# New properties for full convex sets and full convex hulls

Fabien Feschet[1] and Jacques-Olivier Lachaud[2*†]

[1]Université Clermont Auvergne, CNRS, ENSMSE, LIMOS, Clermont-Ferrand, F-63000, France.
0000-0001-5178-0842.
[2*]Université Savoie Mont Blanc, CNRS, LAMA, Chambéry, F-73000, France.
0000-0003-4236-2133.

*Corresponding author(s). E-mail(s): jacques-olivier.lachaud@univ-smb.fr;
Contributing authors: fabien.feschet@u-auvergne.fr;
†These authors contributed equally to this work.

**Abstract**

*Full convexity* has been recently proposed as an alternative definition of digital convexity in $\mathbb{Z}^n$. In contrast to classical definitions, fully convex sets are always connected and even simply connected whatever the dimension, while remaining digitally convex in the usual sense. In order to better understand the properties of full convexity, we present here two new and radically different characterizations of full convexity. The first one mimicks the usual continuous convexity via *segments* inclusion. We show an equivalence of full convexity with this segment convexity in dimensions $\mathbf{1}$ and $\mathbf{2}$, and counterexamples starting from dimension $\mathbf{3}$. If we now ask that the cells touched by all $\boldsymbol{d}$-simplices (instead of just $\mathbf{2}$-simplices aka segments) are within the cells touched by the digital set, we achieve an equivalence in arbitrary dimension $\boldsymbol{d}$. The second characterization is recursive with respect to the dimension, and relies on convexity of its axis-aligned projections. We provide several applications of these characterizations: the full convexity of balls and subsets of the hypercube, a natural measure of full convexity for digital sets, a new and faster algorithm to check the full convexity of digital sets. Finally, we study the main drawback of full convexity: its envelope operator may not be an increasing operator. We characterize fully convex sets that have anti-monotonous subsets, and we show that they must be thin in a precised sense.

**Keywords:** Digital geometry, Digital convexity, Full convexity, convex hull operator

## 1 Introduction

Convexity is a fundamental tool for analyzing functions and shapes. For digital spaces $\mathbb{Z}^d$, digital convexity was first defined as the intersection of real convex sets of $\mathbb{R}^d$ with $\mathbb{Z}^d$ (e.g. see survey of Ronse (1989)). Although easy to formulate, resulting digital convex sets may not be digitally connected in general. This deficiency prevents local shape analysis, so many works have tried to constrain the connectedness of such sets, for instance by relying on digital lines (Kim and Rosenfeld (1982b); Eckhardt (2001)) or on extensions of digital functions (see Kiselman (2004, 2022)). Most works are limited to 2D, and 3D extensions do not solve all geometric issues as shown by Kim and Rosenfeld (1982a). Webster (2001) defines digital convexity from the intersection of Euclidean convex sets with the lattice cubical complex, but it remains unclear how to determine if a given cell complex is indeed convex. We take an interest here in a recent alternative definition of digital convexity, called *full convexity* proposed in Lachaud (2021, 2022), which guarantees the connectedness and even the simple connectedness of fully convex sets. It is important to note that classical digital primitives like standard and naive lines and planes are indeed fully convex, so most of the classical tools of digital geometry fall into this setting.

Full convexity has already proven to be a fruitful framework for analyzing the geometry of digital shapes, as illustrated in Lachaud (2022); Feschet and Lachaud (2022, 2023): local characterization of convex, concave and planar parts, geodesics and visibility problems, reversible and tight reconstruction of triangulated surfaces, digital polyhedral models. The purpose of this paper is mainly to study its core properties and to exhibit new characterizations of full convexity. Such results are important both from a theoretical perspective (new characterizations lead to better understanding of what is digital convexity) and from a practical algorithmic point of view. For instance the characterization by idempotence of some cell operations (Feschet and Lachaud, 2023, Theorem 2) lead to an enveloppe operator that builds a fully convex hull. The first morphological characterization (Lachaud, 2021, Theorem 5) provides an exact algorithm to check full convexity involving $2^d - 1$ convex hull computations and lattice point enumeration; the second characterization in terms of maximal cells of (Feschet and Lachaud, 2023, Theorem 5) lead to an exact algorithm that requires only one convex hull computations and lattice point enumeration.

This article is an extension of Feschet and Lachaud (2024), with several improvements and additions. After recalling some essential background related to full convexity (Section 2), we study in Section 3 if we can define a digital analogue of convexity through its classical formulation of "inclusion of every straight segment". Originally studied by Minsky and Papert (1969), their definition was far too unrestrictive since it included many digital weird sets. We propose here a more reasonable analogue ($S$-convexity) that we prove equivalent to full convexity in $\mathbb{Z}$ and $\mathbb{Z}^2$, but not in higher dimension. We then extend this definition to an analogue of "inclusion of every $d$-simplices" ($S^d$-convexity) to get another characterization of full convexity in $\mathbb{Z}^d$. Compared to Feschet and Lachaud (2024), we have simplified a few lemmas and completed an ellipsis in the proof of Theorem 4. Then we propose in Section 4 a recursive convexity ($P$-convexity): a digital set is $P$-convex whenever it is digitally convex and each one of its projections along axis is $P$-convex. Quite surprisingly, we show that $P$-convexity is indeed equivalent to full convexity. This clearly opens new perspective for studying digital sets, and we provide in Section 5 two straightforward applications: a proof that digitized balls and subsets of the lattice hypercube are always fully convex, and a measure for digital sets that characterizes fully convex sets.

As extensions to Feschet and Lachaud (2024), we show in Section 6 that the $P$-convexity leads to the fastest algorithm for checking full convexity in practice, whatever the dimension and the properties of the digital set. It is also implemented in the DGtal library, in module Digital convexity, full convexity and P-convexity. Furthermore, in the prospect of bringing closer full convexity and continuous convexity, we study the monotonicity property of the fully convex enveloppe, proposed in Feschet and Lachaud (2022, 2023). Indeed, axiomatic approaches to convexity (e.g. see the book of van De Vel (1993), or Roy and Stell (2003) for a perspective more adapted to digital spaces) define convex structures through a hull operator, which gives the smallest convex set embracing a given set. Although the envelope operator $\mathrm{FC}(\cdot)$ of full convexity meets several axioms (like idempotence), it fails to be an increasing operator in some cases (meaning $X \subset Y$ implies $\mathrm{FC}(X) \subset \mathrm{FC}(Y)$). Section 7 studies in details the anti-monotonicity within subsets of fully convex sets, and shows that monotonicity is achieved for thick enough sets. Finally, we conclude and describe a few perspectives to this work in Section 8.

## 2 Useful definitions and properties

We introduce here basic definitions and properties needed in the rest of the paper (they can be found in Lachaud (2022); Feschet and Lachaud (2023)). In the sequel, $C^d$ is the cubical cell complex induced by $\mathbb{Z}^d$. Its 0-dimensional cells are identified to points of $\mathbb{Z}^d$. The set $C^d_k$ is the set of open $k$-dimensional cells of $C^d$.

The *(topological) boundary* $\partial Y$ of a subset $Y$ of $\mathbb{R}^d$ is the set of points in its closure but not in its interior. The star of a cell $\sigma$ in $C^d$, denoted by $\mathrm{Star}(\sigma)$, is the set of cells of $C^d$ whose boundary contains $\sigma$, plus the cell $\sigma$ itself. The closure $\mathrm{Cl}(\sigma)$ of $\sigma$ contains $\sigma$ and all the cells in its boundary. In this paper, the cell boundary operator, also denoted by $\partial$, maps a $k$-cell to all its proper faces, that is all its $k'$-cells, $0 \leqslant k' < k$, and not only its $(k-1)$-cells.

A subcomplex $K$ of $C^d$ with $\mathrm{Star}(K) = K$ is *open*, while being *closed* when $\mathrm{Cl}(K) = K$. The *body* of a subcomplex $K$, i.e. the union of its cells in $\mathbb{R}^d$, is written $\|K\|$.

For any real subset $Y$ of $\mathbb{R}^d$, we denote by $\bar{C}_k^d[Y]$ the set of $k$-cells whose topological closure intersects $Y$, i.e. $\bar{C}_k^d[Y] = \{c \in C_k^d, \bar{c} \cap Y \neq \emptyset\}$, where $\bar{c} = \|\mathrm{Cl}\,(c)\,\|$ for any cell $c$. For any subset $Y \subset \mathbb{R}^d$, it is natural to define $\mathrm{Star}\,(Y) := \bar{C}^d[Y]$. Last, the set $\mathrm{CvxH}\,(Y)$ is the *convex hull* of $Y$ in $\mathbb{R}^d$.

**Definition 1** (Full convexity)**.** *A non empty subset $X \subset \mathbb{Z}^d$ is* digitally $k$-convex *for $0 \leqslant k \leqslant d$ whenever*

$$\bar{C}_k^d[X] = \bar{C}_k^d[\mathrm{CvxH}\,(X)]. \tag{1}$$

*Subset $X$ is* fully (digitally) convex *if it is digitally $k$-convex for all $k, 0 \leqslant k \leqslant d$.*

The following two characterizations will be useful:

**Lemma 1** ((Feschet and Lachaud, 2022, Lemma 4))**.** *A digital set $X$ is fully convex iff* $\mathrm{Star}\,(X) = \mathrm{Star}\,(\mathrm{CvxH}\,(X))$.

**Lemma 2** ((Feschet and Lachaud, 2023, Theorem 2))**.** *A digital set $X$ is fully convex iff $X = \mathrm{FC}(X)$, with* $\mathrm{FC}(X) := \mathrm{Extr}\,(\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(X))))$.

