How to peel fully convex digital sets

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Abstract. Full convexity is an interesting alternative to classical digital convexity since it guarantees connectedness and even simple connectedness in digital spaces $\mathbb{Z}^d$, for any dimension d. This paper aims at giving a better understanding of the monotonicity properties of fully convex digital sets, since earlier works showed that the question was difficult for thin fully convex sets. To decipher the hierarchy of fully convex sets ordered by inclusion, we study how we can peel a fully convex set progressively while keeping its full convexity. We provide a characterization of peelable points and fast algorithms to identify them. Furthermore we show that fully convex set can be peeled one point at a time till reduced to the empty set, similarly to digitally convex sets in the classical sense. The peeling of a fully convex set can be seen as an analog to homotopic thinning processes, but with an additional geometric property.

Keywords: Digital convexity · Homotopic thinning · Digital geometry

1 Introduction

Convexity is a fundamental tool to analyse the geometry of shapes in Euclidean spaces, and it is the basis of many other fields like convex optimization. Our aim is to get a better understanding of the properties of full convexity. Full convexity is a recent alternative definition to digital convexity in the digital lattice $\mathbb{Z}^d$ [17,18], which has the considerable advantage of guaranteeing in arbitrary dimension the digital connectedness of its elements and even the simple connectedness of their geometric realization. This is in stark contrast with classical digital convexity and to most other adaptations of digital convexity (see [16] for a recent overview). Indeed, other geometric definitions of digital convexity add the digital connectedness as an external property [14,13,12,22,7], but this trick does not give any insight at what is a true digital analog to convexity and is not helpful in dimension greater or equal to three. Besides, axiomatic definitions of digital convexity [27,19] or more functional versions [20,23,15] lose the Euclidean geometric intuition of convexity and its relation to digital planarity.

Full convexity is already showing a strong potential in digital geometry analysis [17,18,8,9]: local characterizations of convex and concave parts, unambiguous notion of visibility, exact geodesics, tangent planes, computation time comparable to classical convexity. Fully convex shapes also includes standard digital primitives like digital planes and balls. One can design a fully convex envelope operator idempotent on fully convex set, which gives a proper definition of polyhedral model. Full convexity also enjoys two new nice characterizations [10,11]. However these works have highlighted a difficulty that is proper to the fully convex envelope operator, and which is certainly related to the particular connectedness properties of fully convex sets: the envelope operator may not be increasing, and this may happen around "thin" digital sets.

In order to understand better this peculiar hierarchical structure of fully convex sets and regular digital sets in-between, we propose in this paper to study the *peeling* of a fully convex set. More precisely, we determine when the full convexity property is preserved after removing one point, or even removing several points in parallel. This can be seen as a generalization of the works of Tarsissi *et al.* [24,25,26], which addresses the case of convex polyominoes (hence 4-connected digital convex objects of $\mathbb{Z}^2$) and which focuses on the word combinatorics formulation of these objects [3]. There is also a strong relation with the fruitful works on simple points and homotopic thinning [2,5], which has led to many applications in 3D imaging [21], see also [1,6] for parallel homotopic thinning. In a sense, peeling is a more restrictive homotopic thinning, since it preserves homotopy and convexity, and that kind of property could be of interest in many skeletonization or deformation algorithms.

The paper is organized as follows. Section 2 presents the necessary background information, notations and results. Section 3 provides sufficient conditions for peelability, valid in a sequential or in a parallel process. Section 4 focuses on the characterization of peelability. It is shown to be locally decidable in $\mathbb{Z}^2$, but no more starting from $\mathbb{Z}^3$. We also show that any fully convex set is peelable one point at a time till reduced to the empty set, confirming the hierarchical (yet not arborescent) nature of the space of fully convex sets. Section 5 summarizes the results and outlines some perspectives to this work.

2 Basic notions

We introduce here basic definitions and properties needed in the rest of the paper (they can be found in [18,9,11]). In the sequel, $\mathcal{C}^d$ is the cubical cell complex induced by $\mathbb{Z}^d$. Its 0-dimensional cells are identified to points of $\mathbb{Z}^d$. The set $\mathcal{C}^d_k$ is the set of open k-dimensional cells of $\mathcal{C}^d$.

The *(topological) boundary* ∂Y of a subset Y of $\mathbb{R}^d$ is the set of points in its closure but not in its interior. The star of a cell σ in $\mathcal{C}^d$, denoted by $\mathrm{Star}(\sigma)$, is the set of cells of $\mathcal{C}^d$ whose boundary contains σ, plus the cell σ itself. The closure $\mathrm{Cl}(\sigma)$ of σ contains σ and all the cells in its boundary. In this paper, the cell boundary operator, also denoted by ∂, maps a k-cell to all its proper faces, that is all its k'-cells, $0 \leqslant k' < k$, and not only its $(k-1)$-cells.

A subcomplex K of $\mathcal{C}^d$ with $\mathrm{Star}\,(K) = K$ is *open*, while being *closed* when $\mathrm{Cl}\,(K) = K$. The *body* of a subcomplex K, i.e. the union of its cells in $\mathbb{R}^d$, is written $\|K\|$.

