How to peel fully convex digital sets ?

Fabien Feschet[1] Jacques-Olivier Lachaud[2]

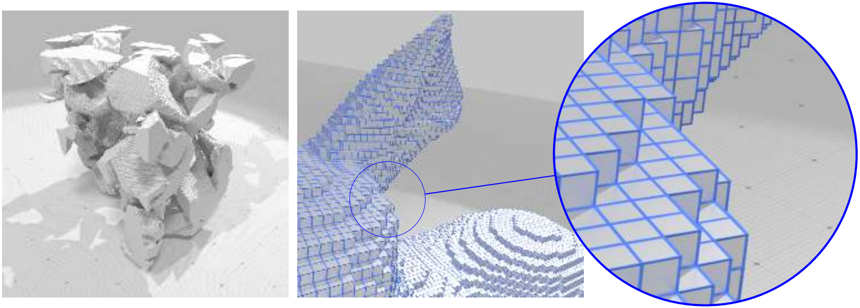
 

[1]LIMOS, University Clermont Auvergne
[2]LAMA, University Savoie Mont Blanc

November 4th, 2025
Discrete Geometry and Mathematical Morphology (DGMM2025)
University of Groningen

How to peel fully convex digital sets ?

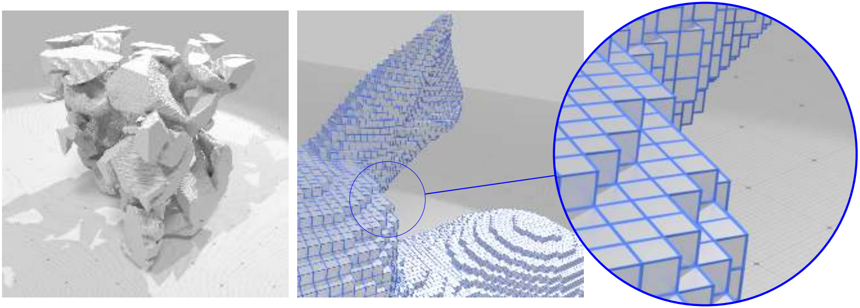
Why digital convexity ?



- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes

$\Rightarrow$ convex/concave/flat/saddle regions, piecewise linear geometry

Why digital convexity ?



- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes

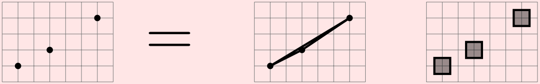
$\Rightarrow$ convex/concave/flat/saddle regions, piecewise linear geometry

- convexity = foundation of convex analysis, linear programming
- digital convexity = foundation of digital convex analysis, integer linear programming ?

Full convexity versus classical digital convexity ?

Classical digital convexity does not imply connectedness !

- $X \subset \mathbb{Z}^d$ is 0-*convex* iff $\mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d = X$



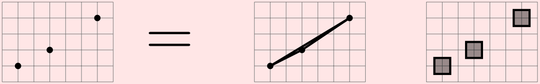
$X \quad = \quad \mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$ pixel view

0-*convex* $\not\Rightarrow$ **connected**

Full convexity versus classical digital convexity ?

Classical digital convexity does not imply **connectedness** !

- $X \subset \mathbb{Z}^d$ is 0-*convex* iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$

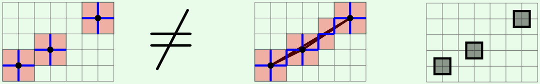


X = $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d$ pixel view

0-*convex* $\not\Rightarrow$ **connected**

Full convexity implies **digital** and **simple connectedness**

- $X \subset \mathbb{Z}^d$ is *fully convex* iff $\mathrm{Cvxh}(X) \cap \bar{\mathcal{C}}^d = X \cap \bar{\mathcal{C}}^d$, where $\bar{\mathcal{C}}^d$ is the unit cellular grid decomposition of $\mathbb{R}^n$ [L. 2021].



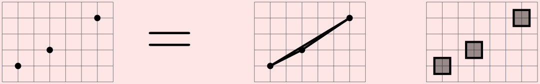
$X \cap \bar{\mathcal{C}}^d$ $\neq$ $\mathrm{Cvxh}(X) \cap \bar{\mathcal{C}}^d$ pixel view

0-convex but not *fully convex* $\Rightarrow$ **not connected**

Full convexity versus classical digital convexity ?

Classical digital convexity does not imply **connectedness** !

- $X \subset \mathbb{Z}^d$ is 0-*convex* iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$

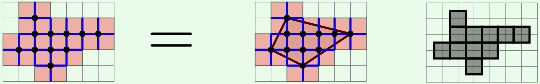


X $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d$ pixel view

0-*convex* $\not\Rightarrow$ **connected**

Full convexity implies **digital** and **simple connectedness**

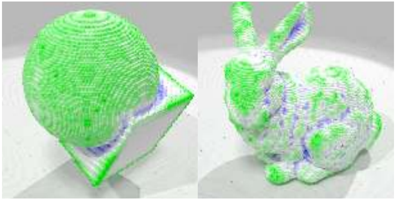
- $X \subset \mathbb{Z}^d$ is *fully convex* iff $\mathrm{Cvxh}(X) \cap \bar{\mathcal{C}}^d = X \cap \bar{\mathcal{C}}^d$, where $\bar{\mathcal{C}}^d$ is the unit cellular grid decomposition of $\mathbb{R}^n$ [L. 2021].