The Skel operator builds the *skeleton of a set of cells*, which is defined as the intersection of all cell complexes whose star includes the set of cells. The *extreme operator* Extr maps a set of cells to their set of vertices, which is a digital set. The skeleton can be characterized as follows.

**Lemma 3** ((Feschet and Lachaud, 2023, Lemma 12))**.** *Let us consider $Y \subset \mathbb{R}^d$. For any $c \in C^d$, $c \in$* $\mathrm{Skel}\,(\mathrm{Star}\,(Y)) \iff \|c\| \cap Y \neq \emptyset$ *and* $\|\partial c\| \cap Y = \emptyset$.

For a cell $c \in C^d$, we say that a convex set $K$ is *framed* within $c$ if $K \cap \|c\| \neq \emptyset$ and $K \cap \|\partial c\| = \emptyset$. So, considering Lemma 3 applied to $\mathrm{CvxH}\,(X)$, $c \in C^d$ belongs to $\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(X)))$ iff $\|c\| \cap \mathrm{CvxH}\,(X) \neq \emptyset$ and $\|\partial c\| \cap \mathrm{CvxH}\,(X) = \emptyset$. In other words, a cell $c$ is in the skeleton of a convex hull if and only if the convex hull is framed within $c$. Note that if the dimension of $c$ is zero, i.e. $c$ is a lattice point, then $\partial c$ is empty, so a convex set is framed within a lattice point if and only if this convex set contains it.

Let us remark that for any set $K$, $\mathrm{FC}(K)$ is the union of $\mathrm{CvxH}(K) \cap \mathbb{Z}^d$ and, using the Extr() operator, of $\bigcup_{c \in \mathrm{Skel}^+(\mathrm{Star}(\mathrm{CvxH}(K)))} \mathrm{Extr}\,(c)$ (denoted by $\mathrm{Ex}\,(K)$), where $\mathrm{Skel}^+\,(\mathrm{Star}\,(\mathrm{CvxH}\,(K)))$ is the set of cells of positive dimension belonging to $\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(K)))$. Those two sets are necessarily disjoint. Indeed by Lemma 3, if $c \in C^d$ belongs to $\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(K)))$ we must have $\|\partial c\| \cap \mathrm{CvxH}\,(K) = \emptyset$ and so in particular $\|\mathrm{Extr}\,(c)\,\| \cap \mathrm{CvxH}\,(K) = \emptyset$ when $\dim c > 0$. We will write $\mathrm{FC}(K) = \left(\mathrm{CvxH}\,(K) \cap \mathbb{Z}^d\right) \bigsqcup \mathrm{Ex}\,(K)$ for the disjoint union.

As $\mathrm{CvxH}\,(\|\partial c\|)$ is the topological closure of $\mathrm{CvxH}\,(\|c\|) = \|c\|$, if $K$ is not framed within $c$, then $K$ must either intersect only the boundary of $c$ or $c$ and its boundary. Moreover if $K$ is closed, then $K \cap \|c\|$ is closed. The framed properties cannot happen when $K$ is the convex hull of points in $\mathbb{Z}^d$ and $\dim c = d$. Indeed, if $K$ is framed within $c$ then obviously $K$ is entirely included in $\|c\|$. But this is impossible since $\|c\|$ does not contain any points. So, to have the framed property, we must have $\dim c \leq d - 1$.

Finite convex sets $Y$ in $\mathbb{R}^d$ are intended as $Y = \mathrm{CvxH}\,(\mathrm{Extrm}\,(Y))$ where $\mathrm{Extrm}\,(Y)$ is the set of extreme points of $Y$. For all finite subsets $Z$ of $\mathbb{Z}^d$, $\mathrm{CvxH}\,(Z)$ is a bounded convex set.

## 3 Segment convexity and generalizations

In the Euclidean space, convexity is defined through inclusion of every straight line segment joining two points of the set. Minsky and Papert (1969), in their famous book on perceptrons, proposed a digital analogue of segment convexity, phrased "A [digital] set $X$ fails to be convex if and only if there exists three [digital] points such that $q$ is in the line segment joining $p$ and $r$ and, $p \in X$, $q \notin X$, $r \in X$." This definition of digital convexity is unfortunately not at all equivalent to the digital convexity (see Figure 1), and even less to full convexity.

Therefore we propose the following digital analogues of "segment" convexity, which are much closer to digital convexity.

**Definition 2** ($S$-convexity and $S^k$-convexity)**.** *We say that a* finite *digital set $X \subset \mathbb{Z}^d$ is* $S$-convex *whenever $\forall p \in X, \forall q \in X$,* $\mathrm{Star}\,(\mathrm{CvxH}\,(\{p, q\})) \subset \mathrm{Star}\,(X)$. *Furthermore, for $k \geq 2$, the set $X$ is* $S^k$-convex *whenever for any $k$-tuple of points $T$ of $X$ (not necessarily distinct), we have* $\mathrm{Star}\,(\mathrm{CvxH}\,(T)) \subset \mathrm{Star}\,(X)$.

Otherwise said for $S$-convexity (resp. $S^k$-convexity), any pair of points of $X$ (resp. any $k$-tuple of points of $X$) must be tangent to $X$ in the terminology of Lachaud (2022). It is obvious that $S^2$-convexity is the $S$-convexity and that $S^{k+1}$-convexity implies $S^k$-convexity. We establish the following results in this section.

**Theorem 1.** *For $d \geqslant 1$, $k \geqslant 2$, full convexity implies $S^k$-convexity.*

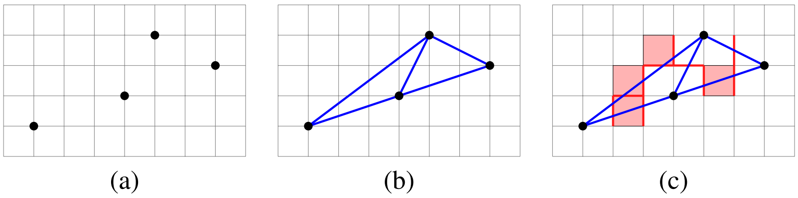



**Fig. 1** Minsky-Papert segment convexity versus $S$-convexity: (a) MP-convex set $X$, since (b) each segment does not touch any other lattice point. But $X$ is not $S$-convex, since (c) these segments touch 1-cells and 2-cells that are not in $\mathrm{Star}(X)$ (in red).

*Proof.* Let us consider a fully convex set $X$. If we consider a $k$-tuple $T$ in $X$ then $\mathrm{CvxH}(T) \subset \mathrm{CvxH}(X)$ because the convex hull operator $\mathrm{CvxH}()$ is increasing. But $\mathrm{Star}()$ is also an increasing operator hence $\mathrm{Star}(\mathrm{CvxH}(T)) \subset \mathrm{Star}(\mathrm{CvxH}(X)) = \mathrm{Star}(X)$ (the latter equality given by Lemma 1). So $X$ is $S^k$-convex. □

**Theorem 2.** *$S$-convexity is equivalent to full convexity in $\mathbb{Z}^1$ and $\mathbb{Z}^2$.*

**Theorem 3.** *$S$-convexity is not equivalent to full convexity starting from $\mathbb{Z}^3$.*

**Theorem 4.** *$S^d$-convexity is equivalent to full convexity in $\mathbb{Z}^d$, for $d \geqslant 2$.*

Some preliminary lemmas will be used to extract impossible configurations for the $S^k$-convexity. With such situations, we are then able to relate full convexity and $S^k$-convexity. Theorem 3 is proven by a counter-example.

**Lemma 4** (Grid lemma)**.** *Let us consider a finite $S^k$-convex digital set $X \subset \mathbb{Z}^d$ with $k \geqslant 2$. Let us denote by $\mathbb{Z}[e_j]$ any line in $\mathbb{Z}^d$ directed by the canonical basis vector $e_j$. Then, $\mathbb{Z}[e_j] \cap X$ is digitally connected.*

*Proof.* Let $\mathbb{Z}[e_j]$ be any line in $\mathbb{Z}^d$ directed by the canonical basis vector $e_j$. If $\mathbb{Z}[e_j] \cap X$ is empty, then it is trivially connected. Otherwise, let $p$ and $q$ be two points of this set. Since $p$ and $q$ belong to a 1-dimensional line, $\mathrm{Star}(\mathrm{CvxH}(\{p,q\})) \cap \mathbb{Z}^d$ is a connected digital sequence of points, hence a connected path $P$ from $p$ to $q$, which also belongs to $\mathbb{Z}[e_j]$. Since $S^k$-convexity implies $S^2$-convexity, the $S^2$-convexity itself induces that

$$\mathrm{Star}(\mathrm{CvxH}(\{p,q\})) \subset \mathrm{Star}(X),$$

which implies by intersection that:

$$\mathrm{Star}(\mathrm{CvxH}(\{p,q\})) \cap \mathbb{Z}^d \cap \mathbb{Z}[e_j] \subset \mathrm{Star}(X) \cap \mathbb{Z}^d \cap \mathbb{Z}[e_j].$$

The left-hand side is equal to $P$, the right hand side is equal to $X \cap \mathbb{Z}[e_j]$ (from $\mathrm{Star}(X) \cap \mathbb{Z}^d = X$). We achieve then $P \subset X \cap \mathbb{Z}[e_j]$. We have just proved that for any two points of $X \cap \mathbb{Z}[e_j]$, there is a connected path between them within this set. □

**Lemma 5** (Star lemma)**.** *Let us consider a finite $S^d$-convex digital set $X \subset \mathbb{Z}^d$. Then $\forall c \in C^d$, $\dim c > 0$, if $\mathrm{CvxH}(X)$ is framed within $c$ then $\exists e \in \partial c$, $\dim e = 0, e \in X$.*