For any real subset Y of $\mathbb{R}^d$, we denote by $\bar{\mathcal{C}}^d_k[Y]$ the set of k-cells whose topological closure intersects Y, i.e. $\bar{\mathcal{C}}^d_k[Y] = \{c \in \mathcal{C}^d_k, \bar{c} \cap Y \neq \emptyset\}$, where $\bar{c} = \|\mathrm{Cl}\,(c)\,\|$ for any cell c. For any subset $Y \subset \mathbb{R}^d$, it is natural to define $\mathrm{Star}\,(Y) := \bar{\mathcal{C}}^d[Y] = \cup_{0 \leqslant k \leqslant d} \bar{\mathcal{C}}^d_k[Y]$. Last, the set $\mathrm{CvxH}\,(Y)$ is the *convex hull* of Y in $\mathbb{R}^d$.

Definition 1 (Full convexity). *A subset $X \subset \mathbb{Z}^d$ is* digitally k-convex *for $0 \leqslant k \leqslant d$ whenever*

$$\bar{\mathcal{C}}^d_k[X] = \bar{\mathcal{C}}^d_k[\mathrm{CvxH}\,(X)]. \tag{1}$$

Subset X is fully (digitally) convex *if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.*

The digital 0-convexity is the classical digital convexity. The following characterization will be useful:

Lemma 1 ([8, Lemma 4]). *A digital set X is fully convex iff* $\mathrm{Star}\,(X) = \mathrm{Star}\,(\mathrm{CvxH}\,(X))$.

Finite convex sets Y in $\mathbb{R}^d$ are intended as $Y = \mathrm{CvxH}\,(\mathrm{Vtcs}\,(Y))$ where $\mathrm{Vtcs}\,(Y)$ is the set of extreme points of Y, also called *vertices* in this paper. By definition, $\mathrm{Vtcs}\,(Y) \subset Y$. Moreover, any one of the vertices of Y cannot be written as a convex linear combination of other points of Y. Since we are considering finite digital sets, the convex set $\mathrm{CvxH}\,(Z)$ is always a bounded polytope of $\mathbb{R}^d$ for all finite subsets Z of $\mathbb{Z}^d$. A vertex z of a convex set $\mathrm{CvxH}\,(Y)$ is such that z does not belong to any open segment $]u, w[$ with $u, w \in \mathrm{CvxH}\,(Y)$. We can specialize this property for digital sets. Let us denote by $z[\mathbf{e}]$ the line with direction $\mathbf{e}$ passing through z. It is the union of two rays $z^+[\mathbf{e}]$ and $z^-[\mathbf{e}]$ which are both semi-infinite in the direction of $+\mathbf{e}$ and resp. $-\mathbf{e}$, and which contains z. We then have the following property.

Lemma 2. *For $X \subset \mathbb{Z}^d$, $z \in \mathrm{Vtcs}\,(\mathrm{CvxH}\,(X))$ and $\varepsilon \in \{+, -\}$, $z^\varepsilon[\mathbf{e}] \cap X \neq \emptyset$ $\Rightarrow z^{-\varepsilon}[\mathbf{e}] \cap X = \emptyset$.*

Proof. Indeed, if both intersections are not empty then z is a convex combination of others points of X and cannot be a vertex of $\mathrm{CvxH}\,(X)$. □

Fully convex sets are stable with respect to orthogonal projections along axes in $\mathbb{R}^d$. Let $I_d = \{1, \ldots, d\}$ and for $j \in I_d$, let π_j denote the orthogonal projector associated to the j-th axis, which consists in omitting the j-th coordinates for all points of $\mathbb{R}^d$. The image of a point in $\mathbb{Z}^d$ by any axis projector is a point in $\mathbb{Z}^{d-1}$. Let us recall the definition of P-convexity [11] and its relation with full convexity.

Definition 2 (P-convexity). *Let $X \subset \mathbb{Z}^d$ be a digital set. The set X is P-*convex *if and only if X is digitally 0-convex (i.e. $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d = X$) and when $d > 1$, for any $j \in I_d$, $\pi_j(X)$ is P-convex in $\mathbb{Z}^{d-1}$.*

Theorem 1 ([11], theorem 5.). *For arbitrary dimension $d \geqslant 1$, for any $X \subset \mathbb{Z}^d$, X is fully convex if and only if X is P-convex.*

The property below recalls that the vertices of an axis projected convex set are to be found among the projection of the vertices of the original convex set.

Proposition 1. *Let X be a 0-convex set. Then for each direction $i \in I_d$, $\pi_i(X)$ is 0-convex and $\forall z \in \mathrm{Vtcs}\left(\mathrm{CvxH}\left(\pi_i(X)\right)\right)$, $\exists z' \in \mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$, $z = \pi_i(z')$.*

3 Sufficient conditions for sequential and parallel peelability

We present in this part two possible ways to peel fully convex sets until reaching a unique point. A first way considers the vertices of the convex hull of the digital set and their possible sequential and parallel elimination. A second way is to peel along the bounding box slice per slice.

The following lemma tells that we can peel a digitally convex set by its vertices. This does not however guarantee to maintain full convexity.