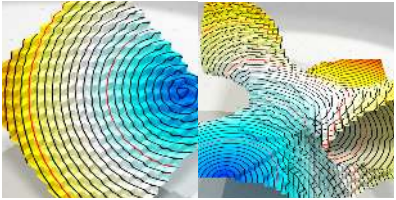
$X \cap \bar{\mathcal{C}}^d$ $\mathrm{Cvxh}(X) \cap \bar{\mathcal{C}}^d$ pixel view

fully convex $\Rightarrow$ **digitally and simply connected**

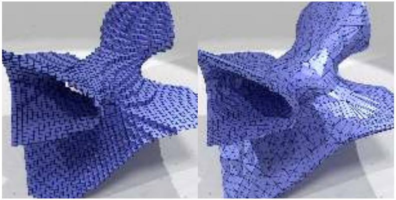
Applications of full convexity to digital shape analysis



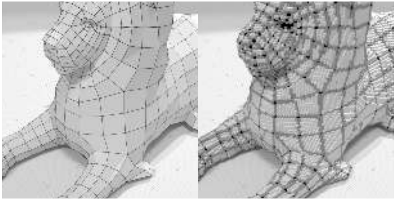
exact local shape analysis
(convex, concave, planar (white))



geodesics
(Euclidean distance in digital planes)

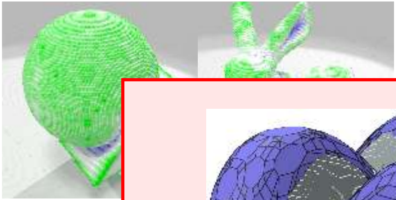


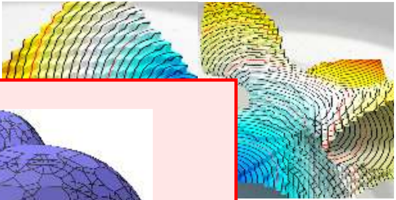
polyhedrization
(Hausdorff 1-close and reversible)



digital polyhedron
(cells are fully convex)

Applications of full convexity to digital shape analysis

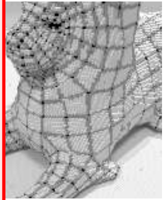




exact l

(convex, c

digital planes)

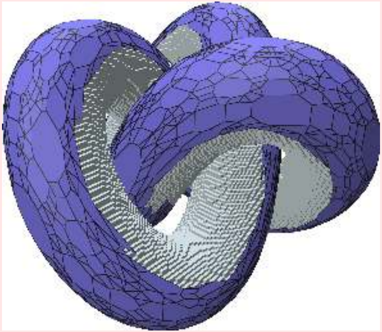


p

(Hausdorff 1-close and reversible)

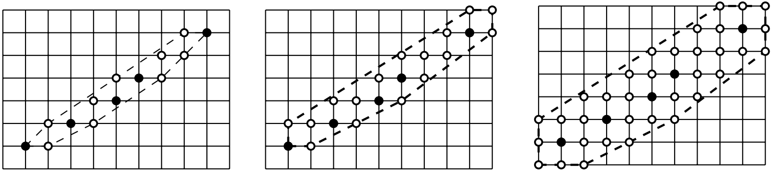
dron

(cells are fully convex)



Lachaud, Roussillon, Convex and concave decomposition of digitized shapes using plane probing and visibility, DGMM'25.

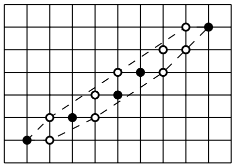
But is there a good fully convex hull operator ?



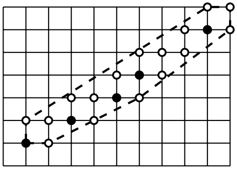
$\mathrm{FC}^*(X)$ $\quad (\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$ $\quad \mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$

operator	$\mathrm{FC}^*(X)$	$(\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$	$\mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$
fully conv.	yes	yes	yes
idempotence	yes	no	no
symmetry	yes	no	yes
$\#(Out)/\#(In)$	low	medium	high
efficiency	iterative	yes	yes
Increasing	not always	yes	yes

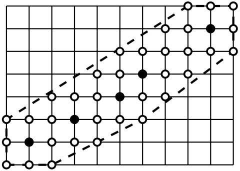
But is there a good fully convex hull operator ?