*Proof.* Since $\mathrm{CvxH}(X) \cap c \neq \emptyset$ but $\mathrm{CvxH}(X) \cap \partial c = \emptyset$, it follows that $\partial \mathrm{CvxH}(X) \cap c \neq \emptyset$. Since $X$ is finite, $\mathrm{CvxH}(X)$ has a simplicial subdivision. It induces a simplicial subdivision $T$ on $\partial\mathrm{CvxH}(X)$, whose faces have maximal dimension $d-1$. Since $\partial\mathrm{CvxH}(X) \cap c \neq \emptyset$, there is a face $\sigma$ of $T$ that touches $c$. From the simplicial subdivision, $\sigma$ is the convex hull of a subset $A$ of $X$, with $d$ elements or less. By $S^d$-convexity, we have $\mathrm{Star}(\mathrm{CvxH}(A)) \subset \mathrm{Star}(X)$. But $\sigma \subset \|\mathrm{Star}(\sigma)\| = \|\mathrm{Star}(\mathrm{CvxH}(A))\|$ and $\sigma$ touches $c$. So we have $c \cap \|\mathrm{Star}(X)\| \neq \emptyset$. Since $c$ is an element of $C^d$ and $\mathrm{Star}(X)$ is a union of cells of $C^d$. It follows that $c \in \mathrm{Star}(X)$. By definition the 0-cells of $\mathrm{Star}(X)$ form the set $X$ itself, and every cell of $\mathrm{Star}(X)$ has a 0-cell of $\mathrm{Star}(X)$ in its boundary. So the cell $c$ touches a 0-cell of $\mathrm{Star}(X)$, hence a 0-cell $e$ belonging to $X$. □

**Lemma 6.** *Let us consider a digital set $X \subset \mathbb{Z}^d$. If $X$ is $S^d$-convex then $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$.*

*Proof.* Let us consider an $S^d$-convex digital set $X \subset \mathbb{Z}^d$. Let us consider a cell $c$ in $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$. Suppose that $\dim c > 0$. From Lemma 3, this means that $\mathrm{CvxH}(X)$ is framed within $c$. Using now Lemma 5 since $X$ is $S^d$-convex, $\exists e \in \partial c$, $\dim e = 0, e \in X$. This implies that $\mathrm{Star}(c) \subset \mathrm{Star}(e)$. Hence, $c$ cannot belong to $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$, but $e$ does. So we got a contradiction with $\|\partial c\| \cap \mathrm{CvxH}(X) = \emptyset$. It follows that $c$ must be a 0-dimensional cell, that is a point. This implies that the skeleton of $\mathrm{Star}(\mathrm{CvxH}(X))$ only contains points such that $\mathrm{Extr}(\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))) = \mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$. But since we only have 0-dimensional cells in the skeleton, we get $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X))) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. We conclude that $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. □

*Proof of Theorem 2.* Using Theorem 1, we only study the case of an $S$-convex set.

Let us consider an $S$-convex set $X$ in $\mathbb{Z}^1$. Using Lemma 4, we have that $X$ must be an interval of points. Hence $X$ is fully convex because $\mathrm{Star}(\mathrm{CvxH}(X)) = \mathrm{Star}(X)$.

The case of an $S$-convex set $X$ in $\mathbb{Z}^2$ is a corollary of Theorem 4, since $S$-convexity is $S^2$-convexity. □

*Proof of Theorem 3.* A counter-example is given on Fig. 2. We should note that large random constructions of $S^2$-convex sets by simulation did not lead to any counter-examples. In fact, problematic examples correspond to sets for which there exists an integer point in the relative interior of a maximal face which does not belong to any segments of the face, and are thus very unlikely to be generated randomly. □

The main result in dimension 2 for $S$-convexity is that for an $S$-convex set $X$, we necessarily have $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^2$. As Theorem 3 states it, this property failed to be true in higher dimension. This explains why we must rely on the more restrictive $S^d$-convexity when increasing the dimension, and the main part of the proof below consists in showing that $S^d$-convexity implies 0-convexity.

*Proof of Theorem 4.* Let us consider a finite $S^d$-convex set $X$. Its convex hull $\mathrm{CvxH}(X)$ can be decomposed into a simplicial complex $T$, which is a collection of $N$ $d$-dimensional simplices $\{C_i\}_{i=1,\dots,N}$ in $\mathbb{Z}^d$. Note that each simplex $C_i$ of $T$ has its $d+1$ vertices in $X$. Let us show that $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. Suppose that $(\mathrm{CvxH}(X) \cap \mathbb{Z}^d) \setminus X$ is not empty and let $z$ be a lattice point in this set. If $z$ belongs to a $k$-dimensional face of $T$, with vertices $V$ and $\#(V) < d+1$, then the $S^d$-convexity tells that $\mathrm{Star}(\mathrm{CvxH}(T)) \subset \mathrm{Star}(X)$. It follows that $z \in \mathrm{Star}(X)$, but also $z \in X$ since $z$ is a lattice point. This is a contradiction.

Therefore $z$ must belong to the interior of a $d+1$-dimensional facet $F$ of $T$. We can consider an arbitrary axis-aligned line $\mathbb{Z}[e_j]$ going through $z$. By convexity of $F$, the straight line $\mathbb{Z}[e_j]$ must cross at least two $d-1$-dimensional faces $F^-$ and $F^+$ of $F$ at respective Euclidean points $x^-$ and $x^+$. Point $x^-$ either is a lattice point $z^-$ (different from $z$) or belongs to a 1-cell $c^-$ of $C^d$. In the latter case, by $S^d$-convexity, $c^- \in \mathrm{Star}(F^-) \subset \mathrm{Star}(X)$ so $c^-$ has at least one of its two boundary lattice point which belongs to $X$. We denote it $z^-$ too. The same reasoning can be applied to face $F^+$, and we build a lattice point $z^+$ which belongs to $X$ and to $\mathbb{Z}[e_j]$ too. Note that the 1-cell $c^-$ is necessary different from the 1-cell $c^+$ and that $z$ must lie somewhere between them.

We apply now Lemma 4 on the line $\mathbb{Z}[e_j]$. It tells that $\mathbb{Z}[e_j] \cap X$ is digitally connected. From the above construction, $\mathbb{Z}[e_j] \cap X$ is not empty since it contains at least $z^-$ and $z^+$. Since the lattice point $z$ lies in the interior of facet $F$, this point is strictly in-between $z^-$ and $z^+$. $\mathbb{Z}[e_j] \cap X$ being digitally connected, every point in-between $z^-$ and $z^+$ must also belong to $X$, so $z \in X$, which is a contradiction.

So, $\left(\mathrm{CvxH}(X) \cap \mathbb{Z}^d\right) \setminus X = \emptyset$, otherwise said $\mathrm{CvxH}(X) \cap \mathbb{Z}^d = X$, that is $X$ is 0-convex.

Applying Lemma 6, we get that $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$ too. Gathering the two equalities, we obtain $\mathrm{FC}(X) = X$, which means that $X$ is fully convex from Lemma 2 (characterization of full convexity with $\mathrm{FC}(\cdot)$ operator). □

## 4 Projection convexity

We here study the stability of fully convex sets with respect to orthogonal projections along axes in $\mathbb{R}^d$. We denote by $\pi_j$ the orthogonal projector associated to the $j$-th axis, which consists in omitting the $j$-th coordinates for all points of $\mathbb{Z}^d$. By direct extension, those projectors are defined for cells in $C^d$. $\pi_j$ are called *axis projectors*. Those projectors share the property that the image of a cell $c \in C^d$ is a cell in $C^{d-1}$ and those projections are the only projections for which this property is true. Moreover, the image of a point in $\mathbb{Z}^d$ by any axis projector is a point in $\mathbb{Z}^{d-1}$. Let us define a convexity by projections as follows.

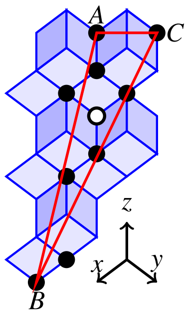
**Definition 3** ($P$-convexity)**.** *Let $X \subset \mathbb{Z}^d$ be a digital set. The set $X$ is $P$-convex if and only if $X$ is digitally 0-convex (i.e. $\mathrm{CvxH}(X) \cap \mathbb{Z}^d = X$) and when $d > 1$, for any $j$, $1 \leqslant j \leqslant d$, $\pi_j(X)$ is $P$-convex in $\mathbb{Z}^{d-1}$.*

Quite surprisingly, we have the equivalence of $P$-convexity with full convexity.

**Theorem 5.** *For arbitrary dimension $d \geqslant 1$, for any $X \subset \mathbb{Z}^d$, $X$ is fully convex if and only if $X$ is $P$-convex.*

*Proof.* The fact that a fully convex $X$ is also $P$-convex directly follows from (i) fully convex sets are in particular digitally 0-convex, (ii) projection $\pi_j(X)$ are fully convex in $\mathbb{Z}^{d-1}$ as shown in (Feschet and Lachaud, 2023, Lemma 23).

When $d = 1$, 0-convexity is equivalent to full convexity (consequence of (Lachaud, 2022, Lemma 4) with $d = 1$), so $P$-convexity implies full convexity for




This is piece of the standard digital plane $P = \{(x, y, z) \in \mathbb{Z}^3, 0 \leqslant x + y + 2z < 4\}$. The set $X$, represented as black disks, is a subset of $P$. The set $Y$ is the union of $X$ with the point $M = (1, 1, -1)$ represented as a white disk. We have $A = (0, 0, 0), B = (4, 2, -3), C = (-1, 1, 0)$. All four points $A, B, C, M$ have remainder 0 in the digital plane. One can check that $M = \frac{1}{3}(A + B + C)$, hence $M \in \mathrm{CvxH}(X)$. But $M$ does not belong to any straight segment between any pair of points of $X$.

