New characterizations of full convexity

Fabien Feschet† & Jacques-Olivier Lachaud‡*

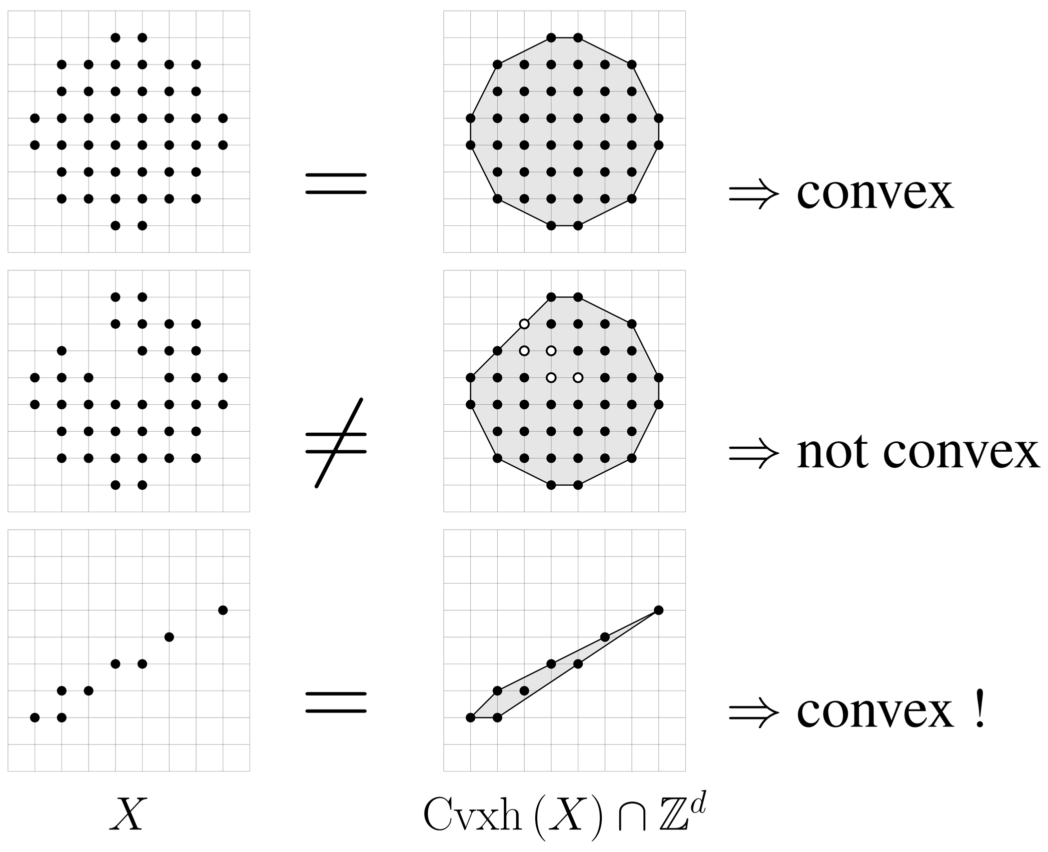
† Université Clermont Auvergne, CNRS, ENSMSE, LIMOS, F-63000 Clermont-Ferrand, France
fabien.feschet@u-auvergne.fr
‡ Université Savoie Mont Blanc, CNRS, LAMA, F-73000 Chambéry, France
jacques-olivier.lachaud@univ-smb.fr

* This work was partly funded by StableProxies ANR-22-CE46-0006 research grant.

Full convexity versus usual digital convexity

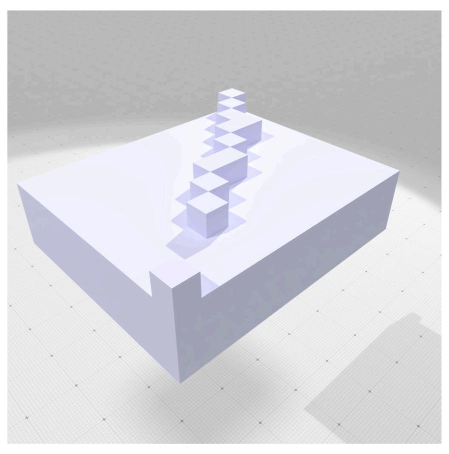
Usual digital convexity (or 0-convexity) [1]

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$



Problem

Digital convexity does not imply **connectedness** in $\mathbb{Z}^d$, and includes weird sets.



digitally convex !

