New characterizations of full convexity*

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Abstract. *Full convexity* has been recently proposed as an alternative definition of digital convexity. In contrast to classical definitions, fully convex sets are always connected and even simply connected whatever the dimension, while remaining digitally convex in the usual sense. Several characterizations were proposed in former works, either based on lattice intersection enumeration with several convex hulls, or using the idempotence of an envelope operator. We continue these efforts by studying simple properties of real convex sets whose digital counterparts remain largely misunderstood. First we study if we can define full convexity through variants of the usual continuous convexity via *segments* inclusion, i.e. "for all pair of points of X, the straight segment joining them must lie within the set X". We show an equivalence of full convexity with this segment convexity in dimension 2, and counterexamples starting from dimension 3. If we consider now d-simplices instead of a segment (2-simplex), we achieve an equivalence in arbitrary dimension d. Secondly, we exhibit another characterization of full convexity, which is recursive with respect to the dimension and uses simple axis projections. This latter characterization leads to two immediate applications: a proof that digital balls are indeed fully convex, and a natural progressive measure of full convexity for arbitrary digital sets.

Keywords: Digital geometry · Digital convexity · Full convexity

1 Introduction

Convexity is a fundamental tool for analyzing functions and shapes. For digital spaces $\mathbb{Z}^d$, digital convexity was first defined as the intersection of real convex sets of $\mathbb{R}^d$ with $\mathbb{Z}^d$ (e.g. see survey [11]). Although easy to formulate, resulting digital convex sets may not be digitally connected in general. This deficiency prevents local shape analysis, so many works have tried to constrain the connectedness of such sets, for instance by relying on digital lines [5,1] or extensions of digital functions [6,7]. Most works are limited to 2D, and 3D extensions do not solve

* This work was partly funded by StableProxies ANR-22-CE46-0006 research grant.

all geometric issues [4]. It is also possible to define digital convexity from the intersection of Euclidean convex sets with the lattice cubical complex [12], but it remains unclear how to determine if a given cell complex is indeed convex. We take an interest here in a recent alternative definition of digital convexity, called *full convexity* [8,9], which guarantees the connectedness and even the simple connectedness of fully convex sets. It is important to note that classical digital primitives like standard and naive lines and planes are indeed fully convex, so most of the classical tools of digital geometry fall into this setting.

Full convexity has already proven to be a fruitful framework for analyzing the geometry of digital shapes [9,2,3]: local characterization of convex, concave and planar parts, geodesics and visibility problems, reversible and tight reconstruction of triangulated surfaces, digital polyhedral models. The purpose of this paper is mainly to study its core properties and to exhibit new characterizations of full convexity. Such results are important both from a theoretical perspective (new characterizations lead to better understanding of what is digital convexity) and from a practical algorithmic point of view. For instance the characterization by idempotence of some cell operations [3, Theorem 2] lead to an enveloppe operator that builds a fully convex hull. The first morphological characterization [8, Theorem 5] provides an exact algorithm to check full convexity involving $2^d - 1$ convex hull computations and lattice point enumeration; the second characterization in terms of maximal cells of [3, Theorem 5] lead to an exact algorithm that requires only one convex hull computations and lattice point enumeration.

After recalling some essential background related to full convexity (Section 2), we study in Section 3 if we can define a digital analogue of convexity through its classical formulation of "inclusion of every straight segment". Originally studied by Minsky and Papert [10], their definition was far too unrestrictive since it included many digital weird sets. We propose here a more reasonable analogue (S-convexity) that we prove equivalent to full convexity in $\mathbb{Z}^2$, but not in higher dimension. We then extend this definition to an analogue of "inclusion of every d-simplices" (S^d-convexity) to get another characterization of full convexity in $\mathbb{Z}^d$. Then we propose in Section 4 a recursive convexity (P-convexity): a digital set is P-convex whenever it is digitally convex and each one of its projections along axis is P-convex. Quite surprisingly, we show that P-convexity is indeed equivalent to full convexity. This clearly opens new perspective for studying digital sets, and we already provide here two immediate applications in Section 5. One is the proof that subsets of the lattice hypercube as well as digital balls are always fully convex, the other is a measure for digital sets that characterizes fully convex sets. We conclude and describe a few perspectives to this work in Section 6.

2 Useful definitions and properties

We introduce here basic definitions and properties needed in the rest of the paper. The references are [9][3]. In the sequel, $\mathscr{C}^d$ is the cubical cell complex induced by $\mathbb{Z}^d$. Its 0-dimensional cells are identified to points of $\mathbb{Z}^d$. The set $\mathscr{C}^d_k$ is the set of open k-dimensional cells of $\mathscr{C}^d$.