**Fig. 2** Counter-example to $S^2$-convexity implies full convexity: set $Y$ is $S^2$-convex and fully convex, while $X$ is $S^2$-convex but not fully convex (and not digitally 0-convex).

this dimension. Let us now show that this implication holds for $d > 1$.

Suppose that $X \subset \mathbb{Z}^d$ is $P$-convex but not fully convex. Since $X$ is 0-convex by definition, we know that $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. So, any 0-dimensional cell of $X$ is in the skeleton $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$ and $\mathrm{CvxH}(X)$ does not contain any other points. So since the fact that $X$ is not fully convex implies that $X \neq \mathrm{FC}(X)$, there exists some cell $c$ in $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$ with $\dim c > 0$. Indeed, the extreme operator only add points for cells of strictly positive dimension. We can characterize $c$ by the framed property: $\mathrm{CvxH}(X) \cap \|c\| \neq \emptyset$ and $\mathrm{CvxH}(X) \cap \|\partial c\| = \emptyset$.

Let $y$ be some point of $\mathrm{CvxH}(X) \cap \|c\|$, which is also not in $X$ since it is not a lattice point. Being in $\mathrm{CvxH}(X)$, by Carathéodory's convexity theorem, there exists at most $d + 1$ extreme points $v_0, v_1, \ldots, v_d$ of $\mathrm{CvxH}(X)$ such that $y$ is a convex linear combination of these points. We thus have $y = \sum_{i=0}^d \lambda_i v_i$, with $\sum_{i=0}^d \lambda_i = 1$ and $\forall i, 0 \leqslant i \leqslant d, \lambda_i \geq 0$. Being extreme points of $\mathrm{CvxH}(X)$, every $v_i$ is a lattice point in $X$.

Let $k := \dim c$. There exists $k$ different directions $J := (j_i)_{i=1,\ldots,k}$ such that $\dim(\pi_{j_i}(c)) = k - 1$. Let $\pi_J$ be the composition of the projections $\pi_{j_1}, \ldots, \pi_{j_k}$ (the order is not important since these operators commute). It follows that $\pi_J(c)$ is a lattice point, say $z$. Since $y \in c$, we have also $z = \pi_J(y)$. It follows that:

$$z = \pi_J\left(\sum_{i=0}^{d} \lambda_i v_i\right) = \sum_{i=0}^{d} \lambda_i(\pi_J(v_i)). \quad \text{(by linearity of } \pi_j)$$

Since every $v_i \in X$, we have shown that $z \in \mathrm{CvxH}(\pi_J(X))$ and $z \in \mathbb{Z}^{d-k}$. Since $X$ is P-convex, its projection $\pi_J(X)$ is 0-convex, so $\mathrm{CvxH}(\pi_J(X)) \cap \mathbb{Z}^{d-k} = \pi_J(X)$ and $z \in \pi_J(X)$.

The last assertion means that there exist a lattice point $x \in X$, such that $z = \pi_J(x)$. Let $C := \pi_J^{-1}(z)$ be the affine $k$-dimensional space containing $z$. It contains in particular $c$, its boundary $\partial c$, the point $y$ and the lattice point $x$. We have $y \in c$ while $x$ cannot be in $c$ since it is a lattice point. Furthermore $\mathrm{CvxH}(X) \cap \|\partial c\| = \emptyset$ implies also $x \notin \partial c$ (because $x \in X$). Now $y \in \mathrm{CvxH}(X)$ by definition, $x \in X \subset \mathrm{CvxH}(X)$, so the straight segment $[y; x]$ must be included in $\mathrm{CvxH}(X)$. It is also included in the $k$-dimensional space $C$ so it is a connected path from the interior of cell $c$ to the exterior of $c$ in $C$: it must cross $\partial c$ at some point $x'$. By convexity we have $x' \in [y; x] = \mathrm{CvxH}(\{x, y\}) \subset \mathrm{CvxH}(X)$. But we have also $x' \in \partial c$ and $\mathrm{CvxH}(X) \cap \|\partial c\| = \emptyset$. This is a contradiction.

Hence $X = FC(X)$ which is equivalent to $X$ fully convex. □

# 5 Applications

We present here two quite immediate applications of the previous characterization of fully convex sets.

## 5.1 New fully convex digital sets

**Proposition 1.** *Let $A$ be a digital set with bounding box defined by a lowest point* $\mathbf{p}$ *and a highest point* $\mathbf{q}$, *with $\forall i, 1 \leqslant i \leqslant d, |q^i - p^i| \leqslant 1$. Then $A$ is fully convex.*

*Proof.* If $A$ is empty then the conclusion holds. In dimension 1, it is clear that any subset of the digital set $\{x, x + 1\}$ is 0-convex hence $P$-convex. Assuming now that the property holds for dimension $d - 1$, let us prove it for $A \subset \mathbb{Z}^d$. Note first that any subset of the hypercube $H$ defined by $\mathbf{p}$ and $\mathbf{q}$ is digitally 0-convex, since any vertex of $\mathrm{CvxH}(A)$ must belong to $A$ since it is a vertex of $H$ too. Each projection $\pi_j(A)$ is also a non-empty subset of a $d - 1$-hypercube $\pi_j(H)$, and is thus $P$-convex by induction hypothesis. The conclusion follows from the equivalence of $P$-convexity with full convexity (Theorem 5). □

A *digital ball of* $\mathbb{Z}^d$ is the intersection of *any* Euclidean $d$-dimensional ball with $\mathbb{Z}^d$. Note that the center of the ball may by any Euclidean point of $\mathbb{R}^d$ and the radius may be any real non negative value.

**Proposition 2.** *Any digital ball of* $\mathbb{Z}^d$ *is fully convex.*

*Proof.* We show that this is true by induction on the dimension $d$. For $d = 1$, full convexity is equivalent to 0-convexity, and a 1-dimensional digital ball is 0-convex. Let us assume that digital balls are fully convex for dimension $d-1$, $d \geq 2$, and let us prove that this assertion is true for dimension $d$.

By Theorem 5, it is equivalent to show that $d$-dimensional digital balls are $P$-convex. Let $X$ be some digital ball of center $\mathbf{c} \in \mathbb{R}^d$ and radius $r$, i.e. $X = B_r(\mathbf{c}) \cap \mathbb{Z}^d$. First of all, $X$ is 0-convex since it is the intersection of a real convex set with the grid $\mathbb{Z}^d$. We have to show that, for any axis direction $j$, $1 \leqslant j \leqslant d$, $\pi_j(X)$ is $P$-convex.

We write the proof for $j = d$ for simplicity of writings, but the proof is the same for the other directions. The main argument is that the projection of a digital ball is itself a $d-1$-digital ball but possibly with a slightly lower radius.

For $\mathbf{x} \in X$, we have $\|\pi_d(\mathbf{x}) - \pi_d(\mathbf{c})\|^2 = \|\mathbf{x} - \mathbf{c}\|^2 - |x_d - c_d|^2 \leq r^2 - |x_d - c_d|^2$, where $x_d$ and $c_d$ are the $d$-th coordinate of their respective point. It is obvious that, for any $z \in \mathbb{Z}$, $|z - c_d| \geq |c_d - \lfloor c_d \rceil| =: \alpha$, where $\lfloor \cdot \rceil$ is the round operator. It follows that $\|\pi_d(\mathbf{x}) - \pi_d(\mathbf{c})\|^2 \leq r^2 - \alpha^2 =: \rho^2$. We have just shown that $\pi_d(X) \subset B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1}$.

Reciprocally, let us now pick a point $\mathbf{y} \in B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1}$. It follows that $\|\mathbf{y} - \pi_d(\mathbf{c})\|^2 \leq r^2 - \alpha^2$. Let us build a $d$-dimensional lattice point $\mathbf{z}$ as $\mathbf{z} = (y_1, \ldots, y_{d-1}, \lfloor c_d \rceil)$. We have:

$$
\begin{aligned}
\|\mathbf{z} - \mathbf{c}\|^2 &= \|\mathbf{y} - \pi_d(\mathbf{c})\|^2 + \mid \lfloor c_d \rceil - c_d \mid^2 \\
&\leqslant r^2 - \alpha^2 + \alpha^2 = r^2.
\end{aligned}
$$

This proves that $\mathbf{z} \in X$. Since $\pi_d(\mathbf{z}) = \mathbf{y}$, it holds that $\mathbf{y} \in \pi_d(X)$. It follows that $B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1} \subset \pi_d(X)$.

So $\pi_d(x)$ is a $d-1$-dimensional digital ball, hence is fully convex or, equivalently, $P$-convex. Since all projections are $P$-convex, it holds that $X$ is $P$-convex or, equivalently, fully convex. □

Note that the argument does not work for an arbitrary ellipsoid since some projections of digital ellipsoids might not be digital ellipsoids: this is due to possible missing points when the ellipsoid is too thin.

## 5.2 A progressive measure for full convexity

Sometimes it is useful to quantify a property over a set in a progressive manner. For instance there exists measures of circularity, convexity, straightness, disconnectedness, and so on (e.g. Zunić and Rosin (2004); Zunić et al. (2010, 2018)). We would like here to define a *full convexity measure* over a digital set, that has value exactly 1 for fully convex sets, while decreasing to zero as the digital set looks less and less like a fully convex set.