Full convexity [2]

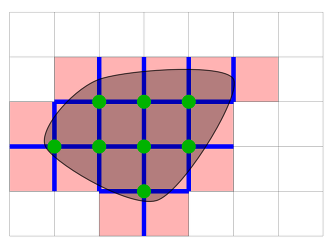
A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{Cvxh}(X)]. \quad (1)$$

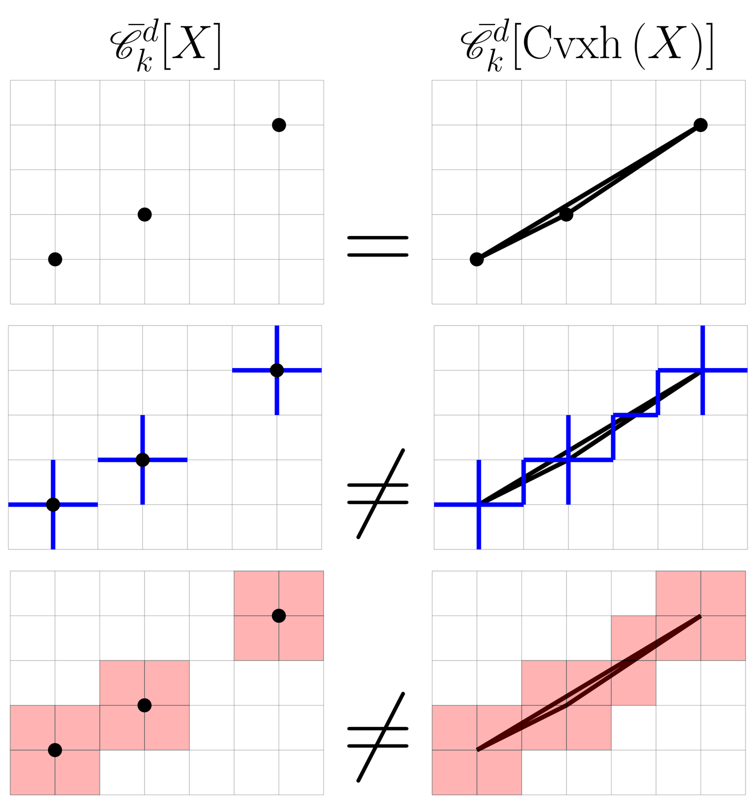
Subset X is **fully convex** if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

Grid intersection complex of $Y \subset \mathbb{R}^d$

$$\bar{\mathscr{C}}^d_k[Y] := \{c \in \mathscr{C}^d_k, \bar{c} \cap Y \neq \emptyset\}$$



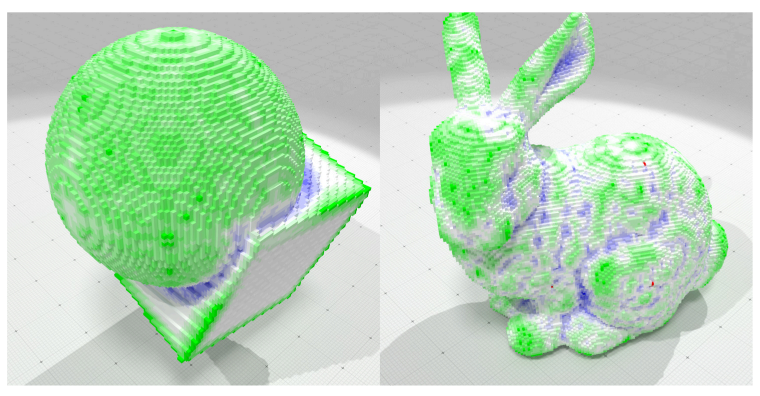
cells $\bar{\mathscr{C}}^d_0[Y]$, $\bar{\mathscr{C}}^d_1[Y]$, $\bar{\mathscr{C}}^d_2[Y]$



X is 0-convex, but **not fully convex** !