$\mathrm{FC}^*(X)$



$(\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$



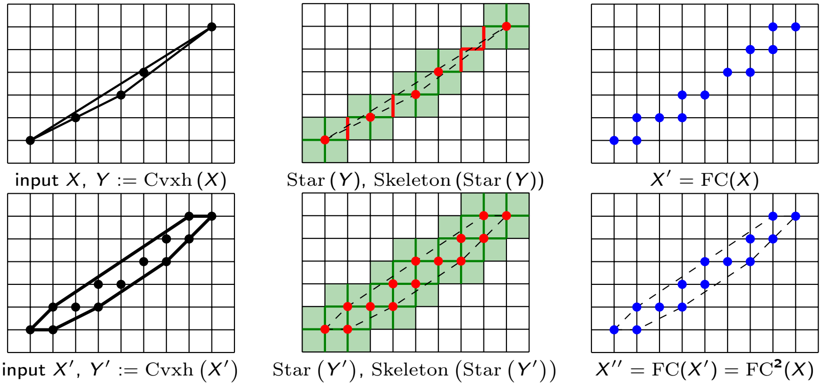
$\mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$

operator	$\mathrm{FC}^*(X)$	$(\mathrm{Cvxh}(X) \oplus H^+) \cap \mathbb{Z}^d$	$\mathrm{Extr}(\mathrm{Star}(\mathrm{Cvxh}(X)))$
fully conv.	yes	yes	yes
idempotence	yes	no	no
symmetry	yes	no	yes
$\#(Out)/\#(In)$	low	medium	high
efficiency	iterative	yes	yes
Increasing	not always	yes	yes

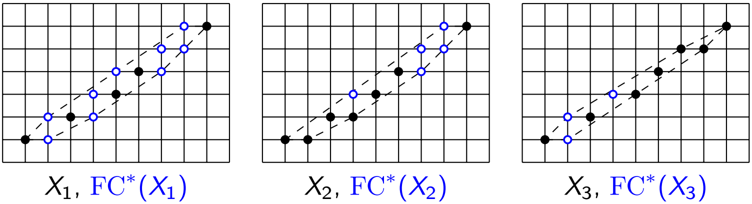
Best compromise: fully convex envelope operator $\mathrm{FC}^*(X)$...
but iterative and sometimes not an increasing operator.

Fully convex envelope $\mathrm{FC}^*(X)$

- Iterative method for computing a fully convex envelope [Feschet,L. '23]
- Let $\mathrm{FC}(X) := \mathrm{Extr}\left(\mathrm{Skeleton}\left(\mathrm{Star}\left(\mathrm{Cvxh}\left(X\right)\right)\right)\right)$
- Iterative composition $\mathrm{FC}^n(X) := \underbrace{\mathrm{FC} \circ \cdots \circ \mathrm{FC}}_{n \text{ times}}(X)$
- *Fully convex envelope* of X is $\mathrm{FC}^*(X) := \lim_{n\to\infty} \mathrm{FC}^n(X)$.



Cons of fully convex envelope $\mathrm{FC}^*(X)$?



- $X_1 \subset X_2$ and $\mathrm{FC}^*(X_2) \subset \mathrm{FC}^*(X_1)$
- $X_1 \subset X_3$ and $\mathrm{FC}^*(X_3) \subset \mathrm{FC}^*(X_1)$
- $\mathrm{FC}^*(\cdot)$ is **not increasing** in general for *thin* sets !
- $\mathrm{FC}^*(\cdot)$ is **increasing** for *thick* sets [Feschet, L. 2023], like $(X \oplus [-1,1]^d) \cap \mathbb{Z}^d$ for any digital set X.
- also **hard** to bound the number of iterations of $\mathrm{FC}^*(\cdot)$

Peeling fully convex sets

Objective of the paper

Better understanding of the **inclusion hierarchy** of *fully convex sets* in arbitrary dimension d.

Method

- **Peeling a fully convex set** X:
  find the $x \in X$ such that $X \setminus \{x\}$ is still fully convex.

Similar works

- contractions and dilations of digital convex sets
  [Tarsissi *et al.*, '19, '22, '23]
  - arithmetic approach focused on 2D convexity
- reminiscent of **simple points** detection
  [Bertrand, Couprie, Kong, and many others since the '90]
  - one of the most successful results of digital topology
  - peelable points preserve **convexity and topology**

How to peel fully convex digital sets ?

“Peeling” conditions for digitally convex sets

Lemma 1 (vertices are peelable points of 0-convex sets)

Let $X \subset \mathbb{Z}^d$ be a 0-convex set, i.e. $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$. Then, for any subset of vertices Z of $\mathrm{Cvxh}(X)$, $X \setminus Z$ is 0-convex.

Lemma 2 (a peelable point is a vertex of $\mathrm{Cvxh}(X)$)

Let $X \subset \mathbb{Z}^d$ be a 0-convex set. Let $z \in X$. Then $X \setminus \{z\}$ is 0-convex if and only if z is a vertex of $\mathrm{Cvxh}(X)$.

Conclusion

Peelable points of a fully convex set are among its vertices.

[Reminder] Full convexity is equivalent to P-convexity

Let π_j be the orthogonal projector associated to the j-th axis.

Projection convexity (P-convexity)

$X \subset \mathbb{Z}^d$ is *P-convex* (in $\mathbb{Z}^d$) iff:

(i) X is digitally 0-convex,

(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex (in $\mathbb{Z}^{d-1}$).

[Reminder] Full convexity is equivalent to P-convexity

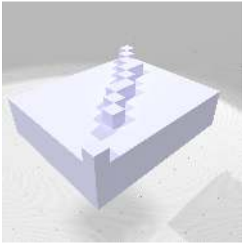
Let π_j be the orthogonal projector associated to the j-th axis.