Let $M_d(A)$ be any $d$-dimensional digital convexity measure of digital set $A$. A choice could be for instance for finite sets:

$$
M_d(A) := \frac{\#(A)}{\#(\mathrm{CvxH}(A) \cap \mathbb{Z}^d)}, \quad M_d(\emptyset) = 1. \quad (2)
$$

The *full convexity measure* $M_d^F$ for $A \subset \mathbb{Z}^d$, $A$ finite, is then:

$$
\begin{aligned}
M_1^F(A) &:= M_1(A) && \text{for } d = 1, && (3)\\
M_d^F(A) &:= M_d(A) \prod_{k=1}^{d} M_{d-1}^F(\pi_k(A)) && \text{for } d > 1. && (4)
\end{aligned}
$$

It coincides with the digital convexity measure in dimension 1, but may differ starting from dimension 2.

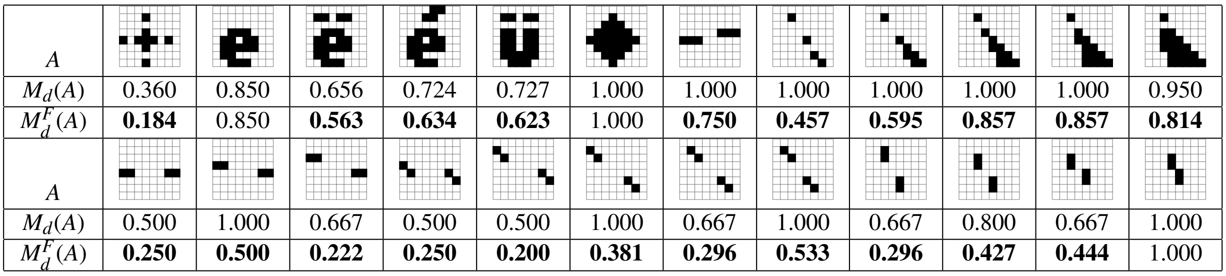
**Theorem 6.** *Let* $A \subset \mathbb{Z}^d$ *finite. Then* $M_d^F(A) = 1$ *if and only if* $A$ *is fully convex and* $0 < M_d^F(A) < 1$ *otherwise. Besides* $M_d^F(A) \leqslant M_d(A)$ *in all cases.*

*Proof.* Immediate from the equivalence of $P$-convexity with full convexity. □

Figure 3 illustrates the links and the differences between the two convexity measures $M_d$ and $M_d^F$ on simple 2D examples. As one can see, the usual convexity measure may not detect disconnectedness, is sensitive to specific alignments of pixels, while full convexity is globally more stable to perturbation and is never 1 when sets are disconnected.

# 6 Efficient computation of full convexity

There exists now multiple equivalent characterizations of full convexity. Some of them induce algorithms for



| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | | | | | | | | | | | | |
| $M_d(A)$ | 0.360 | 0.850 | 0.656 | 0.724 | 0.727 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.950 |
| $M_d^F(A)$ | **0.184** | 0.850 | **0.563** | **0.634** | **0.623** | 1.000 | **0.750** | **0.457** | **0.595** | **0.857** | **0.857** | **0.814** |
| $A$ | | | | | | | | | | | | |
| $M_d(A)$ | 0.500 | 1.000 | 0.667 | 0.500 | 0.500 | 1.000 | 0.667 | 1.000 | 0.667 | 0.800 | 0.667 | 1.000 |
| $M_d^F(A)$ | **0.250** | **0.500** | **0.222** | **0.250** | **0.200** | **0.381** | **0.296** | **0.533** | **0.296** | **0.427** | **0.444** | 1.000 |

**Fig. 3** Common points and differences of convexity measure $M_d$ and full convexity measure $M_d^F$ on small 2D digital sets.

checking if a digital set is indeed fully convex, but the question "which is the fastest way" remains. Let us recap the different characterizations for $X \subset \mathbb{Z}^d$ being full convex:

*digital k-convexities,* $0 \leqslant k \leqslant d$ (Lachaud, 2021, Def. 1). $\forall k, 0 \leqslant k \leqslant d$, it must hold that $\bar{C}_k^d[X] = \bar{C}_k^d[\mathrm{CvxH}(X)]$: the convex hull of $X$ must intersect the same *cells* of the lattice grid as $X$ does. It is the original definition and a natural extension to classical digital convexity, which consider only 0-cells. This method is not implemented directly, since it requires to compute intersections with cell.
*restricted digital k-convexities,* $0 \leqslant k < d$ (Lemma 4 of the same paper). $\forall k, 0 \leqslant k < d$, it must hold that $\bar{C}_k^d[X] = \bar{C}_k^d[\mathrm{CvxH}(X)]$. It is unnecessary to check intersection with $d$-cells, being implied by the others.
*morphological characterization* (Lachaud, 2022, Th. 5). Denoting $\mathcal{S}_i := \{t\mathbf{e}_i, t \in [0,1]\}$ the unit segments along each axis $i$, and, for every subset of directions $\alpha \subset \{1,\ldots,d\}$, denoting $\mathcal{S}_\alpha := \oplus_{i\in\alpha}\mathcal{S}_i$ all their possible Minkowski sums, it must hold that $(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{CvxH}(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d$. This definition is used for full convexity computations in low dimension for simple digital sets (like a segment or a simplex), where it is easy to compute explicitely the Minkowski sum.
*discrete morphological characterization* (Lachaud, 2022, Th. 6). Denoting now $U_\emptyset(X) := X$, and for any subset of directions $\alpha$ and a direction $i \in \alpha$, $U_\alpha(X) := U_{\alpha\setminus\{i\}}(X) \cup \mathbf{e}_i(U_{\alpha\setminus\{i\}}(X))$ (the later being the unit translation of the set along direction $i$), it must hold for all $\alpha$ that $U_\alpha(X) = \mathrm{CvxH}(U_\alpha(X)) \cap \mathbb{Z}^d$. This definition is indeed implementable in arbitrary dimension. It is historically the first one implemented in DGtal.
*cellular characterization* (Feschet and Lachaud, 2023, Lem. 13). One must check $\mathrm{Star}(X) = \mathrm{Star}(\mathrm{CvxH}(X))$. Furthermore, Theorem 5 of the same paper tells that $\mathrm{Star}(\mathrm{CvxH}(X))$ is directly computable from $\mathrm{CvxH}(U_{\{1,\ldots,d\}}(X))$, so one convex hull computation is sufficient. This definition is also implemented in DGtal.
*envelope idempotence* (Feschet and Lachaud, 2023, Th. 2). It holds that $X = FC(X)$, where $FC(X) := \mathrm{Extr}(\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X))))$. Otherwise said, the fully convex envelope of a fully convex set is the set itself. From a computational point of view, this definition is closely related to the previous one, and is more often used to build a fully convex envelope out of an arbitrary set.
*$S^d$-convexity* [here, Th. 4]. For any $d$-tuple of points $T$ of $X$ (not necessarily distinct), we must have $\mathrm{Star}(\mathrm{CvxH}(T)) \subset \mathrm{Star}(X)$. This is a mathematically interesting characterization, but it is not an efficient one (its complexity is obviously greater than $\Omega(n^d)$).
*P-convexity* [here, Th. 5]. $X$ must be digitally 0-convex and when $d > 1$, for any $j$, $1 \leqslant j \leqslant d$, the projection $\pi_j(X)$ of $X$ along direction $j$ must be $P$-convex in $\mathbb{Z}^{d-1}$. This recursive definition is also coded in DGtal.

Now, how do these different characterizations compare in terms of speed of computation ? Following the different remarks above, we focus on methods that (i) can check the full convexity of digital sets with arbitrary many points, (ii) work in arbitrary dimension.

From the above list, we compare three characterizations which are implemented in DGtal (see module Digital convexity, full convexity and P-convexity):

- discrete morphological characterization (as diamonds ◇ and ◆ in figures), method `DigitalConvexity::isFullyConvex`,
- cellular characterization (as disks ○ and ● on figures), method `DigitalConvexity::isFullyConvexFast`,
- P-convexity characterization (as squares □ and ■ on figures), method `PConvexity::isPConvex`.

All three methods are tested for dimension 2, 3, and 4. In the first set of experiments (left of figures) we compare them on generic digital sets, which are randomly generated in a given range with a target density from 10% to 90%. In the second set of experiments (right of figures), we limit our comparison to digital sets that are either digitally 0-convex or fully convex. We then distinguish the timings for checking full convexity depending on the output full convexity property of the input set, since it influences the computation time (typically increases it). Figures 4, 5 and 6 sum up the different computation times.

All approaches follow more or less a quasi linear complexity $O(n \log n)$ (tests are limited to dimension lower or equal to 4). However, we distinguish three cases:

- if $X$ is not even convex, both the $P$-convexity and the discrete morphological characterization are the fastest with similar results, since both their first step involves checking 0-convexity. The cellular characterization is the slowest since it computes directly a convex hull with additional vertices.
- if $X$ is convex but not fully convex, then the fastest method remains the $P$-convexity, followed by the discrete morphological characterization, and the slowest is the cellular characterization.
- if $X$ is fully convex, then the fastest method is again the $P$-convexity, followed by the cellular characterization, and the slowest is the discrete morphological characterization (especially when increasing the dimension).

Overall the $P$-convexity characterization is almost always the fastest way to check the full convexity of a given range of digital points, with speed-up from 2 to 100 times faster especially when increasing the dimension. This new characterization of full convexity is thus for now the best way to check if a digital set is fully convex.

**Remark 1.** *One could be surprised by the last graphs in 4D (Figure 6). Indeed we expect convex hull computations in $O(n^2)$ in 4D. We observe here timings close to $\Theta(n \log n)$. This is due to the fact that the digital sets we are considering are 0-convex, so "full of digital points". Convex hull computations by QuickHull algorithm are thus faster than expected since many points are "hidden" in the shape. It can also be seen on Figure 6, left, that timings depend more on the range of random points than on the density (e.g. see horizontal strokes of circles, corresponding to density increases), which confirms the above explanation.*

# 7 Anti-monotonic behaviour of full convexity

This section is interested in the following question of anti-monotonicity of the FC() operator.

