}_{\varepsilon}^{0,1}$
$\mathrm{AT}_{\varepsilon}^{2,0}$

g (PSNR = 26.1 dB)

u (PSNR = 29.9 dB)

$\mathrm{AT}_{\varepsilon}^{2,0}$

$\mathrm{AT}_{\varepsilon}^{0,1}$

u (PSNR = 29.36 dB)

TV

$\mathrm{AT}_{\varepsilon}^{2,0}$

g (PSNR = 20.23 dB) u (PSNR = 29.03 dB)

g (PSNR = 20.83 dB)

u (PSNR = 27.29 dB)

Conclusion

- Standard discretization schemes, such as finite differences or finite elements, lead to numerical difficulties
- Discrete calculus enables to reconstruct 1D contours and smooth regions

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Perspectives

- Add anisotrpy
- Work on non-image data
- Other applications : deblurring, tomography, 3D surface reconstruction, ...

Digital Geometry Tools and Algorithms - http://dgtal.org

Références I

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