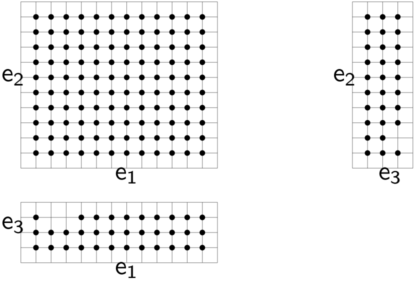
Projection convexity (P-convexity)

$X \subset \mathbb{Z}^d$ is *P-convex* (in $\mathbb{Z}^d$) iff:

(i) X is digitally 0-convex,

(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex (in $\mathbb{Z}^{d-1}$).





[Reminder] Full convexity is equivalent to P-convexity

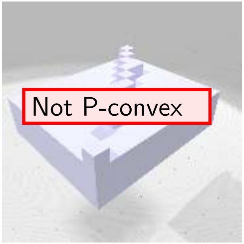
Let π_j be the orthogonal projector associated to the j-th axis.

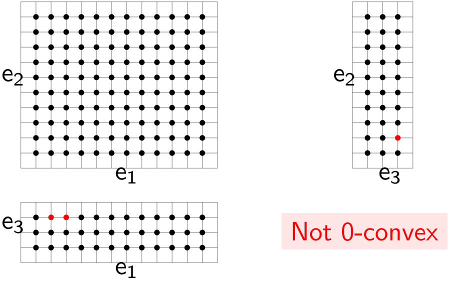
Projection convexity (P-convexity)

$X \subset \mathbb{Z}^d$ is *P-convex* (in $\mathbb{Z}^d$) iff:

(i) X is digitally 0-convex,

(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex (in $\mathbb{Z}^{d-1}$).





[Reminder] Full convexity is equivalent to P-convexity

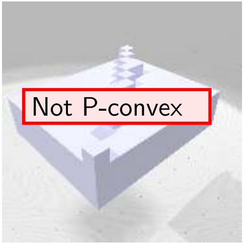
Let π_j be the orthogonal projector associated to the j-th axis.

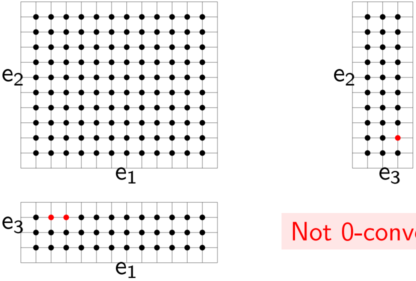
Projection convexity (P-convexity)

$X \subset \mathbb{Z}^d$ is *P-convex* (in $\mathbb{Z}^d$) iff:

(i) X is digitally 0-convex,

(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex (in $\mathbb{Z}^{d-1}$).





Not 0-convex

Theorem [Feschet, L. '24]

P-convexity is equivalent to full convexity.

Sufficient conditions for peelability I

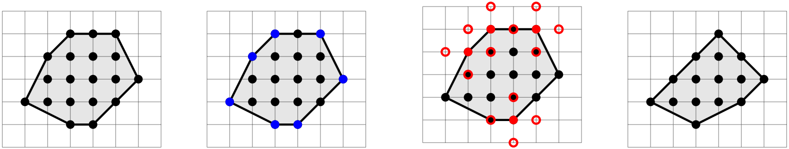
Theorem (sufficient conditions)

Let X, finite, non-empty, fully convex set of $\mathbb{Z}^d$. Let Z be a subset of the vertices of $\mathrm{Cvxh}(X)$.
Assume that, for every direction $i \in I_d$, for every $z \in Z$, we have $\{z - \mathrm{e}_i, z + \mathrm{e}_i\} \cap (X \setminus Z) \neq \emptyset$. Then $X \setminus Z$ is fully convex.

Sufficient conditions for peelability I

Theorem (sufficient conditions)

Let X, finite, non-empty, fully convex set of $\mathbb{Z}^d$. Let Z be a subset of the vertices of $\mathrm{Cvxh}\,(X)$.
Assume that, for every direction $i \in I_d$, for every $z \in Z$, we have $\{z - \mathrm{e}_i, z + \mathrm{e}_i\} \cap (X \setminus Z) \neq \emptyset$. Then $X \setminus Z$ is fully convex.

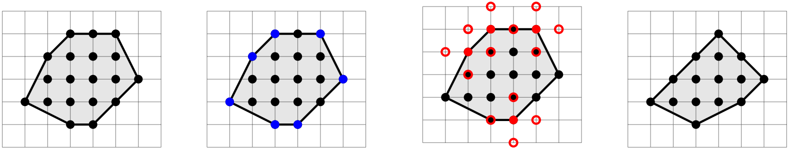


X and $\mathrm{Cvxh}\,(X)$ | $\mathrm{Extr}\,(\mathrm{Cvxh}\,(X))$ | Z and $Z \pm \mathrm{e}_i$ as $\circ$ | $X \setminus Z$

Sufficient conditions for peelability I

Theorem (sufficient conditions)

Let X, finite, non-empty, fully convex set of $\mathbb{Z}^d$. Let Z be a subset of the vertices of $\mathrm{Cvxh}(X)$.
Assume that, for every direction $i \in I_d$, for every $z \in Z$, we have $\{z - \mathrm{e}_i, z + \mathrm{e}_i\} \cap (X \setminus Z) \neq \emptyset$. Then $X \setminus Z$ is fully convex.