Full convexity

- eliminates thin digital convex sets
- implies digital connectedness
- implies simple connectedness
- includes digital lines and planes
- is computable efficiently in $\mathbb{Z}^d$
- has many applications



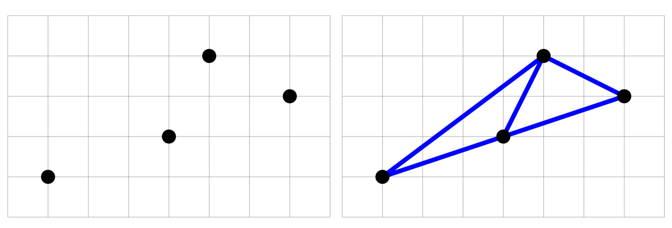
exact local geometric analysis

Segment convexities S- and S^k- and generalizations

"Natural" segment convexity

Convexity in $\mathbb{R}^d$ $X \subset \mathbb{R}^d$ is convex iff $\forall p, q \in X$, then $[pq] \subset X$

MP-convexity in $\mathbb{Z}^d$ $X \subset \mathbb{Z}^d$ is MP-convex iff $\forall p, q \in X$, then $[pq] \cap \mathbb{Z}^d \subset X$ [3]

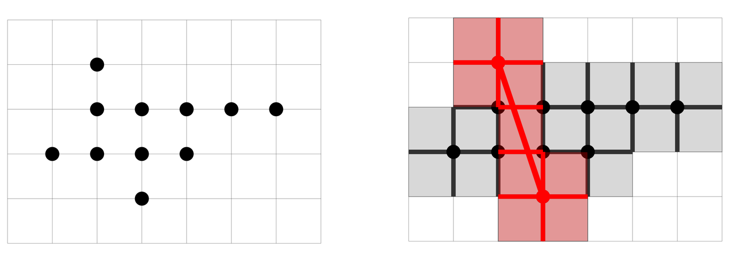


MP-convex ! but not S-convex

Let $\mathrm{Star}(Y) := \cup_{k=0}^d \bar{\mathscr{C}}^d_k[Y]$

S-convexity in $\mathbb{Z}^d$

$X \subset \mathbb{Z}^d$ is **S-convex** iff $\forall p, q \in X, \mathrm{Star}([pq]) \subset \mathrm{Star}(X)$



X S-convex $\mathrm{Star}([pq]) \subset \mathrm{Star}(X)$

S^k-convexity in $\mathbb{Z}^d$

$X \subset \mathbb{Z}^d$ is **S^k-convex** iff $\forall p_1, \dots, p_k \in X, \mathrm{Star}(\mathrm{Cvxh}(p_1, \dots, p_k)) \subset \mathrm{Star}(X)$

References

[1] Ulrich Eckhardt. Digital lines and digital convexity. In *Digital and Image Geometry, Advanced Lectures*, volume 2243 of *LNCS*, page 209–228, 2001.

[2] Jacques-Olivier Lachaud. An alternative definition for digital convexity. *J. Math. Imaging Vis.*, 64(7):718–735, 2022.

[3] Marvin Minsky and Seymour A Papert. *Perceptrons: an introduction to computational geometry*. MIT press, 1969.

Equivalence of full convexity and S^k-convexity

Theorem 1

For $d \geqslant 1$, $k \geqslant 2$, full convexity implies S^k-convexity.

Proof. Let us consider a fully convex set X. Let T a k-tuple in X.

$$\mathrm{Cvxh}(T) \subset \mathrm{Cvxh}(X)$$
$$\Rightarrow \mathrm{Star}(\mathrm{Cvxh}(T)) \subset \mathrm{Star}(\mathrm{Cvxh}(X))$$
$$\Leftrightarrow \mathrm{Star}(\mathrm{Cvxh}(T)) \subset \mathrm{Star}(X)$$

□

Theorem 2

S-convexity implies full convexity in $\mathbb{Z}^2$

Theorem 3

S-convexity does not imply full convexity in $\mathbb{Z}^d$, $d \geq 3$.