**Open Problem 1.** *Let us consider two finite sets $X$ and $Y$ in $\mathbb{Z}^d$, what can be said when*

$$Y \subsetneq X \quad \text{and} \quad \mathrm{FC}(X) \subsetneq \mathrm{FC}(Y) \qquad ?$$

## 7.1 Pierceable cells

We start by exposing an extension principle for anti-monotonicity.

**Lemma 7.** *Let us suppose that $Y \subsetneq X \subset \mathbb{Z}^d$ and $\exists e \in \mathbb{Z}^d$ with $e \notin \mathrm{FC}(X)$ but $e \in \mathrm{FC}(Y)$. Then $\exists c \in C^d$ with $e \in \mathrm{Extr}\,(c)$ such that $\mathrm{CvxH}\,(Y)$ is framed within $c$, $\mathrm{CvxH}\,(X)$ is not framed within $c$.*

*Proof.* We have $e \in \mathrm{FC}(Y)$ with $\mathrm{FC}(Y) = \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right) \bigsqcup \mathrm{Ex}\,(Y)$. But since $\mathrm{CvxH}\,(Y) \subset \mathrm{CvxH}\,(X)$ we cannot have $e \in \mathrm{CvxH}\,(Y)$. So $e \in \mathrm{Ex}\,(Y)$. Then $\exists c \in \mathrm{Skel}^+\,(\mathrm{Star}\,(\mathrm{CvxH}\,(Y)))$ with $e \in \mathrm{Extr}\,(c)$. By Lemma 3, this means that $\mathrm{CvxH}\,(Y)$ is framed within c. Moreover since $e \notin \mathrm{FC}(X)$, $\mathrm{CvxH}\,(X)$ is not framed within $c$. □

The cells in $\mathrm{Star}\,(\partial \mathrm{CvxH}\,(X))$ which are such that $\mathrm{CvxH}\,(X)$ is not framed within $c$ and $\exists Y \subsetneq X$ with $\mathrm{CvxH}\,(Y)$ framed within $c$ are called *pierceable* (see Figure 7). Those cells are fundamental in Problem 1. Indeed, as CvxH () is a monotonous operator the subset $Y$ cannot capture more points than $X$. So the difference between $\mathrm{FC}(X)$ and $\mathrm{FC}(Y)$ can only be obtained through Ex (). But the involved cells in Ex () must belong to the skeleton of the set and so it is possible to capture more points using Extr () for $Y$ only when $X$ was not framed within a cell for which $Y$ is framed within. We precisely have the following.

**Lemma 8.** *Let us suppose that $Y \subsetneq X \subset \mathbb{Z}^d$ is a solution of Problem 1, then $\mathrm{Ex}\,(X) \subsetneq \mathrm{Ex}\,(Y)$.*

*Proof.* Let us consider $c \in \mathrm{Skel}^+\,(\mathrm{Star}\,(\mathrm{CvxH}\,(X)))$. By Lemma 3, $\|c\| \cap \mathrm{CvxH}\,(X) \neq \emptyset$ and $\|\partial c\| \cap \mathrm{CvxH}\,(X) = \emptyset$. Since $Y \subset X$, we have $\mathrm{CvxH}\,(Y) \subset \mathrm{CvxH}\,(X)$. Hence, $\|\partial c\| \cap \mathrm{CvxH}\,(Y) = \emptyset$.

(a) Let us suppose that $\|c\| \cap \mathrm{CvxH}\,(Y) = \emptyset$. We know that $\mathrm{Extr}\,(c) \subset \mathrm{FC}(X)$. So $\mathrm{Extr}\,(c) \subset \mathrm{FC}(Y)$ because $\mathrm{FC}(X) \subset \mathrm{FC}(Y)$. But $\|\partial c\| \cap \mathrm{CvxH}\,(Y) = \emptyset$, thus $\mathrm{Extr}\,(c) \cap \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right) = \emptyset$. Hence, $\mathrm{Extr}\,(c) \subset \mathrm{Ex}\,(Y)$.

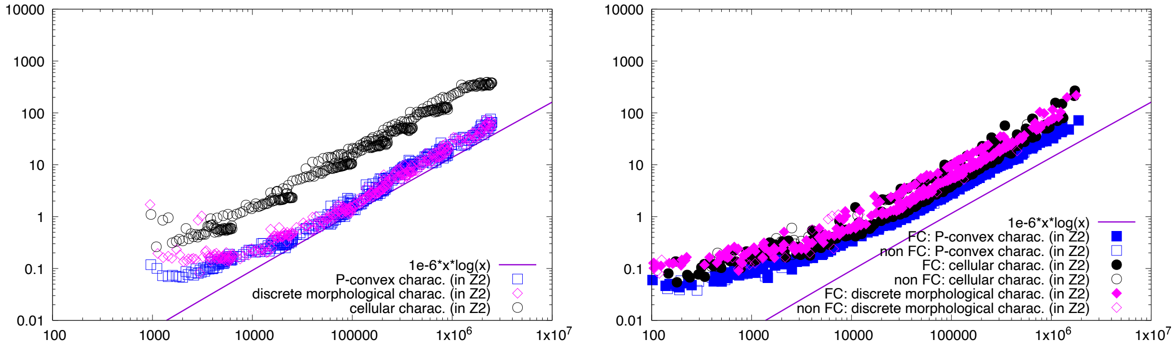



**Fig. 4** This figure displays the respective computation times (ms) in $\mathbb{Z}^2$ of P-convexity (as squares), discrete morphological characterization (as diamonds) and cellular characterization (as disks), as a function of the cardinal of the digital set. **Left**: digital sets are randomly generated in a given range with a target point density from 10% to 90%. **Right**: digital sets are randomly generated so that they are either 0-convex or fully convex.

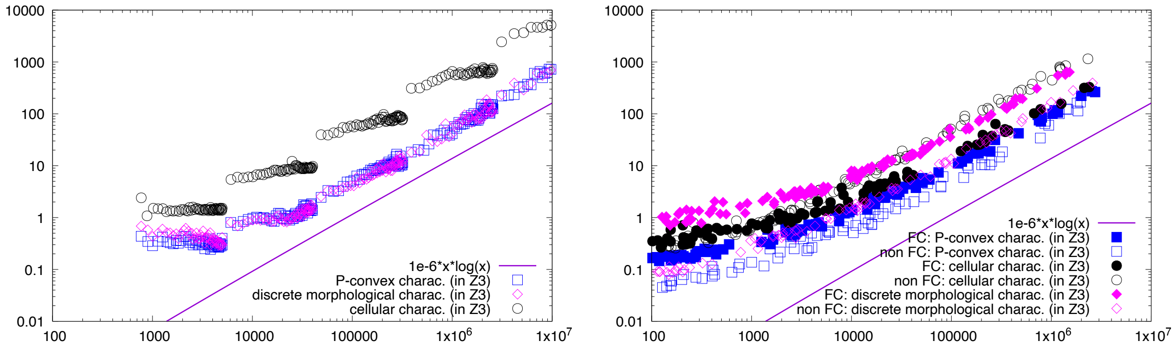



**Fig. 5** This figure displays the respective computation times (ms) in $\mathbb{Z}^3$ of P-convexity (as squares), discrete morphological characterization (as diamonds) and cellular characterization (as disks), as a function of the cardinal of the digital set. **Left**: digital sets are randomly generated in a given range with a target point density from 10% to 90%. **Right**: digital sets are randomly generated so that they are either 0-convex or fully convex.

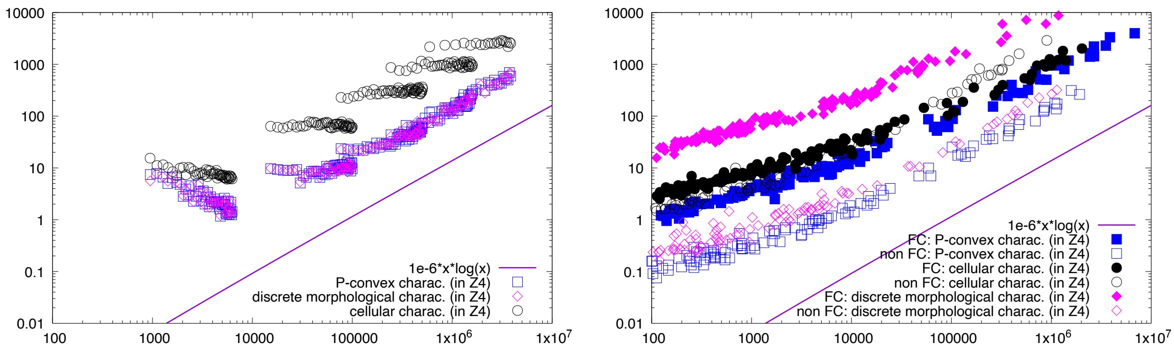



**Fig. 6** This figure displays the respective computation times (ms) in $\mathbb{Z}^4$ of P-convexity (as squares), discrete morphological characterization (as diamonds) and cellular characterization (as disks), as a function of the cardinal of the digital set. **Left**: digital sets are randomly generated in a given range with a target point density from 10% to 90%. **Right**: digital sets are randomly generated so that they are either 0-convex or fully convex.