X and $\mathrm{Cvxh}(X)$ | $\mathrm{Extr}(\mathrm{Cvxh}(X))$ | Z and $Z \pm \mathrm{e}_i$ as $\circ$ | $X \setminus Z$

Proof.

1. $X \setminus Z$ is 0-convex because $Z \subset \mathrm{Extr}(\mathrm{Cvxh}(X))$ (Lemma 1).
2. Since $\{z - \mathrm{e}_i, z + \mathrm{e}_i\} \cap (X \setminus Z) \neq \emptyset$, then $\pi_i(X \setminus Z) = \pi_i(X)$.
3. So $\pi_i(X \setminus Z)$ is P-convex and use equivalence with full convexity.

□

Sufficient conditions for peelability II

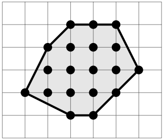
Proposition (side removal)

Let X be any fully convex set of $\mathbb{Z}^d$. Let $H = \{\mathsf{x} \in \mathbb{R}^d, \mathsf{x} \cdot \mathsf{e}_i \leqslant \alpha\}$ be any half-space of axis-aligned normal vector e_i and intercept α. Then $X \setminus H$ is fully convex.

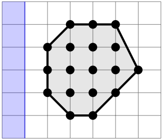
Sufficient conditions for peelability II

Proposition (side removal)

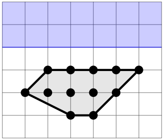
Let X be any fully convex set of $\mathbb{Z}^d$. Let $H = \{\mathrm{x} \in \mathbb{R}^d, \mathrm{x} \cdot \mathrm{e}_i \leqslant \alpha\}$ be any half-space of axis-aligned normal vector e_i and intercept α. Then $X \setminus H$ is fully convex.



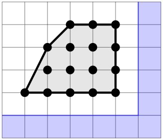
X and $\mathrm{Cvxh}\,(X)$



$X \setminus H$



$X \setminus H$

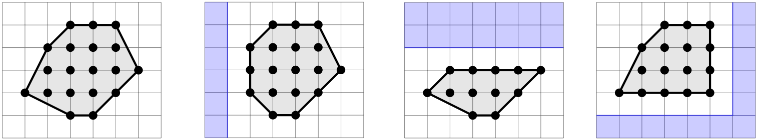


$X \setminus (H \cup H')$

Sufficient conditions for peelability II

Proposition (side removal)

Let X be any fully convex set of $\mathbb{Z}^d$. Let $H = \{x \in \mathbb{R}^d, x \cdot e_i \leqslant \alpha\}$ be any half-space of axis-aligned normal vector e_i and intercept α. Then $X \setminus H$ is fully convex.



X and $\mathrm{Cvxh}(X)$ — $X \setminus H$ — $X \setminus H$ — $X \setminus (H \cup H')$

Proof.

Half-space $H^c = \{x \in \mathbb{R}^d, x \cdot e_i \geqslant \alpha + 1\}$ is *stable*, i.e. it is convex and for any cell σ of $\mathcal{C}^d$, $H^c \cap \sigma \neq \emptyset \Rightarrow \bar{\sigma} \subset H^c$. Lemma 5 [L. 2022] concludes with $X \cap H^c$. □

How to peel fully convex digital sets ?

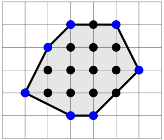
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in X* when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



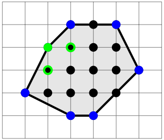
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



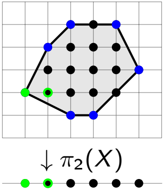
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - \mathrm{e}_i \in X$ or $z + \mathrm{e}_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



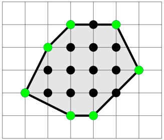
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - \mathrm{e}_i \in X$ or $z + \mathrm{e}_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



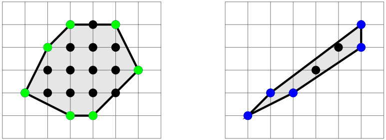
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - \mathrm{e}_i \in X$ or $z + \mathrm{e}_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



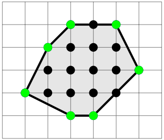
Definition of peelability

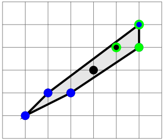
Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)





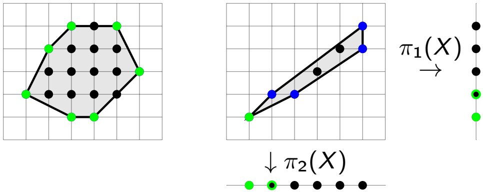
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in* X when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



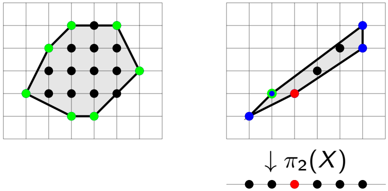
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in X* when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