The *(topological) boundary* ∂Y of a subset Y of $\mathbb{R}^d$ is the set of points in its closure but not in its interior. The star of a cell σ in $\mathscr{C}^d$, denoted by $\mathrm{Star}(\sigma)$, is the set of cells of $\mathscr{C}^d$ whose boundary contains σ, plus the cell σ itself. The closure $\mathrm{Cl}(\sigma)$ of σ contains σ and all the cells in its boundary. In this paper, the cell boundary operator, also denoted by ∂, maps a k-cell to all its proper faces, that is all its k'-cells, $0 \leqslant k' < k$, and not only its $(k-1)$-cells.

A subcomplex K of $\mathscr{C}^d$ with $\mathrm{Star}(K) = K$ is *open*, while being *closed* when $\mathrm{Cl}(K) = K$. The *body* of a subcomplex K, i.e. the union of its cells in $\mathbb{R}^d$, is written $\|K\|$.

For any real subset Y of $\mathbb{R}^d$, we denote by $\bar{\mathscr{C}}^d_k[Y]$ the set of k-cells whose topological closure intersects Y, i.e. $\bar{\mathscr{C}}^d_k[Y] = \{c \in \mathscr{C}^d_k, \bar{c} \cap Y \neq \emptyset\}$, where $\bar{c} = \|\mathrm{Cl}(c)\|$ for any cell c. For any subset $Y \subset \mathbb{R}^d$, it is natural to define $\mathrm{Star}(Y) := \bar{\mathscr{C}}^d[Y]$. Last, the set $\mathrm{CvxH}(Y)$ is the *convex hull* of Y in $\mathbb{R}^d$.

Definition 1 (Full convexity). *A non empty subset $X \subset \mathbb{Z}^d$ is* digitally k-convex *for $0 \leqslant k \leqslant d$ whenever*

$$\bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{CvxH}(X)]. \tag{1}$$

Subset X is fully (digitally) convex *if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.*

The following two characterizations will be useful:

Lemma 1 ([2, Lemma 4]). *A digital set X is fully convex iff* $\mathrm{Star}(X) = \mathrm{Star}(\mathrm{CvxH}(X))$.

Lemma 2 ([3, Theorem 2]). *A digital set X is fully convex iff $X = \mathrm{FC}(X)$, with* $\mathrm{FC}(X) := \mathrm{Extr}(\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X))))$.

The Skel operator builds the *skeleton of a set of cells*, which is defined as the intersection of all cell complexes whose star includes the set of cells. The *extreme operator* Extr maps a set of cells to their set of vertices, which is a digital set. The skeleton can be characterized as follows.

Lemma 3 ([3], Lemma 12). *Let us consider $Y \subset \mathbb{R}^d$. For any $c \in \mathscr{C}^d$, $c \in \mathrm{Skel}(\mathrm{Star}(Y)) \iff \|c\| \cap Y \neq \emptyset$ and $\|\partial c\| \cap Y = \emptyset$.*

For a cell $c \in \mathscr{C}^d$, we say that a convex set K is *framed* within c if $K \cap \|c\| \neq \emptyset$ and $K \cap \|\partial c\| = \emptyset$. So, considering Lemma 12 in [3] applied to $\mathrm{CvxH}(X)$, $c \in \mathscr{C}^d$ belongs to $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$ iff $\|c\| \cap \mathrm{CvxH}(X) \neq \emptyset$ and $\|\partial c\| \cap \mathrm{CvxH}(X) = \emptyset$. In other words, a cell c is in the skeleton of a convex hull if and only if the convex hull is framed within c. Note that if the dimension of c is zero, i.e. c is a lattice point, then ∂c is empty, so a convex set is framed within a lattice point if and only if this convex set contains it.

As $\mathrm{CvxH}(\|\partial c\|)$ is the topological closure of $\mathrm{CvxH}(\|c\|) = \|c\|$, if K is not framed within c, then K must either intersect only the boundary of c or c and its boundary. Moreover if K is closed, then $K \cap \|c\|$ is closed. The framed properties cannot happen when K is the convex hull of points in $\mathbb{Z}^d$ and $\dim c = d$. Indeed, if K is framed within c then obviously K is entirely included in $\|c\|$. But this is impossible since $\|c\|$ does not contain any points. So, to have the framed property, we must have $\dim c \leq d - 1$.

3 Segment convexity and generalizations

In the Euclidean space, convexity is defined through inclusion of every straight line segment joining two points of the set. Minsky and Papert, in their famous book on perceptrons [10], proposed a digital analogue of segment convexity, phrased "A [digital] set X fails to be convex if and only if there exists three [digital] points such that q is in the line segment joining p and r and, $p \in X$, $q \notin X$, $r \in X$." This definition of digital convexity is unfortunately not at all equivalent to the digital convexity (see Figure 1), and even less to full convexity.

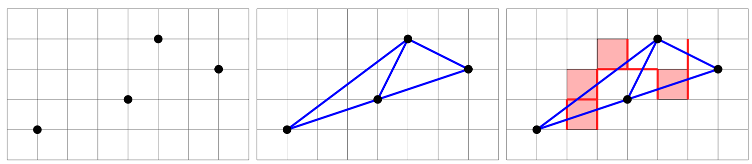


Fig. 1. Minsky-Papert segment convexity versus S-convexity: (a) MP-convex set X, since (b) each segment does not touch any other lattice point. But X is not S-convex, since (c) these segments touch 1-cells and 2-cells that are not in $\mathrm{Star}\,(X)$ (in red).