Lemma 3. *Let $X \subset \mathbb{Z}^d$ be a 0-convex set, i.e. $\mathrm{CvxH}\left(X\right) \cap \mathbb{Z}^d = X$, and $V := \mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$. Then for any subset of vertices $Z \subset V$, $X \setminus Z$ is 0-convex.*

Proof. Let $X' := X \setminus Z$. We have $\mathrm{CvxH}\left(X'\right) \cap \mathbb{Z}^d \supset X'$ since $\mathrm{CvxH}\left(\cdot\right)$ is increasing and $X' \subset \mathbb{Z}^d$. Let us thus show that $\mathrm{CvxH}\left(X'\right) \cap \mathbb{Z}^d \subset X'$. This is true when $X' = \emptyset$ so we can suppose that $X' \neq \emptyset$, which implies that $\mathrm{CvxH}\left(X'\right) \cap \mathbb{Z}^d \neq \emptyset$.

We reason by contradiction: let suppose that $\exists x \in \mathrm{CvxH}\left(X'\right) \cap \mathbb{Z}^d$, while $x \notin X'$. We have also $x \in \mathrm{CvxH}\left(X\right) \cap \mathbb{Z}^d$ (since $\mathrm{CvxH}\left(\cdot\right)$ is increasing and $X' \subset X$). The fact that X is 0-convex implies $x \in X$. It follows that $x \in X \setminus X' = Z \subset V$. Since $x \notin X'$ and since $\mathrm{Vtcs}\left(\mathrm{CvxH}\left(X'\right)\right) \subset X'$, $x \notin \mathrm{Vtcs}\left(\mathrm{CvxH}\left(X'\right)\right)$. So, $x \in \mathrm{CvxH}\left(X'\right) \cap \mathbb{Z}^d$ implies that x is a convex combination of $\mathrm{Vtcs}\left(\mathrm{CvxH}\left(X'\right)\right)$. But $X' \subset X$ implies that x is a convex combination of $\mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$, which is a contradiction with $x \in V = \mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$. □

It is necessary to peel convex hull vertices if we peel X one point at a time.

Lemma 4. *Let $X \subset \mathbb{Z}^d$ be a 0-convex set. Let $z \in X$. Then $X \setminus \{z\}$ is 0-convex if and only z is a vertex of $\mathrm{CvxH}\left(X\right)$.*

Proof. ($\Leftarrow$) by using Lemma 3.

($\Rightarrow$) Assume $X \setminus \{z\}$ is 0-convex, i.e. $\mathrm{CvxH}\left(X \setminus \{z\}\right) \cap \mathbb{Z}^d = X \setminus \{z\}$. Hence, $z \notin \mathrm{CvxH}\left(X \setminus \{z\}\right)$ because $z \in \mathbb{Z}^d$. But since $\mathrm{CvxH}\left(X\right) = \mathrm{CvxH}\left(\mathrm{Vtcs}\left(X\right)\right)$, $\mathrm{CvxH}\left(X \setminus \{z\}\right) \subsetneq \mathrm{CvxH}\left(X\right)$ implies that $\mathrm{Vtcs}\left(\mathrm{CvxH}\left(X \setminus \{z\}\right)\right) \neq \mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$. So at least one point of $\mathrm{Vtcs}\left(\mathrm{CvxH}\left(X\right)\right)$ has been removed. Only z was removed so z is a vertex of $\mathrm{CvxH}\left(X\right)$. □

We show below that a local test is sufficient to determine if removing a vertex of a fully convex set keeps the full convexity property.

Theorem 2. *Let X be a non empty finite fully convex set of $\mathbb{Z}^d$ and let $Z \subset \mathrm{Vtcs}(\mathrm{CvxH}(X))$ be a subset of its convex hull vertices. Assume that, for every direction $i \in I_d$, for every $z \in Z$, we have $\{z - \mathbf{e}_i, z + \mathbf{e}_i\} \cap (X \setminus Z) \neq \emptyset$. Then $X \setminus Z$ is fully convex.*

Proof. We prove that $X \setminus Z$ is P-convex. First of all, the result is trivial if Z is empty. Otherwise, the first requirement for P-convexity is that $X \setminus Z$ is 0-convex, which is a consequence of Lemma 3. This is enough to conclude if $d = 1$.

Otherwise, if $d \geqslant 2$, we have to prove that for every direction $i \in I_d$, $\pi_i(X \setminus Z)$ is P-convex. Let $X' := X \setminus Z$. Let $Y := \pi_i(Z)$. We consider the two subsets of X and X' that projects exactly onto Y, i.e. $C := \pi_i^{-1}(Y) \cap X$ and $C' := \pi_i^{-1}(Y) \cap X'$. It is clear that C is non empty since Z is included in C, and that $\pi_i(C) = Y$ ($Z \subset C$, $\pi_i(Z) \subset \pi_i(C) \subset Y$ and $\pi_i(Z) = Y$ by definition).