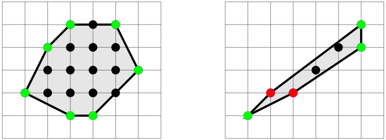
Definition of peelability

Peelability

Let X fully convex. The point $z \in X$ is *peelable in X* when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - e_i \in X$ or $z + e_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)



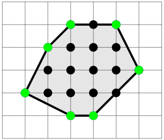
Definition of peelability

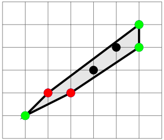
Peelability

Let X fully convex. The point $z \in X$ is *peelable in X* when z is a vertex of $\mathrm{Cvxh}(X)$ and, when $d \geqslant 2$, $\forall i \in \{1, \ldots, d\}$, either

(1) $z - \mathrm{e}_i \in X$ or $z + \mathrm{e}_i \in X$, (We say that $i \in \mathrm{Easy}_X[z]$)

or (2) $\pi_i(z)$ is peelable in $\pi_i(X)$. (We say that $i \in \mathrm{Hard}_X[z]$)





We postpone here the proof that z is peelable in X iff $X \setminus \{z\}$ is fully convex.

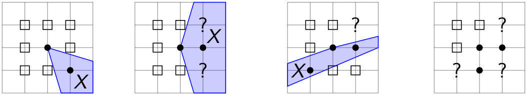
Vertex peelability is locally decidable in 2D

Lemma (vertex peelability is local in 2D)

Let $X \subset \mathbb{Z}^2$, finite and fully convex, and $N_X(z) := \{y \in X, \|z - y\|_\infty \leqslant 1\}$ the neighborhood of z in X.
A vertex z of $\mathrm{Cvxh}\,(X)$ is peelable in X iff z is peelable in $N_X(z)$.

Proof.

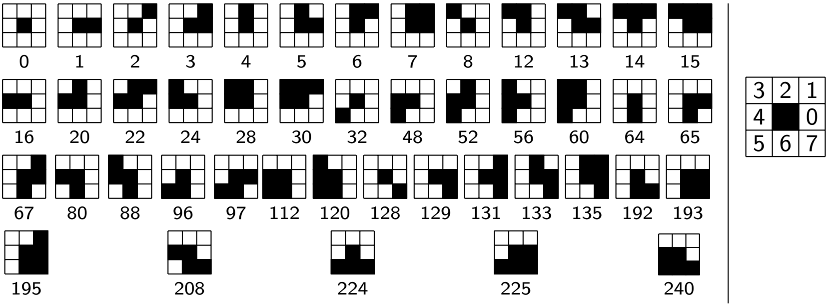
By enumeration on $\#(X \cap (z[\mathrm{e}_1] \cup z[\mathrm{e}_2]))$:



$N = 0$	$N = 1$	$N = 1$	$N = 2$
peelable	peelable	not peelable	peelable

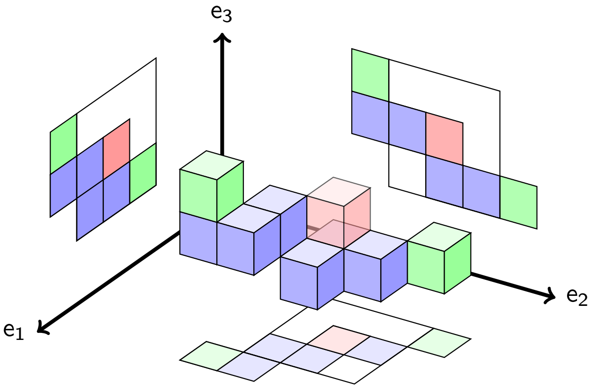
□

Peelable configurations and algorithm in 2D




```
// input:  cfg  the encoded neighborhood of the point (in {0,...,255})
// output: true iff the point is peelable wrt its neighborhood
// prerequisite: the point should be a vertex of CvxH(X)
bool isLocallyPeelable( unsigned int cfg )
{ // 45 configurations are peelable
  static const unsigned int p[ 8 ]
    = { 0x5151f1ff, 0x11110001, 0x00101000b, 0x01010003,
        0x000000ab, 0x00000000, 0x00001000b, 0x00010003 };
  return p[ cfg >> 5 ] & ( 1 << (cfg & 0x1f) );
}
```

Peelability is not locally decidable starting from 3D



- Point z is a vertex of $\mathrm{Cvxh}(X)$
- it is peelable in $\pi_i(X)$ for the 3 directions
- it is *not* a vertex of $\pi_1(X)$ (not detectable in $N_X(z)$)
- thus z is **not peelable** in X

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

Proof.

$\Rightarrow$ z is peelable in X so z is a vertex of $\mathrm{Cvxh}\,(X)$. By Lemma 1, $X \setminus \{z\}$ is 0-convex. Then, recursively along projections using P-convexity, if i is easy, $\pi_i(X \setminus \{z\})$ is P-convex. If i is hard, $\pi_i(X \setminus \{z\})$ is 0-convex by peelability and we can go on recursively.

□

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

Proof.