Theorem 4

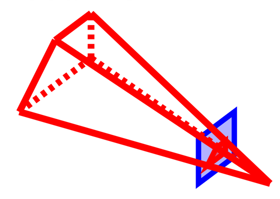
S^d-convexity implies full convexity in $\mathbb{Z}^d$, $d \geq 2$.

Key lemma for Theorem 4

If X is S^d-convex and $\mathrm{Cvxh}(X) \cap c \neq \emptyset$ for a cell $c \in \mathscr{C}^d$, then $\mathrm{Cvxh}(X)$ must touch a 0-cell $e \in \partial c \cap X$.

Proof by contradiction. Assume

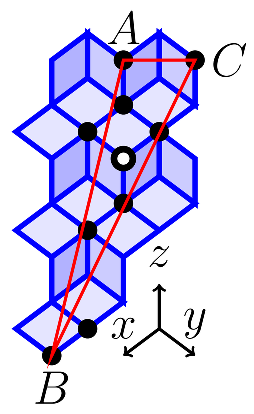
$\mathrm{Cvxh}(X) \cap \partial c = \emptyset$

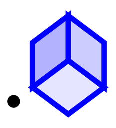


- ∃ a supporting $d-1$-hyperplane of $\partial\mathrm{Cvxh}(X)$ touching c
- ∃ a d-tuple T of X on this hyperplane
- so $\mathrm{Star}(\mathrm{Cvxh}(T)) \subset \mathrm{Star}(X)$ by S^d-convexity, hence $c \in \mathrm{Star}(X)$
- thus $\exists e \in X$ and $e \in \partial c$.

□

Proof of Theorem 3. Counterexample:



-  is a digital plane $\mathfrak{P}$
- Set $X \subset \mathfrak{P} \subset \mathbb{Z}^3$ as •
- Points $A, B, C \in X$ lie on top of $\mathfrak{P}$
- Point ∘ $= \frac{1}{3}(A + B + C)$ also but $\notin X$

X is S-convex but not 0-convex, hence not fully convex. □

Characterization 1

S^d-convexity is equivalent to full convexity in $\mathbb{Z}^d$

Projection convexity is equivalent to full convexity

P-convexity

Let $\mathscr{P}_j$ be the projector orthogonal to j-th axis. $X \subset \mathbb{Z}^d$ is **P-convex** iff:

(i) X is 0-convex, and

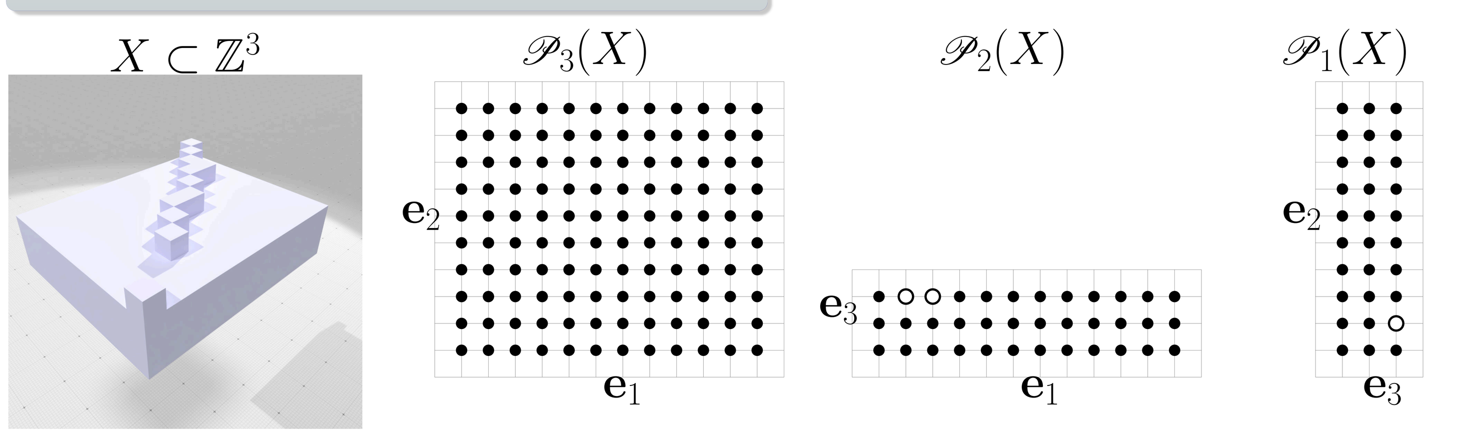
(ii) when $d > 1$, $\forall j, 1 \leqslant j \leqslant d$, $\mathscr{P}_j(X)$ is P-convex.

