Full convexity for polyhedral models in digital spaces

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Université de Strasbourg

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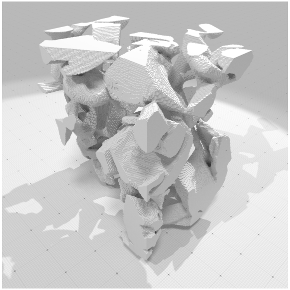
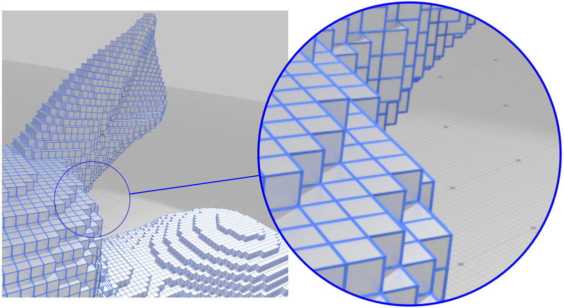
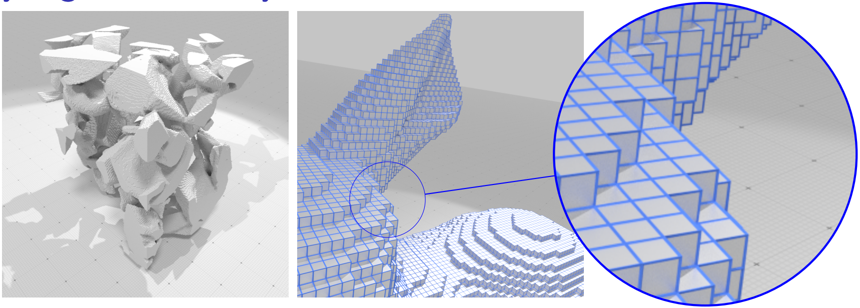
Why digital convexity ?





- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes
- identifies convex/concave/flat/saddle regions
- gives locally its piecewise linear geometry
- facets give normal estimations

Why digital convexity ?



- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes
- identifies convex/concave/flat/saddle regions
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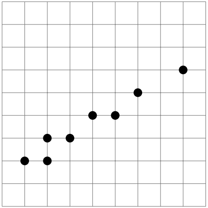
- convexity = foundation of convex analysis, linear programming
- digital convexity = foundation of digital convex analysis, integer linear programming ?

Natural digital convexity is not satisfactory

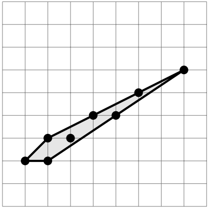
Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d = X$



X

$=$



$\mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex !

Digital convexity does not imply digital connectedness !

Summary of digital convexity properties

properties	H-convexity
simple, generic	$+$ (indeed, $X = \mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$)
classical convex objects	$\approx$ (but weird sets are convex)
connectedness	$-$ (many convex sets are disconnected)
simple connectedness	$-$ (of course no)
intersection property	$+$
fast convexity test	$+$ (quickhull+lattice enumeration)

Usual digital convexity adds connectedness

properties	H-convexity	H-convexity + connectedness
simple, generic	$+$	$-$
classical convex objects	$\approx$	$\approx$
connectedness	$-$	$\approx$ (slices unconnected)
simple connectedness	$-$	$-$ (unclear)
intersection property	$+$	$-$
fast convexity test	$+$	$+$

[Minsky, Papert 88], [Kim 82], [Kim, Rosenfeld 82], [Hübler, Klette, Voss89], [Ronse 89], [Eckhardt 01] . . .

Proposal: full convexity

properties	H-convexity	H-convexity + connect.	Full convexity
simple, generic	$+$	$-$	$+$
classical convex objects	$\approx$	$\approx$	$+$
connectedness	$-$	$\approx$	$+$
simple connectedness	$-$	$-$	$+$
intersection property	$+$	$-$	$-$ (but...)
fast convexity test	$+$	$+$	$+$

Proposal: full convexity

properties	H-convexity	H-convexity + connect.	Full convexity
simple, generic	$+$	$-$	$+$
classical convex objects	$\approx$	$\approx$	$+$
connectedness	$-$	$\approx$	$+$
simple connectedness	$-$	$-$	$+$
intersection property	$+$	$-$	$-$ (but...)
fast convexity test	$+$	$+$	$+$

Focus of this work

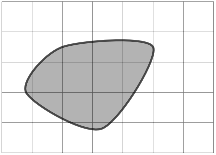
Can we define a fully convex hull operator ?
Can we use it to define polyhedral models ?