$\Rightarrow$ z is peelable in X so z is a vertex of $\mathrm{Cvxh}\,(X)$. By Lemma 1, $X \setminus \{z\}$ is 0-convex. Then, recursively along projections using P-convexity, if i is easy, $\pi_i(X \setminus \{z\})$ is P-convex. If i is hard, $\pi_i(X \setminus \{z\})$ is 0-convex by peelability and we can go on recursively.

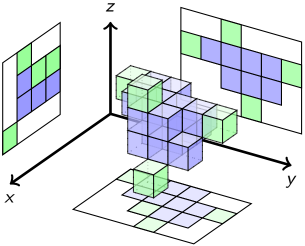
$\Leftarrow$ X and $X \setminus \{z\}$ fully convex implies z is a vertex of $\mathrm{Cvxh}\,(X)$.
If $\mathrm{Easy}_z[X] = \{1, \dots, d\}$, z is peelable in X. Else, let $i \in \mathrm{Hard}_z[X]$.
We look at $\mathcal{Z}_i := z[\mathrm{e}_i] \cap \mathrm{Cvxh}\,(X)$ and **assume** that $\mathcal{Z}_i \neq \{z\}$.
Let F^ϵ be farthest facet of $\mathrm{Cvxh}\,(X)$ along $\epsilon \mathrm{e}_i$ touching $\mathcal{Z}_i$. It is also a facet of $\mathrm{Cvxh}\,(X \setminus \{z\})$ and separates z from $z + \epsilon \mathrm{e}_i$.
The 1-cell $c =]z, z + \epsilon \mathrm{e}_i[$ touches F^ϵ and $\mathrm{Cvxh}\,(X \setminus \{z\})$, so by full convexity $\mathrm{Extr}\,(c) \cap \mathrm{Cvxh}\,(X \setminus \{z\}) \neq \emptyset$, a **contradiction**.
So $\mathcal{Z}_i = \{z\}$, and this implies that z is a vertex of $\pi_i(X)$.

$\square$

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

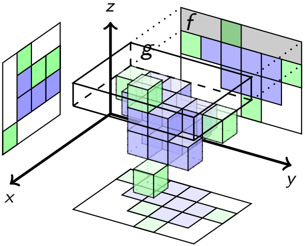
$\boxed{d = 1}$ X has 1 or 2 peelable points.

□

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

$\boxed{d = 1}$ X has 1 or 2 peelable points.

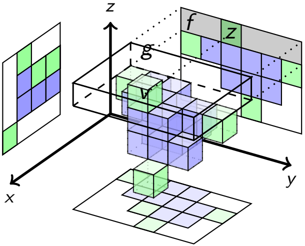
$\boxed{d > 1}$ Let f be a facet of $X_i := \pi_i(X)$.
It is a projection of a facet g of $\mathrm{bbox}(X)$.

□

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

$\boxed{d = 1}$ X has 1 or 2 peelable points.

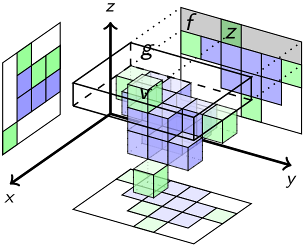
$\boxed{d > 1}$ Let f be a facet of $X_i := \pi_i(X)$.
It is a projection of a facet g of $\mathrm{bbox}(X)$.
By recurs. hyp. $\exists z \in f$ peelable point of X_i.
Then $\exists v$ a vertex of $\mathrm{Cvxh}\,(X)$ with $v \in F$ and $\pi_i(v) = z$.

□

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

$\boxed{d = 1}$ X has 1 or 2 peelable points.

$\boxed{d > 1}$ Let f be a facet of $X_i := \pi_i(X)$.
It is a projection of a facet g of $\mathrm{bbox}(X)$.
By recurs. hyp. $\exists z \in f$ peelable point of X_i.
Then $\exists v$ a vertex of $\mathrm{Cvxh}\,(X)$ with $v \in F$ and $\pi_i(v) = z$.

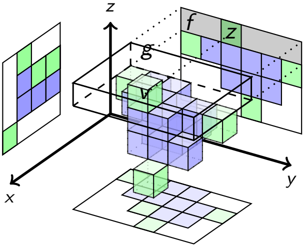
Hard directions $j \neq i$ for v are stable by projections, but z is peelable, so it remains only i.

□

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

$\boxed{d = 1}$ X has 1 or 2 peelable points.

$\boxed{d > 1}$ Let f be a facet of $X_i := \pi_i(X)$. It is a projection of a facet g of $\mathrm{bbox}(X)$. By recurs. hyp. $\exists z \in f$ peelable point of X_i. Then $\exists v$ a vertex of $\mathrm{Cvxh}\,(X)$ with $v \in F$ and $\pi_i(v) = z$.

Hard directions $j \neq i$ for v are stable by projections, but z is peelable, so it remains only i.

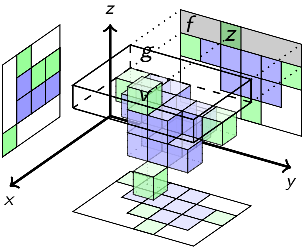
If i easy, we are done. If i is hard, $\pi_i(v) = z$ is a vertex of X_i since it belongs to its bbox.

□

Every fully convex set is peelable

Theorem (every fully convex set is peelable)

Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.