Looking at C', for any $y \in Y$, there is some $z \in Z$ with $z = \pi_i(z)$. By hypothesis $\{z - \mathbf{e}_i, z + \mathbf{e}_i\} \cap (X \setminus Z) \neq \emptyset$. Let $z' \in \{z - \mathbf{e}_i, z + \mathbf{e}_i\} \cap (X \setminus Z)$. Clearly $\pi_i(z') = \pi_i(z \pm \mathbf{e}_i) = y$. Furthermore $z' \in X \setminus Z = X'$ and projects onto Y. So $z' \in C'$. Since for any $y \in Y$, we have found some $z' \in C'$ that projects onto it, we have $\pi_i(C') \supset Y$. And obviously $\pi_i(C') = \pi_i(\pi_i^{-1}(Y) \cap X') \subset Y$. This proves $\pi_i(C') = Y$.

The sets $\pi_i(X \setminus C)$ and $\pi_i(C)$ are disjoint since C contains all the elements of X that projects onto Y. The same is true for $\pi_i(X' \setminus C')$ and $\pi_i(C')$. The equality $X \setminus C = X' \setminus C'$ holds (using $\pi_i(x) \notin Y = \pi_i(Z)$ implies $x \notin Z$) since:

$$x \in X' \setminus C' \Leftrightarrow x \in X, x \notin Z, \pi_i(x) \notin Y \Leftrightarrow x \in X, \pi_i(x) \notin Y \Leftrightarrow x \in X \setminus C.$$

$$\begin{aligned}
\text{Then} \quad \pi_i(X') &= \pi_i(X' \setminus C') \cup \pi_i(C') && \text{(disjoint union and } C' \subset X') \\
&= \pi_i(X' \setminus C') \cup \pi_i(C) && \text{(since } \pi_i(C') = \pi_i(C) = Y) \\
&= \pi_i(X \setminus C) \cup \pi_i(C) && (X \setminus C = X' \setminus C') \\
&= \pi_i(X). && \text{(since } C \subset X)
\end{aligned}$$

Since X was fully convex by hypothesis, it is also P-convex, so its projection $\pi_i(X)$ is P-convex. So $\pi_i(X')$ is P-convex and Theorem 1 concludes. □

The previous theorem induces a very simple algorithm to thin a fully convex set X sequentially up to a given constraint set Y: (i) extract vertices of the convex hull of X, (ii) check if one vertex that is not in Y satisfies the theorem hypothesis, (iii) remove it from X and loop to (i) till no such vertex can be found. However it requires to compute or update the convex hull of X at each step. The same theorem shows sufficient conditions under which we can peel X in parallel. The convex hull must still be updated but for generally less iterations.

It is also quite easy to remove points on the bounding box of a fully convex set X. We recall [18, Lemma 5]: if X is fully convex and $Y \subset \mathbb{R}^d$ is *stable*, then $X \cap Y$ is fully convex. Now [18, Lemma 6] states that any half-space of integer intercept and axis normal vector is stable, and any intersection of stable sets is itself stable. The result below is straightforward by choosing P as the half-space just touching one side of the bounding box.

Proposition 2. *Let $X \subset \mathbb{Z}^d$ fully convex, non-empty, finite. Let (l, u) be the lowest and highest points of its axis-aligned bounding box. If P is a $d-1$-hyperplane orthogonal to an axis, with l or u in P, then $X \setminus P$ is fully convex.*

We can thus remove any one of the side of X and yet keep its full convexity.

4 Peelability

Previous peelability results apply only in some circumstances and do not show that one can peel a set one point at a time. To provide a more precise answer on the deletion of points in fully convex sets while maintaining the full convexity, we therefore provide a definition of what peelable really means.

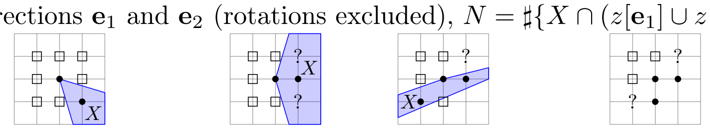
Definition 3. *For a fully convex set X and some $z \in X$, we say that z is* peelable in X *when $z \in \mathrm{Vtcs}\,(\mathrm{CvxH}\,(X))$ and, when $d \geqslant 2$, for every direction $i \in I_d$, either (i) $z - \mathbf{e}_i \in X$ or $z + \mathbf{e}_i \in X$, or (ii) $\pi_i(z)$ is peelable in $\pi_i(X)$.*

When $d = 1$, a point z is peelable as soon as it is a vertex of $\mathrm{CvxH}\,(X)$. Let us also remark that when using projections, a peelable point z must but projected onto a vertex of the convex hull of the projection such that the condition $z \in \mathrm{Vtcs}\,(\mathrm{CvxH}\,(X))$ must be checked again. We now explain why peelability is not a local notion. To do this, we first prove that, for $d \leqslant 2$, peelability of a vertex z is indeed local, and then provides a counter example in $d \geqslant 3$ of local decidability of peelability for z.

For some $z \in \mathbb{Z}^d$, we consider $\mathcal{N}_X(z) = \mathrm{Cl}\,(\mathrm{Star}\,(z)) \cap X$ the neighborhood of z in X. Let us show now that peelability is locally decidable in 2d (see appendix A for the complete list of local configurations and a fast implementation).