Characterization 2

P-convexity is equivalent to full convexity in $\mathbb{Z}^d$

Proof. See paper. □

A recursive definition of full convexity !



not P-convex   0-convex   **not** 0-convex   **not** 0-convex

Corollary 1

Any digital subset of the digital hypercube is fully convex.

Corollary 2

Any intersection of any Euclidean d-dimensional ball with $\mathbb{Z}^d$ is fully convex.

Application: a measure for full convexity

Convexity measure $M_d(X)$

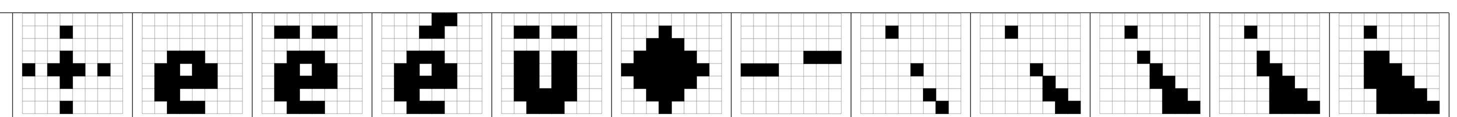
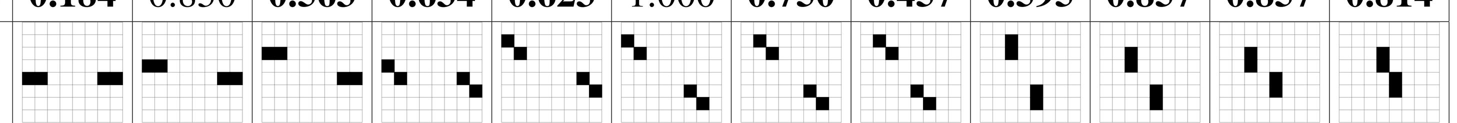
Let $M_d(X)$ be any digital convexity measure of $X \subset \mathbb{Z}^d$, e.g. $M_d(\emptyset) = 1$,

$$M_d(X) := \frac{\#(X)}{\#(\mathrm{Cvxh}(X) \cap \mathbb{Z}^d)}.$$

Full convexity measure $M^F_d(X)$

$$M^F_1(X) := M_1(X), \text{ else for } d > 1$$

$$M^F_d(X) := M_d(X) \prod_{k=1}^{d} M^F_{d-1}(\pi_k(X)).$$

A												
$M_d(A)$	0.360	0.850	0.656	0.724	0.727	1.000	1.000	1.000	1.000	1.000	1.000	0.950
$M^F_d(A)$	**0.184**	0.850	**0.563**	**0.634**	**0.623**	1.000	**0.750**	**0.457**	**0.595**	**0.857**	**0.857**	**0.814**
A												
$M_d(A)$	0.500	1.000	0.667	0.500	0.500	1.000	0.667	1.000	0.667	0.800	0.667	1.000
$M^F_d(A)$	**0.250**	**0.500**	**0.222**	**0.250**	**0.200**	**0.381**	**0.296**	**0.533**	**0.296**	**0.427**	**0.444**	1.000

Theorem

Let $A \subset \mathbb{Z}^d$ finite. Then $M^F_d(A) = 1$ if and only if A is fully convex and $0 < M^F_d(A) < 1$ otherwise. Besides $M^F_d(A) \leqslant M_d(A)$ in all cases.