Proof (by recursion on the dimension d).

$\boxed{d = 1}$ X has 1 or 2 peelable points.

$\boxed{d > 1}$ Let f be a facet of $X_i := \pi_i(X)$.
It is a projection of a facet g of $\mathrm{bbox}(X)$.
By recurs. hyp. $\exists z \in f$ peelable point of X_i.
Then $\exists v$ a vertex of $\mathrm{Cvxh}\,(X)$ with $v \in F$ and $\pi_i(v) = z$.

Hard directions $j \neq i$ for v are stable by projections, but z is peelable, so it remains only i.

If i easy, we are done. If i is hard, $\pi_i(v) = z$ is a vertex of X_i since it belongs to its bbox.

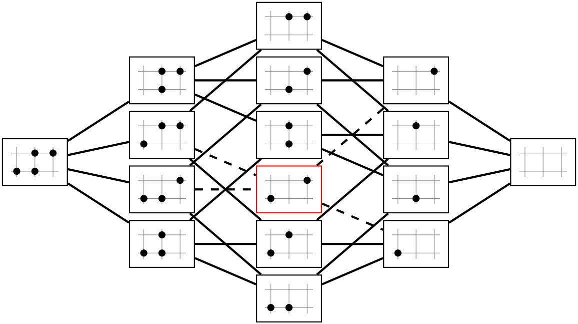
Finally any facet of $\mathrm{bbox}(X)$ is the antecedent of some facet f. $\square$

Conclusion

- better understanding of how to shrink full convex sets
- peelability is a local property in 2D only
- recursive characterization of peelability in arbitrary dimension
- no isolated fully convex set in the hierarchy of fully convex sets

Conclusion

- better understanding of how to shrink full convex sets
- peelability is a local property in 2D only
- recursive characterization of peelability in arbitrary dimension
- no isolated fully convex set in the hierarchy of fully convex sets



Future works

- can we design an envelope operator as thin as $\mathrm{FC}^*(\cdot)$ through peeling, e.g by peeling $\mathrm{Cvxh}\left(X \oplus [-\frac{1}{2}, \frac{1}{2}]^d\right) \cap \mathbb{Z}^d$?
- could it be made increasing ?
- given $Y \subsetneq X$ fully convex sets, can we peel X keeping Y inside ?
- how to define the intersection of two fully convex sets ?

Future works

- can we design an envelope operator as thin as $\mathrm{FC}^*(\cdot)$ through peeling, e.g by peeling $\mathrm{Cvxh}\left(X \oplus [-\frac{1}{2}, \frac{1}{2}]^d\right) \cap \mathbb{Z}^d$?
- could it be made increasing ?
- given $Y \subsetneq X$ fully convex sets, can we peel X keeping Y inside ?
- how to define the intersection of two fully convex sets ?

Future works

- can we design an envelope operator as thin as $\mathrm{FC}^*(\cdot)$ through peeling, e.g by peeling $\mathrm{Cvxh}\left(X \oplus [-\frac{1}{2}, \frac{1}{2}]^d\right) \cap \mathbb{Z}^d$?
- could it be made increasing ?
- given $Y \subsetneq X$ fully convex sets, can we peel X keeping Y inside ?
- how to define the intersection of two fully convex sets ?

Thank you for your attention !

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

Proof.

$\Rightarrow$ z is peelable in X so z is a vertex of $\mathrm{Cvxh}\,(X)$. By Lemma 1, $X \setminus \{z\}$ is 0-convex. Then, recursively along projections using P-convexity, if i is easy, $\pi_i(X \setminus \{z\})$ is P-convex. If i is hard, $\pi_i(X \setminus \{z\})$ is 0-convex by peelability and we can go on recursively.

□

Peelability guarantees full convexity

Theorem (peelability guarantees full convexity)

Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of X iff $X \setminus \{z\}$ is fully convex.

$\Leftarrow$ X and $X \setminus \{z\}$ fully convex implies z is a vertex of $\mathrm{Cvxh}\,(X)$.
If $\mathrm{Easy}_z[X] = \{1, \ldots, d\}$, z is peelable in X. Else, let $i \in \mathrm{Hard}_z[X]$.
We look at $\mathcal{Z}_i := z[\mathrm{e}_i] \cap \mathrm{Cvxh}\,(X)$ and **assume** that $\mathcal{Z}_i \neq \{z\}$.
Let F^ϵ be farthest facet of $\mathrm{Cvxh}\,(X)$ along $\epsilon \mathrm{e}_i$ touching $\mathcal{Z}_i$. It is also a facet of $\mathrm{Cvxh}\,(X \setminus \{z\})$ and separates z from $z + \epsilon \mathrm{e}_i$.
The 1-cell $c =]z, z + \epsilon \mathrm{e}_i[$ touches F^ϵ and $\mathrm{Cvxh}\,(X \setminus \{z\})$, so by full convexity $\mathrm{Extr}\,(c) \cap \mathrm{Cvxh}\,(X \setminus \{z\}) \neq \emptyset$, a **contradiction**.
So $\mathcal{Z}_i = \{z\}$, and this implies that z is a vertex of $\pi_i(X)$.