Therefore we propose the following digital analogues of "segment" convexity, which are much closer to digital convexity.

Definition 2 (S-convexity and S^k-convexity). *We say that a digital set $X \subset \mathbb{Z}^d$ is* S-convex *whenever* $\forall p \in X, \forall q \in X$, $\mathrm{Star}\,(\mathrm{CvxH}\,(\{p, q\})) \subset \mathrm{Star}\,(X)$. *Furthermore, for $k \geq 2$, the set X is* S^k-convex *whenever for any k-tuple of points T of X (not necessarily distinct), we have* $\mathrm{Star}\,(\mathrm{CvxH}\,(T)) \subset \mathrm{Star}\,(X)$.

Otherwise said for S-convexity (resp. S^k-convexity), any pair of points of X (resp. any k-tuple of points of X) must be tangent to X in the terminology of [9]. It is obvious that S^2-convexity is the S-convexity and that S^{k+1}-convexity implies S^k-convexity. We establish the following results in this section.

Theorem 1. *For $d \geqslant 1$, $k \geqslant 2$, full convexity implies S^k-convexity.*

Proof. Let us consider a fully convex set X. If we consider a k-tuple T in X then $\mathrm{CvxH}\,(T) \subset \mathrm{CvxH}\,(X)$ because the convex hull operator $\mathrm{CvxH}\,()$ is increasing. But $\mathrm{Star}\,()$ is also an increasing operator hence $\mathrm{Star}\,(\mathrm{CvxH}\,(T)) \subset \mathrm{Star}\,(\mathrm{CvxH}\,(X)) = \mathrm{Star}\,(X)$ (the latter equality given by Lemma 1). So X is S^k-convex. □

Theorem 2. *S-convexity is equivalent to full convexity in $\mathbb{Z}^1$ and $\mathbb{Z}^2$.*

Theorem 3. *S-convexity is not equivalent to full convexity starting from $\mathbb{Z}^3$.*

Theorem 4. *S^d-convexity is equivalent to full convexity in $\mathbb{Z}^d$, for $d \geqslant 2$.*

Some preliminary lemmas will be used to extract impossible configurations for the S^k-convexity. With such situations, we are then able to relate full convexity and S^k-convexity. Theorem 3 is proven by a counter-example.

Lemma 4 (Blocking lemma). *Let us consider an S^k-convex digital set $X \subset \mathbb{Z}^d$. Let us consider $p \in \mathbb{Z}^d$ but $p \notin X$. If Y is a subset of X such that $p \notin \mathrm{CvxH}(Y)$ and such that $\mathrm{CvxH}(Y)$ has at most $k-1$ vertices, then there exists a pointed convex cone C with apex p such that $C \cap X = \emptyset$.*

Proof. Let us consider an S^k-convex digital set $X \subset \mathbb{Z}^d$. Let us consider $p \in \mathbb{Z}^d \setminus X$, and a subset Y of X such that $p \notin \mathrm{CvxH}(Y)$. Since $\mathrm{CvxH}(Y)$ is a Euclidean convex set and since p does not belong to it, there exists an hyperplane H_p separating $\mathrm{CvxH}(Y)$ from p. We denote by $H_p^{(-)}$ the half-space containing $\mathrm{CvxH}(Y)$ and by $H_p^{(+)}$ the half-space containing p. Let us consider any $(k-1)$-tuple T containing the vertices of $\mathrm{CvxH}(Y)$ with repetition if needed. We denote by $Y^{(-)}$ the pointed convex cone $Y^{(-)} = \mathrm{CvxH}(T \cup \{p\})$. Let us denote by $Y^{(+)}$ the pointed cone obtained by a central symmetry with center p of $Y^{(-)}$. We claim that $Y^{(+)} \cap X = \emptyset$.

Let us hence suppose on the contrary that $Y^{(+)} \cap X \neq \emptyset$ and consider a point x in this intersection. Since T is a $(k-1)$-tuple in X, the set $T_x = T \cup \{x\}$ is a k-tuple in X. Since X is S^k-convex, we have $\mathrm{Star}(\mathrm{CvxH}(T_x)) \subset \mathrm{Star}(X)$. But by the construction with central symmetry with center p, we have $p \in \mathrm{CvxH}(T_x)$. So p is a point in $\mathrm{Star}(X)$, which implies $p \in X$. This is a contradiction. $\square$

For any two points $u,\ v \in \mathbb{Z}^d$, let us denote by $(u; v)$ the line passing through u and v. Let us denote by $(\infty; v)_u$ the semi-line in $(u; v)$ containing u and stopping at v. The semi-line $(v; \infty)_u$ is an infinite semi-line in $(u; v)$ containing v but no other point of $(\infty; v)_u$.