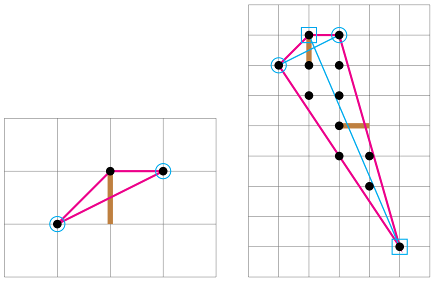


**Fig. 7** Pierceable cells are potentially extensive for FC. (left) $X$ contains solid (black) points, $Y$ contains open (cyan) points and the middle (brown) 1d cell is a pierceable cell. (right) Top left 1d cell is pierceable for $Y$ given by open (cyan) points while middle right 1d cell is pierceable by open (cyan) rectangle set $Y$.

(b) Let us suppose that $\|c\| \cap \mathrm{CvxH}(Y) \neq \emptyset$. Thus $\mathrm{CvxH}(Y)$ is framed within $c$ and so $\mathrm{Extr}(c) \subset \mathrm{Ex}(Y)$.

Considering both cases, we get that $\mathrm{Ex}(X) \subset \mathrm{Ex}(Y)$.

Suppose that $\mathrm{Ex}(X) = \mathrm{Ex}(Y)$ then since $\left(\mathrm{CvxH}(Y) \cap \mathbb{Z}^d\right) \subset \left(\mathrm{CvxH}(X) \cap \mathbb{Z}^d\right)$, we get that $\mathrm{FC}(Y) \subset \mathrm{FC}(X)$ which is a contradiction. □

## 7.2 Extreme points of convex hulls must be checked

When Problem 1 has a solution then at least one extreme point of $\mathrm{CvxH}(X)$ contains a pierceable cell because in $Y$, we must recover the lost extreme points of $X$.

**Lemma 9.** *Let us consider a set $X \subset \mathbb{Z}^d$. If Problem 1 has a solution for $Y \subsetneq X$ then it exists at least one extreme point of* $\mathrm{CvxH}(X)$ *which is not an extreme point of* $\mathrm{CvxH}(Y)$ *and every extreme point of* $\mathrm{CvxH}(X)$ *which is not an extreme point of* $\mathrm{CvxH}(Y)$ *has a pierceable cell by $Y$ in its star.*

*Proof.* For a bounded real set $K$, we have $\mathrm{CvxH}(K) = \mathrm{CvxH}(\mathrm{Extrm}(K))$. So, as soon as the extreme points of two convex hulls are equal, then the convex hulls are equal. From $Y \subsetneq X$, we have that $\mathrm{CvxH}(Y) \subset \mathrm{CvxH}(X)$. The example of $X$ versus $\mathrm{Extrm}(X)$ shows that strict inclusion of sets does not implies strict inclusion of convex hulls. But, $\mathrm{CvxH}(Y) = \mathrm{CvxH}(X) \Rightarrow \mathrm{FC}(Y) = \mathrm{FC}(X)$. So this contradicts that Problem 1 has a solution using $Y$.

We must hence have $\mathrm{CvxH}(Y) \subsetneq \mathrm{CvxH}(X)$. This in particular implies that $\mathrm{Extrm}(\mathrm{CvxH}(X)) \neq \mathrm{Extrm}(\mathrm{CvxH}(Y))$. So $\exists e \in \mathrm{Extrm}(\mathrm{CvxH}(X))$, $e \notin \mathrm{Extrm}(\mathrm{CvxH}(Y))$. Since $e \in \mathrm{FC}(X)$, $\mathrm{FC}(X) \subsetneq \mathrm{FC}(Y)$ implies that $e \in \mathrm{FC}(Y)$. But by construction, $e \notin \mathrm{CvxH}(Y)$. So $e \in \mathrm{Ex}(Y)$ and $\exists c \in$

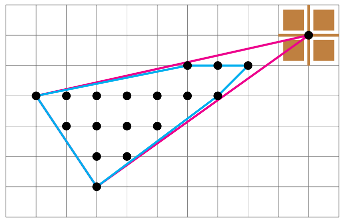


**Fig. 8** Local obstruction while removing an extreme point $e$ (top right). $\mathrm{CvxH}(X \setminus \{e\})$ does not intersect $\|\mathrm{Star}(e)\|$, hence $e$ cannot be avoided in every $Y \subsetneq X$ built for anti-monotonicity of FC.

$\mathrm{Skel}^+(\mathrm{Star}(\mathrm{CvxH}(Y)))$, $e \in \mathrm{Extr}(c)$. So $\mathrm{CvxH}(Y)$ is framed within $c$ by Lemma 3. By $e \in \mathrm{CvxH}(X) \cap \mathbb{Z}^d$ so $c \notin \mathrm{Skel}^+(\mathrm{Star}(\mathrm{CvxH}(X)))$. Hence $\mathrm{CvxH}(X)$ is not framed within $c$. In other word, $c$ is pierceable for $X$ by $Y$. □

**Lemma 10.** *Let us consider a set $X \subset \mathbb{Z}^d$ such that Problem 1 has a solution for $Y \subsetneq X$. Suppose that there exists a pierceable cell by* $\mathrm{CvxH}(Y)$ *in the interior of a facet $F$ of* $\mathrm{CvxH}(X)$ *then one vertex of $F$ has also a pierceable cell by* $\mathrm{CvxH}(Y)$.

*Proof.* Suppose $c \in C^d$ such that for a face $F$ of $\mathrm{CvxH}(X)$ we have $\|c\| \cap F \neq \emptyset$ and $c$ pierceable and problem (1) has a solution. Every vertex of $F$ is on $\mathrm{CvxH}(X)$ so belongs to $\mathrm{FC}(X)$, thus to $\mathrm{FC}(Y)$. Due to Lemma 3, $\mathrm{CvxH}(Y)$ is framed within $\|c\|$ which implies that $\mathrm{CvxH}(Y)$ has a closest point to $\|\partial c\|$ in $\|c\|$ which is separated from $\|\partial c\|$. Let us call $y$ such a point. Suppose now that every vertex $v_i$ of $F$ belongs to $\mathrm{CvxH}(Y)$ then $\mathrm{CvxH}(\{v_i\} \cup \{y\}) \subset \mathrm{CvxH}(Y)$. But this implies a contradiction that $y$ is a closest point since any points between $\mathrm{CvxH}(X)$ and $y$ are in the first convex hull. Hence there exists at least one vertex $v_*$ which does not belong to $\mathrm{CvxH}(Y)$. But since $v_* \in \mathrm{FC}(X)$, we have $v_* \in \mathrm{FC}(Y)$ and so $v_* \in \mathrm{Ex}(Y)$. Any cell in $\mathrm{Skel}^+(\mathrm{CvxH}(Y))$ having $v_*$ as an extreme point is thus pierceable for $X$ using $Y$. □

## 7.3 Local obstruction

We can detect obstructions to a solution to Problem 1 by looking at local configurations.

**Lemma 11** (see Figure 8)**.** *Let us consider a set $X \subset \mathbb{Z}^d$. Let us consider an extreme point $e$ of* $\mathrm{CvxH}(X)$*. If* $\mathrm{CvxH}(X \setminus \{e\}) \cap \|\mathrm{Star}(e)\| = \emptyset$ *then $e$ is an extreme point of any $Y \subsetneq X$ which is a solution to Problem 1.*

*Proof.* For any $e \in X$, $\forall Y \subsetneq X$, $e \notin Y \Rightarrow \mathrm{CvxH}(Y) \subset \mathrm{CvxH}(X \setminus \{e\}) \subset \mathrm{CvxH}(X)$. If $e$ is an extreme point for $\mathrm{CvxH}(X)$ which is not an extreme point for $\mathrm{CvxH}(Y)$ then Lemma 9 say that one cell $c \in$

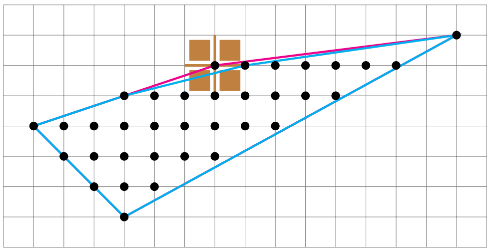


**Fig. 9** Local obstruction while removing an extreme point $e$ (middle top). $\mathrm{CvxH}\,(X \setminus \{e\})$ intersects $\|\mathrm{Star}\,(e)\|$, hence $e$ can be avoided potentially in a $Y \subsetneq X$ built for anti-monotonicity of FC.

$\mathrm{Star}\,(e)$ is pierceable by any solution $Y$ to Problem 1. But if $\mathrm{CvxH}\,(X \setminus \{e\}) \cap \|\mathrm{Star}\,(e)\| = \emptyset$ then by inclusion of convex hulls, we have $\mathrm{CvxH}\,(Y) \cap \|\mathrm{Star}\,(e)\| = \emptyset$ which contradicts Lemma 3 because $\mathrm{CvxH}\,(Y)$ must be framed within $c \in \mathrm{Star}\,(e)$ which implies that $\mathrm{CvxH}\,(Y) \cap \|c\| \neq \emptyset$. □

**Lemma 12** (see Figure 9). *Let us consider a set $X \subset \mathbb{Z}^d$. If Problem 1 has a solution then $\exists e \in \mathrm{Extrm}\,(\mathrm{CvxH}\,(X))$ such that $\mathrm{CvxH}\,(X \setminus \{e\}) \cap \|\mathrm{Star}\,(e)\| \neq \emptyset$.*

*Proof.* If $\forall e \in \mathrm{Extrm}\,(\mathrm{CvxH}\,(X))$ we have $\mathrm{CvxH}\,(X \setminus \{e\}) \cap \|\mathrm{Star}\,(e)\| = \emptyset$ then by using Lemma 11, we know that $\mathrm{Extrm}\,(Y) = \mathrm{Extrm}\,(X)$ such that $\mathrm{CvxH}\,(Y) = \mathrm{CvxH}\,(X)$ which implies that $\mathrm{FC}(Y) = \mathrm{FC}(X)$ which contradicts the fact that $Y$ provides a solution for Problem 1. □

As a first conclusion to identifying solutions to Problem 1, Lemmas 9 and 10 indicate that specific cells around extreme points of $X$ should be examined. Besides Lemma 11 and 12 tell that some extreme points do not need to be examined. However there is yet no easy characterization of anti-monotonicity. Therefore, in the following section, we look for a simple sufficient condition that prevents anti-monotonicity for the full convex hull operator.