Lemma 5. *Let $X \subset \mathbb{Z}^2$, finite and full convex. Then point $z \in \mathrm{Vtcs}\,(\mathrm{CvxH}\,(X))$ is peelable in X if and only if z is peelable in $\mathcal{N}_X(z)$.*

Proof. The cases depend on the number of intersections between X and rays $z[.]$ along directions $\mathbf{e}_1$ and $\mathbf{e}_2$ (rotations excluded), $N = \sharp\{X \cap (z[\mathbf{e}_1] \cup z[\mathbf{e}_2])\}$.



$N = 0$ $N = 1$ $N = 1$ $N = 2$

($N = 0$) z is peelable in X and in $\mathcal{N}_X(z)$ because z is projected at a vertex for any direction $\mathbf{e}_1$ and $\mathbf{e}_2$. **($N = 1$ left)** z is peelable in X and in $\mathcal{N}_X(z)$ because z is only projected using $\mathbf{e}_2$ and it is a vertex both in projection for X and for $\mathcal{N}_X(z)$. **($N = 1$ right)** z is not peelable along direction $\mathbf{e}_2$ because it is covered by an edge of $\mathrm{CvxH}\,(X)$, but it is also not peelable in $\mathcal{N}_X(z)$ along $\mathbf{e}_2$ because it is not a vertex in projection. Note that point $e = z - \mathbf{e}_1 - \mathbf{e}_2$ must belong to X as X is fully convex: otherwise $\mathrm{CvxH}\,(X)$ would intersect a 1d cell c not touching X so $c \notin \mathrm{Star}\,(X)$, but $c \in \mathrm{Star}\,(\mathrm{CvxH}\,(X))$ which is a contradiction to Lemma 1. **($N = 2$)** Point z is peelable in both X and $\mathcal{N}_X(z)$ because it has a neighbor using both direction $\mathbf{e}_1$ and $\mathbf{e}_2$. □

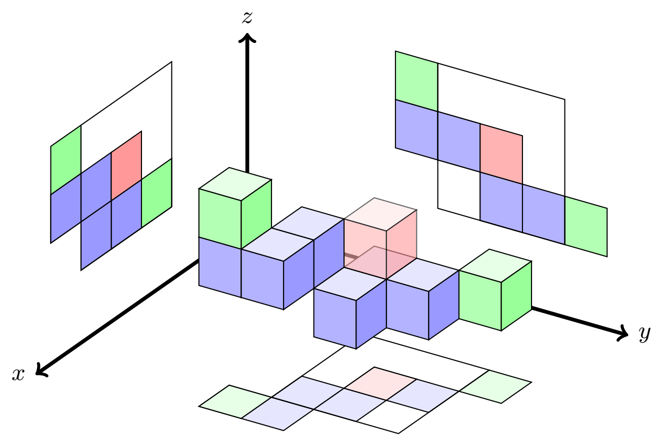


Fig. 1. Counter-example to local decidability of peeling in 3D. The fully convex set X is composed of the blue, green and red points (represented as cubes). The red point is the vertex p. As one can see, p is locally peelable along each projection (see Figure 3 in appendix). However it is not a vertex of the convex hull of the projection of X along axis x into the plane (yz), since it lies in the middle of the edge formed by the projection of the two green points. So $X \setminus \{p\}$ is not P-convex, hence not full convex.

Starting at $d = 3$, peelability cannot be locally decided as explicited in Figure 1. It is quite easy to generalize this example so that it requires arbitrary large neighborhoods to decide the peelability.

We remark first that peelability imposes no specific ordering on projections.

Lemma 6. *For $J = (i_1, \cdots, i_p)$, let $\pi_J := \bigcirc_{k=1..p} \pi_{i_k}$ be the projection composing every axis projections in J, and let σ_J be a permutation of J. Then $\pi_{\sigma_J} = \pi_J$.*

Proof. It is sufficient to show it for two projections with directions $i \neq j$. Let us pick $z = (z_1, \cdots, z_d) \in \mathbb{Z}^d$. $\pi_k(z) = z_{\overline{k}}$ where $\overline{k}$ means omission of the coordinate of index k that is the set $I_d \setminus \{k\}$. So $(\pi_i \bigcirc \pi_j)(z) = z'_{\overline{i}}$ where $z' = z_{\overline{j}}$. Since $i \neq j$, $z'_{\overline{i}} = z_{\overline{i,j}}$. Thus, $(\pi_i \bigcirc \pi_j)(z) = z_{\overline{i,j}} = z_{\overline{j,i}} = (\pi_j \bigcirc \pi_i)(z)$. □

We start with an initial fully convex set X in $\mathbb{Z}^d$. A point $z \in \mathbb{Z}^d$ is peelable only if $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$. If it is the case there exists a set of directions, called *easy directions*, defined by $\mathrm{Easy}_X[z] := \{i \in I_d : \exists \varepsilon_i \in \{+1, -1\}, z + \varepsilon_i \mathbf{e}_i \in X\}$. So to check peelability, projections will only be tested for *hard directions*, defined as $\mathrm{Hard}_X[z] := I_d \setminus \mathrm{Easy}_X[z]$. If $\mathrm{Hard}_X[z] = \emptyset$ then z is a peelable point and all such points are peelable.