Lemma 5 (Line Blocking lemma). *Let us consider an S^k-convex digital set $X \subset \mathbb{Z}^d$. Let us consider $p \in \mathbb{Z}^d,\ p \notin X$. For any $x \in X$, $(p; \infty)_x \cap X = \emptyset$.*

Proof. Let us consider an S^k-convex digital set $X \subset \mathbb{Z}^d$. Let us consider $p \in \mathbb{Z}^d,\ p \notin X$. Let us consider the $(k-1)$-tuple Y by using x, $k-1$ times. Then applying Lemma 4, we get a pointed cone C, which is in fact $(p; \infty)_x$, whose intersection with X is empty. $\square$

Lemma 6 (Grid lemma). *Let us consider a finite S^k-convex digital set $X \subset \mathbb{Z}^d$ with $k \geqslant 2$. Let us denote by $\mathbb{Z}[e_j]$ any line in $\mathbb{Z}^d$ directed by the canonical basis vector e_j. Then, $\mathbb{Z}[e_j] \cap X$ is connected.*

Proof. By using Lemma 5, we known that if we can find a point p outside X then a semi-line will be blocked for X. The same is true for any 1d slice of X, a 1d slice being $\mathbb{Z}[e_j] \cap X$. But since X is finite, there always exists a point p in $\mathbb{Z}[e_j]$ which is outside X. We consider a point p such that one of its neighbor in $\mathbb{Z}[e_j]$ is in X. There are two possible extreme choices for p called $p^{(-)}$ and $p^{(+)}$ and so two neighbors $x^{(-)}$ and $x^{(+)}$. We claim that $\mathbb{Z}[e_j] \cap X = [x^{(-)}; x^{(+)}] \cap \mathbb{Z}^d$. Indeed, if a lattice point q is missing in this interval then it belongs to the convex hull of a $k \geq 2$ tuple in X. Hence it must belong to X. $\square$

Lemma 7 (Star lemma). *Let us consider a digital set $X \subset \mathbb{Z}^d$. If X is S^d-convex then $\forall c \in \mathscr{C}^d$, dim $c > 0$, if $\mathrm{CvxH}(X)$ is framed within c then $\exists e \in \partial c$, dim $e = 0, e \in X$.*

Proof. Let us consider an S^d-convex digital set $X \subset \mathbb{Z}^d$. Let us consider $c \in \mathscr{C}^d$, $\dim c > 0$, $K = \|c\| \cap \mathrm{CvxH}(X) \neq \emptyset$. Suppose that $\mathrm{CvxH}(X)$ is framed within $\|c\|$. In other words, since $\mathrm{CvxH}(X)$ is convex and closed, we have $K \subsetneq \|c\|$. The framed property implies that there exists supporting hyperplanes of $\mathrm{CvxH}(X)$ separating $\|c\|$ from $\|\partial c\|$, but touching $\partial \mathrm{CvxH}(X)$ too. We can pick one of these supporting hyperplanes that contains a face of $\partial \mathrm{CvxH}(X)$: it has an affine dimension $0 \leq k \leq d-1$. This convex face can be decomposed into a set of k-simplexes $\{S_i\}$. At least one of them, say S_{i_c}, intersects $\|c\|$, otherwise K cannot be a subset of $\|c\|$. So $c \in \mathrm{Star}(S_{i_c})$. Now S_{i_c} is the convex hull of a $k+1$-tuple of points of X, with $k+1 \leqslant d$. By definition of the S^d-convexity, we must have $c \in \mathrm{Star}(X)$. But X is a set of points so at least one of the vertices of c must be in X. We have just found some cell $e \in \partial c$, $\dim e = 0, e \in X$.

If we assume now that $\mathrm{CvxH}(X)$ is not framed within c, we have both $\mathrm{CvxH}(X) \cap \|c\| \neq \emptyset$ and $\mathrm{CvxH}(X) \cap \|\partial c\| \neq \emptyset$. So, we can consider any cell f in $\|\partial c\|$. If $\mathrm{CvxH}(X)$ is framed within f, we got a 0-dimensional cell e, otherwise, we choose a lower dimensional cell in the boundary of f. At each step the dimension decreases. So at the end, either we find a 0-dimensional cell e belonging to $\mathrm{CvxH}(X)$ or we find a cell within which $\mathrm{CvxH}(X)$ is framed, and we also obtain a 0-dimensional cell e. $\square$

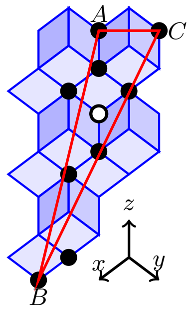
Lemma 8. *Let us consider a digital set $X \subset \mathbb{Z}^d$. If X is S^d-convex then $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$.*