## 7.4 Thick enough sets do not support anti-monotonicity

Let us consider a set $Y \subsetneq X$. We know that $\mathrm{FC}(Y) = \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right) \bigsqcup \mathrm{Ex}\,(Y)$. We also have $\mathrm{Ex}\,(X) \cap \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right) = \emptyset$. So, we can make a partition of $\mathrm{Ex}\,(Y)$ by $\mathrm{Ex}\,(X)$ such that $\mathrm{Ex}\,(Y) = (\mathrm{Ex}\,(Y) \cap \mathrm{Ex}\,(X)) \bigsqcup P(Y;X)$. The complete decomposition is thus

$$\mathrm{FC}(Y) = \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right) \bigsqcup (\mathrm{Ex}\,(Y) \cap \mathrm{Ex}\,(X)) \bigsqcup P(Y;X).$$

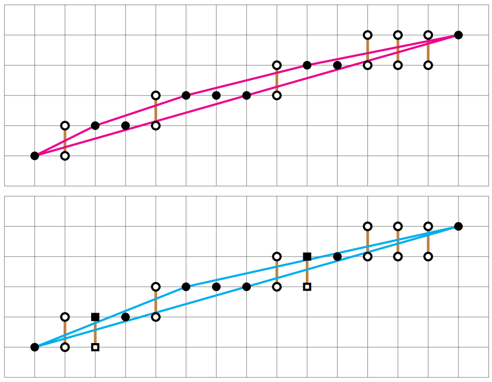


**Fig. 10** *(Top)* $X$ (black disks), $\mathrm{CvxH}\,(X)$ (magenta polyline), $\mathrm{Skel}^+\,(\mathrm{CvxH}\,(X))$ (brown open segments) and $\mathrm{Ex}\,(X)$ (black circles). *(bottom)* $Y$ (black disks), $\mathrm{CvxH}\,(Y)$ (cyan polyline), $\mathrm{Skel}^+\,(\mathrm{CvxH}\,(Y))$ (brown open segments), $\mathrm{Ex}\,(Y) \cap \mathrm{Ex}\,(X)$ (black circles), $P(Y;X)$ (rectangular points).

Points in $P(Y;X)$ are not in $\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d$ but are in $\mathrm{Ex}\,(Y)$ (see Figure 10). As such they are obtained by $\mathrm{Extr}\,(c)$ for some cell $c \in \mathrm{Skel}^+\,(\mathrm{Star}\,(\mathrm{CvxH}\,(Y)))$. However those points are not in $\mathrm{Ex}\,(X)$. So they can only be in $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$ or outside $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$ but associated to cells within $\mathrm{CvxH}\,(X)$ which are not framed. Hence when there exist points in $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$ which are not in $\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d$, they must be in $P(Y;X)$ so as to be in $\mathrm{FC}(Y)$. More precisely we have the following.

**Lemma 13.** *[see Figure 10 filled rectangular points.] Let us consider $Y \subset X$. If $\left(\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d\right) \setminus \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right)$ is not a strict subset of $P(Y;X)$ then we cannot have* $\mathrm{FC}(X) \subset \mathrm{FC}(Y)$.

*Proof.* If $\left(\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d\right) \setminus \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right)$ is not a strict subset of $P(Y;X)$ then we can pick $e \in \left(\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d\right) \setminus \left(\mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d\right)$ with $e \notin P(Y;X)$. Hence by construction we have $e \notin \mathrm{CvxH}\,(Y) \cap \mathbb{Z}^d$. We also have $e \notin \mathrm{Ex}\,(Y) \cap \mathrm{Ex}\,(X)$ because $e \in \mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$ which is disjoint from $\mathrm{Ex}\,(X)$. So since $e \notin P(Y;X)$, we get that $e \notin \mathrm{FC}(Y)$. But $e \in \mathrm{FC}(X)$ and thus, we cannot have $\mathrm{FC}(X) \subset \mathrm{FC}(Y)$. □

Now for a cell $c$ in $\mathrm{Skel}^+\,(\mathrm{CvxH}\,(Y))$, $\mathrm{CvxH}\,(Y)$ cannot capture more points than $\mathrm{Extr}\,(c)$ in $\mathrm{AffH}\,(\|c\|)$. Thus, if $X$ has points outside $\mathrm{Extr}\,(c)$ in $\mathrm{AffH}\,(\|c\|)$, they cannot be captured by $\mathrm{CvxH}\,(Y)$ so they are missing in $\mathrm{FC}(Y)$ while belonging to $\mathrm{FC}(X)$. This obviously implies that $Y$ cannot be a solution of Problem 1. Hence, we can have global obstruction as follows.

**Theorem 7.** *If for any cell $c$ such that* $\mathrm{CvxH}(X) \cap \|c\| \neq \emptyset$, *$X$ contains a point in* $\mathrm{AffH}(c)$ *outside* $\mathrm{Extr}(c)$ *then for any $Y \subsetneq X$ there exists at least a point in* $\mathrm{FC}(X)$ *which is not in* $\mathrm{FC}(Y)$*, meaning that Problem 1 has no solution.*

*Proof.* To have $\mathrm{FC}(X) \subsetneq \mathrm{FC}(Y)$, we must have a point $z \in P(Y; X)$ which is not in $\mathrm{CvxH}(X) \cap \mathbb{Z}^d$ and not in $\mathrm{Ex}(X)$. We consider a cell $c \in \mathrm{Skel}^+(\mathrm{Star}(\mathrm{CvxH}(Y)))$ such that $z \in \mathrm{Extr}(c)$. By Lemma 3, we have $\|c\| \cap \mathrm{CvxH}(Y) \neq \emptyset$. Thus since for $Y \subsetneq X$ we have $\mathrm{CvxH}(Y) \subset \mathrm{CvxH}(X)$, we know that $\|c\| \cap \mathrm{CvxH}(X) \neq \emptyset$. Since we have $z \notin \mathrm{Ex}(X)$, we deduce that $\|\partial c\| \cap \mathrm{CvxH}(X) \neq \emptyset$. However, since $\mathrm{CvxH}(Y)$ is framed within $c$, we have that $\mathrm{AffH}(c) \cap \mathrm{FC}(Y) = \mathrm{Extr}(c)$. Indeed, $\mathrm{CvxH}(Y) \cap \mathrm{AffH}(\|c\|) \subsetneq \|c\|$ by definition. So we already know that if $X$ contains at least one point in $\mathrm{AffH}(\|c\|)$ not in $\mathrm{Extr}(c)$ then this point cannot be captured by $\mathrm{CvxH}(Y)$ using $c_0 = c$. We can now use a chain of cells. Indeed, $z$ must be captured, in $\mathrm{Ex}(Y)$, using another cell $c_1$. But in $\mathrm{AffH}(c_1)$, $\mathrm{FC}(Y)$ cannot capture more points than $\mathrm{Extr}(c_1)$ so if $X$ still contains another point then it cannot be captured by $\mathrm{FC}(Y)$. So we must rely on a cell $c_2$ and so on. We have a contradiction since $X$ is finite. □

One should notice that the argument might be reduced to 1-d cells only. In fact, when considering $\mathrm{Extr}(c)$, it can always be viewed as a collection of $\mathrm{Extr}(f)$ for $f \in \mathrm{Star}(c)$ with $\dim f = 1$. So, as soon as $\mathrm{CvxH}(X)$ intersects a cell $c$, on all its associated 1-d cells $f$, having three points for $X$ on $\mathrm{AffH}(\|f\|)$ is sufficient to guarantee that Lemma 7 applies.

**Corollary 1.** *If for any cell 1-d $c$ such that* $\mathrm{CvxH}(X) \cap \|c\| \neq \emptyset$, *$X$ contains three points in* $\mathrm{AffH}(c)$ *then Problem 1 has no solution.*

This corollary implies a simple and easily checkable criterion on a digital set, which guarantees monotonicity of full convex envelopes. It suffices to verify for any axis-aligned line intersecting the set if there are at least three points in this intersection. As obvious instances satisfying this criterion, we can mention digital sets of the form $(X \oplus [-1, 1]^d) \cap \mathbb{Z}^d$, with $\oplus$ the Minkowski sum.

# 8 Conclusion and perspectives

We have presented two original characterizations of full convexity. The first one gives a nice analogue of the "segment inclusion" definition of convexity with full convexity in dimension 1 and 2, and it shows also that, in higher dimensional spaces, additional continuity constraints are required. The second characterization tells that full convexity requires the digital convexity of the set and all of its shadows along axes.

Both characterizations shed new light on what is really full convexity. They provide alternative algorithms to check full convexity, which have better time complexity both theoretically and practically as shows in the provided experiments. Also, they may help deciding if some digital sets are fully convex, as we show here for digital balls and hypercube subsets. They enable the definition of new measures for digital sets, with a stronger power of categorization.

We extend further the understanding of full convex hulls obtained via the FC operator. This operator can have an anti monotonic behavior for thin sets. We provide a first analysis of this situation with global and local properties as a first step to a complete algorithm for the detection of anti-monotonic behavior. We eventually exhibit a definition of thick enough sets avoiding this problem.

Finally, the characterization of full convexity through projections can be of interest for discrete tomography, as it induces connectedness in a natural way.

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