Hard directions are stable in projection since emptyness of $z[\mathbf{e}_i]$ is stable under π_i and obviously emptyness in projection implies emptyness in π^{-1}. So the following property holds:

Lemma 7. *Suppose X is a non empty fully convex set in $\mathbb{Z}^d$, and let $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$. Assume $i \neq j$ are two different directions of I_d with $\pi_i(z) \in \mathrm{Vtcs}(\mathrm{CvxH}(\pi_i(X)))$. We have $j \in \mathrm{Hard}_{\pi_i(X)}[\pi_i(z)] \Leftrightarrow j \in \mathrm{Hard}_X[z]$.*

Suppose that $\mathrm{Hard}_X[z] \neq \emptyset$. Let us remark that the set $\mathrm{Easy}_X[z]$ is preserved in the sense that for any projection $i \in \mathrm{Hard}_X[z]$ and any $\varepsilon_i \in \{+1, -1\}$, it holds that $z + \varepsilon_i \mathbf{e}_k \in X \Rightarrow \pi_i(z) + \varepsilon_i \pi_i(\mathbf{e}_k) \in \pi_i(X)$ for $k \in \mathrm{Easy}_X[z]$. In other words, when checking the peelability of z and taking projections in $\mathrm{Hard}_X[z]$, we preserve the fact that directions in $\mathrm{Easy}_X[z]$ do not have to be checked. Hence, we have to study what happened when peelability conditions are not verified for hard directions.

Point $z \in X$ is not peelable if and only if either $z \notin \mathrm{Vtcs}(\mathrm{CvxH}(X))$ or $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$ and there exists a direction i such that: (i) $z - \mathbf{e}_i \notin X$ and $z + \mathbf{e}_i \notin X$ and (ii) $\pi_i(z)$ is not peelable in $\pi_i(X)$. So it is clear that the condition stopping the iteration of the projections is that $z \notin \mathrm{Vtcs}(\mathrm{CvxH}(X))$ in some projection. The following lemma explains this case geometrically.

Lemma 8. *Let X fully convex and let $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$ and a direction $i \in \mathrm{Hard}_X[z]$. Then $\pi_i(z) \notin \mathrm{Vtcs}(\pi_i(\mathrm{CvxH}(X))) \Leftrightarrow \exists F^\varepsilon$ face of $\mathrm{CvxH}(X)$ such that $z^\varepsilon[\mathbf{e}_i] \cap \mathrm{relint}(F^\varepsilon) \neq \emptyset$ for some $\varepsilon \in \{+1, -1\}$.*

Proof. (Preamble) Since $z \in X$, $z + \mathbf{e}_i \notin X$ and $z - \mathbf{e}_i \notin X$, we have that $z + k\mathbf{e}_i \notin X$ for any $k \in \mathbb{Z} \setminus \{0\}$ since X is fully convex (0-convexity suffices).

($\Rightarrow$) If $\pi_i(z) \notin \mathrm{Vtcs}(\pi_i(\mathrm{CvxH}(X)))$ then there exists a segment $[u, v]$ in $\pi_i(\mathrm{CvxH}(X))$ such that $\pi_i(z) \in]u, v[$. Considering a segment of the form $]u', v'[$ with $u' \in \pi_i^{-1}(u)$, $v' \in \pi_i^{-1}(v)$ and $u', v' \in \mathrm{CvxH}(X)$, we note that $z[\mathbf{e}_i] \cap]u', v'[\neq \emptyset$ and that $\pi_i(z[\mathbf{e}_i] \cap]u', v'[) = \pi_i(z)$ for all possible u' and v'.

Suppose that $z[\mathbf{e}_i] \cap \mathrm{CvxH}(X) = \{z\}$. For any u' and v', there is a point of $]u', v'[$ in $z[\mathbf{e}_i]$. By hypothesis this point is z. Hence z is a convex combination of u' and v' which is a contradiction. So, $z[\mathbf{e}_i] \cap \mathrm{CvxH}(X)$ is a segment S containing z as a vertex. Consider the other vertex v of S and a facet F^ε containing it. Point v cannot be a vertex of F^ε since this would be a point in $\mathbb{Z}^d$ other than z and belonging to $z[\mathbf{e}_i]$ which is impossible by the preamble. Hence $v \in \mathrm{relint}(F^\varepsilon)$. We get that $z^\varepsilon[\mathbf{e}_i] \cap \mathrm{relint}(F^\varepsilon) \neq \emptyset$.

($\Leftarrow$) The point v in $z^\varepsilon[\mathbf{e}_i] \cap \mathrm{relint}(F^\varepsilon)$ is not in $\mathbb{Z}^d$ with $\varepsilon \in \{+1, -1\}$. So the projections of any vertex of the face F^ε are different from $\pi_i(z)$. Hence $\pi_i(z)$ belongs to the convex hull of the projections of $\mathrm{Vtcs}(\mathcal{F}^\varepsilon)$ because convex combinations are preserved by the linear operator π_i. This implies that $\pi_i(z) \notin \mathrm{Vtcs}(\pi_i(\mathrm{CvxH}(X)))$. □

Lemma 8 tells us that when the projection of a vertex z is covered by the projection of a face then z would not be peelable. This indicates that points on the boundary of X are the points which are the most difficult to cover using projections. We now give the main result for the usage of peelability which legitimates its definition.