Proof. Let us consider an S^d-convex digital set $X \subset \mathbb{Z}^d$. Let us consider a cell c in $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$. Suppose that $\dim c > 0$. Using Lemma 7 since X is S^d-convex, $\exists e \in \partial c$, $\dim e = 0, e \in X$. This implies that $\mathrm{Star}(c) \subset \mathrm{Star}(e)$. Hence, c cannot belong to $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$, but e does. So we got a contradiction with $\|\partial c\| \cap \mathrm{CvxH}(X) = \emptyset$. It follows that c must be a 0-dimensional cell, that is a point. This implies that the Skeleton of $\mathrm{CvxH}(X)$ only contains points such that $\mathrm{Extr}(\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))) = \mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X)))$. But since we only have 0-dimensional cells in the skeleton, we get $\mathrm{Skel}(\mathrm{Star}(\mathrm{CvxH}(X))) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. We conclude that $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$. $\square$

Proof (Theorem 2). Using Theorem 1, we only study the case of an S-convex set.

Let us consider an S-convex set X in $\mathbb{Z}^1$. Using Lemma 6, we have that X must be an interval of points. Hence X is fully convex because $\mathrm{Star}(\mathrm{CvxH}(X)) = \mathrm{Star}(X)$.

Let us consider an S-convex set X in $\mathbb{Z}^2$. Let us suppose that there exists a lattice point z in $\left(\mathrm{CvxH}(X) \cap \mathbb{Z}^2\right) \setminus X$. We first remark that z is strictly interior to $\mathrm{CvxH}(X)$. Indeed, z cannot be a vertex of the convex hull since the vertices are always in X. Furthermore z cannot be on an edge of $\mathrm{CvxH}(X)$ because Lemma 5 would imply a contradiction. We consider the set $L(c)$ of 1-dimensional



This is piece of the standard digital plane $P = \{(x, y, z) \in \mathbb{Z}^3, 0 \leqslant x + y + 2z < 4\}$. The set X, represented as black disks, is a subset of P. The set Y is the union of X with the point $M = (1, 1, -1)$ represented as a white disk. We have $A = (0, 0, 0), B = (4, 2, -3), C = (-1, 1, 0)$. All four points A, B, C, M have remainder 0 in the digital plane. One can check that $M = \frac{1}{3}(A + B + C)$, hence $M \in \mathrm{CvxH}(X)$. But M does not belong to any straight segment between any pair of points of X.

Fig. 2. Counter-example to S^2-convexity implies full convexity: set Y is S^2-convex and fully convex, while X is S^2-convex but not fully convex (and not digitally 0-convex).

cells $c \in \mathscr{C}^d$ in $\mathrm{Star}(z)$. Because z is strictly interior to $\mathrm{CvxH}(X)$, we must have $L(c) \cap \mathrm{CvxH}(X) \neq \emptyset$. So we consider a 1-dimensional cell c in $L(c)$ such that $\|c\| \cap \mathrm{CvxH}(X) \neq \emptyset$. Using Lemma 7, we get a point $e \in \partial c$ with $e \in X$. Let us consider the 1 dimensional cell c_{op} which is on the same axis as z and e but which does not have e on its boundary. We note that $c_{op} = \{f, z\}$. But, $f \notin X$, because of Lemma 6. So we can move on this axis in the direction of f until the 0-cell on the boundary of the 1-cell is not in $\mathrm{CvxH}(X)$. Let us call $c_{\lim}$ this last 1-dimensional cell. We have that $\|c_{\lim}\| \cap \mathrm{CvxH}(X) \neq \emptyset$ and no points in $\partial c_{\lim}$ are in X which is in contradiction with Lemma 7. So, $\left(\mathrm{CvxH}(X) \cap \mathbb{Z}^2\right) \setminus X = \emptyset$ and it follows that $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^2$.

We now use Lemma 8 to get that $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^2$.

We have obtained the equality $\mathrm{FC}(X) = \mathrm{CvxH}(X) \cap \mathbb{Z}^2 = X$ which completes the proof that X is indeed a fully convex set in $\mathbb{Z}^2$. $\square$

Proof (Theorem 3). A counter-example is given on Fig. 2. We should note that large random constructions of S^2-convex sets by simulation did not lead to any counter-examples. In fact, problematic examples correspond to sets for which there exists an integer point in the relative interior of a maximal face which does not belong to any segments of the face, and are thus very unlikely to be generated randomly. $\square$

The main result in dimension 2 for S-convexity is that for an S-convex set X, we necessarily have $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^2$. As Theorem 3 states it, this property failed to be true in higher dimension. This explains why we must rely on the more restrictive S^d-convexity when increasing the dimension.

Proof (Theorem 4). Let us consider an S^d-convex set X. Its convex hull $\mathrm{CvxH}(X)$ can be partitioned into a set of d-dimensional simplices C_i in $\mathbb{Z}^d$ for which we can rely on the S^d-convexity property of X. Indeed, if there exists a point $z \in \left(\mathrm{CvxH}(X) \cap \mathbb{Z}^d\right) \setminus X$, then $\exists i_z$ such that $z \in C_{i_z}$. Since X is S^d-convex, this means that $z \in \mathrm{Star}(\mathrm{CvxH}(C_{i_z})) \subset \mathrm{Star}(X)$. So since z is a point, this implies that z is in X which is a contradiction. So, $\left(\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d\right) \setminus X = \emptyset$, otherwise said $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d = X$, that is X is 0-convex. But applying Lemma 8, we get that $\mathrm{FC}(X) = \mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$ too. Gathering the two equalities, we obtain $\mathrm{FC}(X) = X$, which means that X is fully convex. □

4 Projection convexity

We here study the stability of fully convex sets with respect to orthogonal projections along axes in $\mathbb{R}^d$. We denote by π_j the orthogonal projector associated to the j-th axis, which consists in omitting the j-th coordinates for all points of $\mathbb{Z}^d$. By direct extension, those projectors are defined for cells in $\mathscr{C}^d$. π_j are called *axis projectors*. Those projectors share the property that the image of a cell $c \in \mathscr{C}^d$ is a cell in $\mathscr{C}^{d-1}$ and those projections are the only projections for which this property is true. Moreover, the image of a point in $\mathbb{Z}^d$ by any axis projector is a point in $\mathbb{Z}^{d-1}$. Let us define a convexity by projections as follows.

Definition 3 (P-convexity). *Let $X \subset \mathbb{Z}^d$ be a digital set. The set X is P-*convex *if and only if X is digitally 0-convex (i.e.* $\mathrm{CvxH}\,(X) \cap \mathbb{Z}^d = X$*) and when $d > 1$, for any j, $1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex in $\mathbb{Z}^{d-1}$.*

Quite surprisingly, we have the equivalence of P-convexity with full convexity.

Theorem 5. *For arbitrary dimension $d \geqslant 1$, for any $X \subset \mathbb{Z}^d$, X is fully convex if and only if X is P-convex.*

Proof. The fact that a fully convex X is also P-convex directly follows from (i) fully convex sets are in particular digitally 0-convex, (ii) projection $\pi_j(X)$ are fully convex in $\mathbb{Z}^{d-1}$ as shown in [3, Lemma 23].

When $d = 1$, 0-convexity is equivalent to full convexity (consequence of [9, Lemma 4] with $d = 1$), so P-convexity implies full convexity for this dimension. Let us now show that this implication holds for $d > 1$.

Suppose that $X \subset \mathbb{Z}^d$ is P-convex but not fully convex. Since X is 0-convex by definition, we know that $X = \mathrm{CvxH}\,(X) \cap \mathbb{Z}^d$. So, any 0-dimensional cell of X is in the skeleton $\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(X)))$ and $\mathrm{CvxH}\,(X)$ does not contain any other points. So since the fact that X is not fully convex implies that $X \neq \mathrm{FC}(X)$, there exists some cell c in $\mathrm{Skel}\,(\mathrm{Star}\,(\mathrm{CvxH}\,(X)))$ with $\dim c > 0$. Indeed, the extreme operator only add points for cells of strictly positive dimension. We can characterize c by the framed property: $\mathrm{CvxH}\,(X) \cap \|c\| \neq \emptyset$ and $\mathrm{CvxH}\,(X) \cap \|\partial c\| = \emptyset$.

Let y be some point of $\mathrm{CvxH}\,(X) \cap \|c\|$, which is also not in X since it is not a lattice point. Being in $\mathrm{CvxH}\,(X)$, by Carathéodory's convexity theorem, there exists at most $d+1$ extreme points $v_0, v_1, \ldots, v_d$ of $\mathrm{CvxH}\,(X)$ such that y is a convex linear combination of these points. We thus have $y = \sum_{i=0}^{d} \lambda_i v_i$, with $\sum_{i=0}^{d} \lambda_i = 1$ and $\forall i, 0 \leqslant i \leqslant d, \lambda_i \geq 0$. Being extreme points of $\mathrm{CvxH}\,(X)$, every v_i is a lattice point in X.

Let $k := \dim c$. There exists k different directions $J := (j_i)_{i=1,\dots,k}$ such that $\dim(\pi_{j_i}(c)) = k-1$. Let π_J be the composition of the projections $\pi_{j_1}, \dots, \pi_{j_k}$ (the order is not important since these operators commute). It follows that $\pi_J(c)$ is a lattice point, say z. Since $y \in c$, we have also $z = \pi_J(y)$. It follows that:

$$z = \pi_J\left(\sum_{i=0}^{d} \lambda_i v_i\right) = \sum_{i=0}^{d} \lambda_i(\pi_J(v_i)). \qquad \text{(by linearity of } \pi_j\text{)}$$

Since every $v_i \in X$, we have shown that $z \in \mathrm{CvxH}\,(\pi_J(X))$ and $z \in \mathbb{Z}^{d-k}$. Since X is P-convex, its projection $\pi_J(X)$ is 0-convex, so $\mathrm{CvxH}\,(\pi_J(X)) \cap \mathbb{Z}^{d-k} = \pi_J(X)$ and $z \in \pi_J(X)$.