Theorem 3. *Let X be a non empty finite fully convex set of $\mathbb{Z}^d$. Then, z is a peelable point of $X \Leftrightarrow X \setminus \{z\}$ is fully convex.*

Proof. ($\Rightarrow$) If z is a peelable point of X, then $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$. We know that vertex removal preserves 0-convexity (Theorem 2) when condition (i) of peelability is verified for all directions. Else we can recursively use the peelability along projections, as in the definition of P-convexity. Noticing that 0-convexity is preserved by projections of full convex sets - while it is not true in general for arbitrary 0-convex sets - and using Theorem 1, it is clear that removing peelable points preserves full convexity.

($\Leftarrow$) Suppose that both X and $X \setminus \{z\}$ are full convex. We cannot have $\mathrm{CvxH}(X) = \mathrm{CvxH}(X \setminus \{z\})$ because it would imply that $\mathrm{CvxH}(X) \cap \mathbb{Z}^d = \mathrm{CvxH}(X \setminus \{z\}) \cap \mathbb{Z}^d$ which is impossible because z is in the former set but not in the later one. So since $\mathrm{CvxH}(X) \neq \mathrm{CvxH}(X \setminus \{z\})$ then $\exists z_0 \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$ such that $z_0 \notin \mathrm{Vtcs}(\mathrm{CvxH}(X \setminus \{z\}))$. So in particular $z_0 \notin X \setminus \{z\}$. But $X \setminus (X \setminus \{z\}) = \{z\}$. Hence $z = z_0$ and $z \in \mathrm{Vtcs}(\mathrm{CvxH}(X))$.

If $\mathrm{Easy}_X[z] = I_d$ then z is peelable. Otherwise $\mathrm{Hard}_X[z]$ is not empty and let $i \in \mathrm{Hard}_X[z]$. We consider $\mathcal{Z}_i = z[\mathbf{e}_i] \cap \mathrm{CvxH}(X)$. Assume $\mathcal{Z}_i \neq \{z\}$. So we consider the farthest face F^ε such that $z[\mathbf{e}_i] \cap F^\varepsilon \neq \emptyset$ using direction $\varepsilon \mathbf{e}_i$ from z. By construction F^ε if also a face of $\mathrm{CvxH}(X \setminus \{z\})$. But F^ε separates z from $z + \varepsilon e_i \notin X$. Consider now the 1d cell $c \in \mathcal{C}^d$ whose realization is $\|c\| =]z; z + \varepsilon \mathbf{e}_i[$. It is clear that $\|c\| \cap F^\varepsilon \neq \emptyset$. Since F^ε is a face of $\mathrm{CvxH}(X \setminus \{z\})$, we have $\|c\| \cap \mathrm{CvxH}(X \setminus \{z\}) \neq \emptyset$. So since $X \setminus \{z\}$ is full convex, we must have that $\mathrm{Vtcs}(\|c\|) \cap (X \setminus \{z\}) \neq \emptyset$. This is a contradiction since neither points are in $X \setminus \{z\}$.

So, for any $i \in \mathrm{Hard}_X[z]$, we must have $\mathcal{Z}_i = z[\mathbf{e}_i] \cap \mathrm{CvxH}(X) = \{z\}$. But using Lemma 8, this implies that $\pi_i(z) \in \mathrm{Vtcs}(\pi_i(\mathrm{CvxH}(X)))$ for direction i. Hence z is a peelable point. □

Thanks to Theorem 3, we know that peelable points are necessary and sufficient to peel fully convex sets while maintaining full convexity. However, we have to prove that such points exist for every non empty fully convex set. This is done by the following theorem (see Figure 2 for an illustration).

Theorem 4. *Suppose X is a non empty fully convex set in $\mathbb{Z}^d$. Then, any facet of the bounding box of X contains a peelable point.*

Proof. We are going to prove the theorem by recursion on the dimension. To do this, we need to remark that for any direction $i \in I_d$ the bounding box facets for the projection $\pi_i(X)$ are precisely the projections of the facets of the bounding box of X. Thus, any facet f of the bounding box of $\pi_i(X)$ can be written as $\pi_i(F)$ where F is a facet of the bounding box of X. We can made this fact precise simply by looking at the facets of the bounding box of X whose normal vectors are the $\pm \mathbf{e}_j$ for $j \neq i$. Indeed, these are precisely the facets whose projections define the bounding box of $\pi_i(X)$.

Let us now consider the starting case: $d = 1$. As previously seen, a point z is peelable for $d = 1$ if and only if z is a vertex of X. So X has one peelable point if $X = \{z\}$ and two peelable points when X has two points or more. So the property is true for $d = 1$.