The last assertion means that there exist a lattice point $x \in X$, such that $z = \pi_J(x)$. Let $C := \pi_J^{-1}(z)$ be the affine k-dimensional space containing z. It contains in particular c, its boundary ∂c, the point y and the lattice point x. We have $y \in c$ while x cannot be in c since it is a lattice point. Furthermore $\mathrm{CvxH}\,(X) \cap \|\partial c\| = \emptyset$ implies also $x \notin \partial c$ (because $x \in X$). Now $y \in \mathrm{CvxH}\,(X)$ by definition, $x \in X \subset \mathrm{CvxH}\,(X)$, so the straight segment $[y; x]$ must be included in $\mathrm{CvxH}\,(X)$. It also included in the k-dimensional space C so it is a connected path from the interior of cell c to the exterior of c in C: it must cross ∂c at some point x'. By convexity we have $x' \in [y; x] = \mathrm{CvxH}\,(\{x, y\}) \subset \mathrm{CvxH}\,(X)$. But we have also $x' \in \partial c$ and $\mathrm{CvxH}\,(X) \cap \|\partial c\| = \emptyset$. This is a contradiction.

Hence $X = FC(X)$ which is equivalent to X fully convex. □

5 Applications

We present here two quite immediate applications of the previous characterization of fully convex sets.

5.1 New fully convex digital sets

Proposition 1 *Let A be a digital set with bounding box defined by a lowest point $\mathbf{p}$ and a highest point $\mathbf{q}$, with $\forall i, 1 \leqslant i \leqslant d, |q^i - p^i| \leqslant 1$. Then A is fully convex.*

Proof. If A is empty then the conclusion holds. In dimension 1, it is clear that any subset of the digital set $\{x, x+1\}$ is 0-convex hence P-convex. Assuming now that the property holds for dimension $d-1$, let us prove it for $A \subset \mathbb{Z}^d$. Note first that any subset of the hypercube H defined by $\mathbf{p}$ and $\mathbf{q}$ is digitally 0-convex, since any vertex of $\mathrm{CvxH}\,(A)$ must belong to A since it is a vertex of H too. Each projection $\pi_j(A)$ is also a non-empty subset of a $d-1$-hypercube $\pi_j(H)$, and is thus P-convex by induction hypothesis. The conclusion follows from the equivalence of P-convexity with full convexity (Theorem 5). □

A *digital ball of $\mathbb{Z}^d$* is the intersection of *any* Euclidean d-dimensional ball with $\mathbb{Z}^d$. Note that the center of the ball may by any Euclidean point of $\mathbb{R}^d$ and the radius may be any real non negative value.

Proposition 2 *Any digital ball of $\mathbb{Z}^d$ is fully convex.*

Proof. We show that this is true by induction on the dimension d. For $d = 1$, full convexity is equivalent to 0-convexity, and a 1-dimensional digital ball is 0-convex. Let us assume that digital balls are fully convex for dimension $d - 1$, $d \geq 2$, and let us prove that this assertion is true for dimension d.

By Theorem 5, it is equivalent to show that d-dimensional digital balls are P-convex. Let X be some digital ball of center $\mathbf{c} \in \mathbb{R}^d$ and radius r, i.e. $X = B_r(\mathbf{c}) \cap \mathbb{Z}^d$. First of all, X is 0-convex since it is the intersection of a real convex set with the grid $\mathbb{Z}^d$. We have to show that, for any axis direction j, $1 \leqslant j \leqslant d$, $\pi_j(X)$ is P-convex.

We write the proof for $j = d$ for simplicity of writings, but the proof is the same for the other directions. The main argument is that the projection of a digital ball is itself a $d-1$-digital ball but possibly with a slightly lower radius.

For $\mathbf{x} \in X$, we have $\|\pi_d(\mathbf{x}) - \pi_d(\mathbf{c})\|^2 = \|\mathbf{x} - \mathbf{c}\|^2 - |x_d - c_d|^2 \leq r^2 - |x_d - c_d|^2$, where x_d and c_d are the d-th coordinate of their respective point. It is obvious that, for any $z \in \mathbb{Z}$, $|z - c_d| \geq |c_d - \lfloor c_d \rceil| =: \alpha$, where $\lfloor \cdot \rceil$ is the round operator. It follows that $\|\pi_d(\mathbf{x}) - \pi_d(\mathbf{c})\|^2 \leq r^2 - \alpha^2 =: \rho^2$. We have just shown that $\pi_d(X) \subset B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1}$.

Reciprocally, let us now pick a point $\mathbf{y} \in B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1}$. It follows that $\|\mathbf{y} - \pi_d(\mathbf{c})\|^2 \leq r^2 - \alpha^2$. Let us build a d-dimensional lattice point $\mathbf{z}$ as $\mathbf{z} = (y_1, \ldots, y_{d-1}, \lfloor c_d \rceil)$. We have:

$$\|\mathbf{z} - \mathbf{c}\|^2 = \|\mathbf{y} - \pi_d(\mathbf{c})\|^2 + |\lfloor c_d \rceil - c_d|^2 \leqslant r^2 - \alpha^2 + \alpha^2 = r^2.$$

This proves that $\mathbf{z} \in X$. Since $\pi_d(\mathbf{z}) = \mathbf{y}$, it holds that $\mathbf{y} \in \pi_d(X)$. It follows that $B_\rho(\pi_d(\mathbf{c})) \cap \mathbb{Z}^{d-1} \subset \pi_d(X)$.

So $\pi_d(x)$ is a $d-1$-dimensional digital ball, hence is fully convex or, equivalently, P-convex. Since all projections are P-convex, it holds that X is P-convex or, equivalently, fully convex. $\square$

Note that the argument does not work for an arbitrary ellipsoid since some projections of digital ellipsoids might not be digital ellipsoids: this is due to possible missing points when the ellipsoid is too thin.

5.2 A progressive measure for full convexity

Sometimes it is useful to quantify a property over a set in a progressive manner. For instance there exists measures of circularity, convexity, straightness, disconnectedness, and so on (e.g. [15,13,14]). We would like here to define a *full convexity measure* over a digital set, that has value exactly 1 for fully convex sets, while decreasing to zero as the digital set looks less and less like a fully convex set.

Let $M_d(A)$ be any d-dimensional digital convexity measure of digital set A. A choice could be for instance for finite sets:

$$M_d(A) := \frac{\#(A)}{\#(\mathrm{CvxH}(A) \cap \mathbb{Z}^d)}, \qquad M_d(\emptyset) = 1. \tag{2}$$

The *full convexity measure* M_d^F for $A \subset \mathbb{Z}^d$, A finite, is then:

$$M_1^F(A) := M_1(A) \qquad \text{for } d = 1, \tag{3}$$

$$M_d^F(A) := M_d(A) \prod_{k=1}^{d} M_{d-1}^F(\pi_k(A)) \qquad \text{for } d > 1. \tag{4}$$

It coincides with the digital convexity measure in dimension 1, but may differ starting from dimension 2.

Theorem 6. *Let $A \subset \mathbb{Z}^d$ finite. Then $M_d^F(A) = 1$ if and only if A is fully convex and $0 < M_d^F(A) < 1$ otherwise. Besides $M_d^F(A) \leqslant M_d(A)$ in all cases.*

Proof. Immediate from the equivalence of P-convexity with full convexity. $\square$

Figure 3 illustrates the links and the differences between the two convexity measures M_d and M_d^F on simple 2D examples. As one can see, the usual convexity measure may not detect disconnectedness, is sensitive to specific alignments of pixels, while full convexity is globally more stable to perturbation and is never 1 when sets are disconnected.

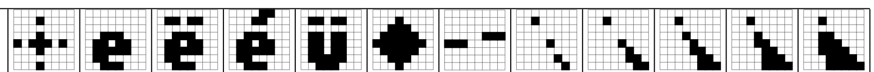
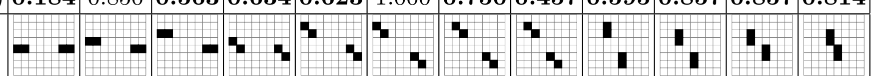
A												
$M_d(A)$	0.360	0.850	0.656	0.724	0.727	1.000	1.000	1.000	1.000	1.000	1.000	0.950
$M_d^F(A)$	**0.184**	0.850	**0.563**	**0.634**	**0.623**	1.000	**0.750**	**0.457**	**0.595**	**0.857**	**0.857**	**0.814**
A												
$M_d(A)$	0.500	1.000	0.667	0.500	0.500	1.000	0.667	1.000	0.667	0.800	0.667	1.000
$M_d^F(A)$	**0.250**	**0.500**	**0.222**	**0.250**	**0.200**	**0.381**	**0.296**	**0.533**	**0.296**	**0.427**	**0.444**	1.000

Fig. 3. Common points and differences of convexity measure M_d and full convexity measure M_d^F on small 2D digital sets.

6 Conclusion and perspectives

We have presented two original characterizations of full convexity. The first one gives a nice analogue of the "segment inclusion" definition of convexity with full convexity in dimension 1 and 2, and it shows also that, in higher dimensional spaces, additional continuity constraints are required. The second characterization tells that full convexity requires the digital convexity of the set and all of its shadows along axes.

Both characterizations shed new light on what is really full convexity. They may provide alternative algorithms to check full convexity, which may have better time complexity. They may help deciding if some digital sets are fully convex, as we show here for digital balls and hypercube subsets. They enable the definition of new measures for digital sets, with a stronger power of categorization. Finally, the characterization of full convexity through projections can be of interest for discrete tomography, as it induces connectedness in a natural way.

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