Let us suppose that for any dimension less than d, the property is true. We consider a direction $i \in I_d$. Let us denote by $X_i := \pi_i(X)$. Let us consider a facet f of the bounding box of X_i. As recalled previously, $f = \pi_i(F)$ where F is

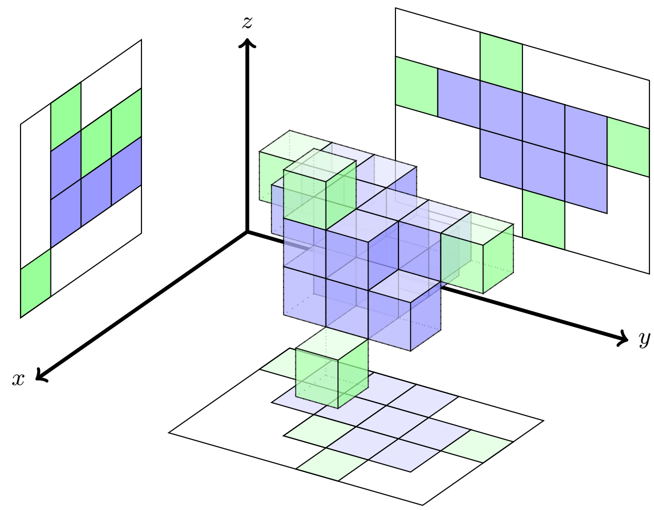


Fig. 2. Any facet of the bounding box of a fully convex set X contains a peelable point (displayed in green).

a facet of the bounding box of X. By hypothesis, the dimension of X_i is strictly lower than the dimension of X. Thus applying the hypothesis since $\pi_i(X)$ is fully convex, we get that in f, there exists a peelable point z_f. Using Lemma 1, we now that, z_f being a vertex of $\mathrm{CvxH}\,(X_i)$ in f, it is the projection of a vertex z_F of $\mathrm{CvxH}\,(X)$. By construction $z_F \in F$. But since we know that hard directions are stable by projection (Lemma 7), we know that all directions which are hard in projections behave well because z_f is peelable. The peelability of z_F is thus solely govern by the remaining direction i. There are two cases. If $i \in \mathrm{Easy}_X[z_F]$ then by definition z_F is peelable. If $i \in \mathrm{Hard}_X[z_F]$, then we note that $z_f \in \mathrm{Vtcs}\,(X_i)$ and thus direction i behave well in the sense that $\pi_i(z_F) = z_f$ is a vertex of X_i. Hence again z_F is peelable. Overall we obtain that the face F has a peelable point z_F. But, since the all the facets of the bounding box of X are constructible by the facets of the bounding box of X_i (by considering all the directions $i \in I_d$), we obtain that the property of true for dimension d. $\square$

5 Conclusion and perspectives

In this paper we have studied the inclusion hierarchy of full convex sets in $\mathbb{Z}^d$ by examining if and how one can peel a full convex set and keep this property throughout the process. Such sets must be peeled necessary at vertices. Using the equivalence of full convexity with P-convexity, we have exhibited easy sufficient conditions for parallel peeling. We have then characterized the peelability conditions through projections and proved that any full convex set is peelable at least one point at a time, which may be found lying on its bounding box.

Our next objective is to prove that a full convex set X is peelable under the constraint of leaving the full convex subset $Y \subset X$ unchanged. It could lead to a new method for computing the fully convex envelope of a digital set, by Minkowski dilation, then peeling. The current envelope operator [8,9] is indeed an iterative finite growing process but with unclear bound: a peeling method would be much easier to bound. A related problem is defining a meaningful intersection for full convex sets, and peelability could help in this endeavour.

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A Peelable configurations in 2D

```
// Check peelability of some point in a set X, given its neighborhood
// input:  cfg  the encoded neighborhood of the point (in {0,...,255})
// output: true iff the point is peelable wrt its neighborhood
// prerequisite: the point should be a vertex of CvxH(X)
bool isLocallyPeelable( unsigned int cfg )
{ // 45 configurations are peelable
  static const unsigned int p[ 8 ]
    = { 0x5151f1ff, 0x11110001, 0x00101000b, 0x01010003,
        0x000000ab, 0x00000000, 0x00001000b, 0x00010003 };
  return p[ cfg >> 5 ] & ( 1 << (cfg & 0x1f) );
}
```

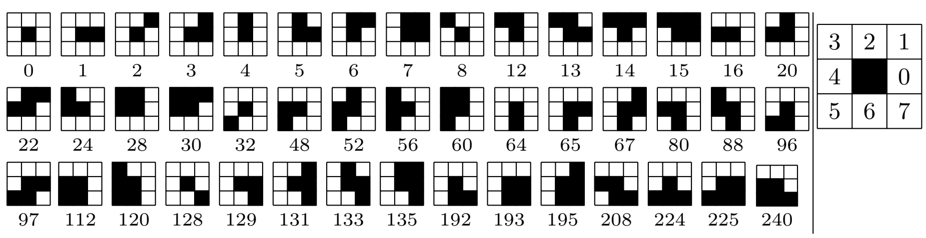

Fig. 3. The 45 configurations peelable in 2D. Each configuration is encoded as an 8-bit integer, each bit b being set to 1 whenever the b-th neighbor (as displayed on the right) is present in the digital set.