Full convexity for polyhedral models in digital spaces

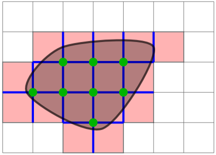
Cubical grid, intersection complex

- cubical grid complex $\mathcal{C}^d$
  - $\mathcal{C}_0^d$ vertices or 0-cells $= \mathbb{Z}^d$
  - $\mathcal{C}_1^d$ edges or 1-cells $=$ open unit segment joining 0-cells
  - $\mathcal{C}_2^d$ faces or 2-cells $=$ open unit square joining 1-cells
  - ...
- *intersection complex* of $Y \subset \mathbb{R}^d$

$$\bar{\mathcal{C}}_k^d[Y] := \{c \in \mathcal{C}_k^d, \bar{c} \cap Y \neq \emptyset\}$$



Y



cells $\bar{\mathcal{C}}_0^d[Y]$, $\bar{\mathcal{C}}_1^d[Y]$, $\bar{\mathcal{C}}_2^d[Y]$

What is full convexity ?

Definition (Full convexity [L. 2021])
A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}^d_k[X] = \bar{\mathcal{C}}^d_k[\mathrm{Cvxh}\,(X)]. \quad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

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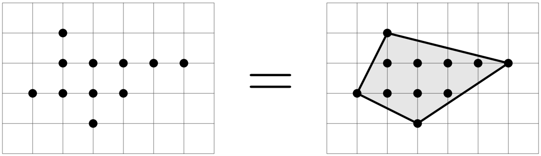

$\bar{\mathcal{C}}^d_0[X]$ $=$ $\bar{\mathcal{C}}^d_0[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex

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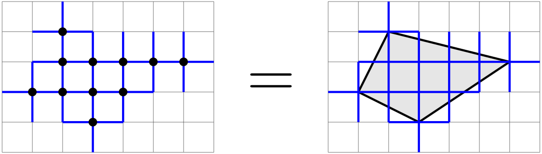

$\bar{\mathcal{C}}^d_1[X] = \bar{\mathcal{C}}^d_1[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex

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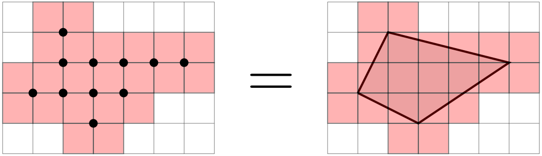

$\bar{\mathcal{C}}^d_2[X]$ $=$ $\bar{\mathcal{C}}^d_2[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex, and 2-convex, hence fully convex.

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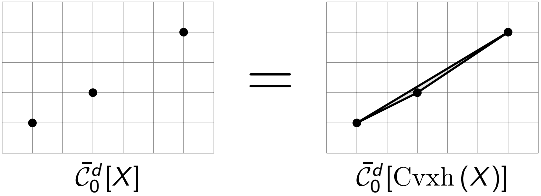
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



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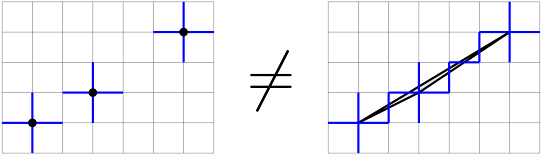

$\bar{\mathcal{C}}_1^d[X]$ $\quad$ $\bar{\mathcal{C}}_1^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex

What is full convexity ?

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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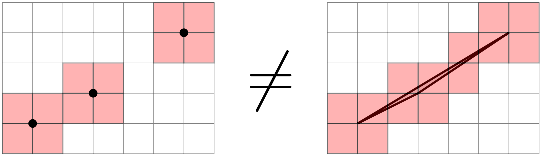

$\bar{\mathcal{C}}^d_2[X] \neq \bar{\mathcal{C}}^d_2[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex, nor 2-convex.

What is full convexity ?

Definition (Full convexity [L. 2021])

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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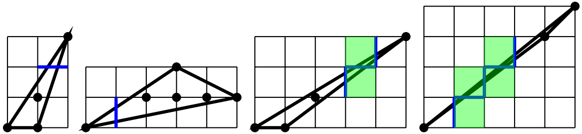
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

Full convexity eliminates too thin digital convex sets in arbitrary dimension.



Some properties of full convexity

Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then X is d-connected.

Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is **simply** *connected.*

Theorem

Verifying if a digital set is fully convex requires one convex hull computation and one lattice polytope enumeration.

Full convexity for polyhedral models in digital spaces

What about a digital convex hull ?

- digital convex hull $\mathrm{Cvxh}_{\mathbb{Z}^d}(A) := \mathrm{Cvxh}(A) \cap \mathbb{Z}^d$

properties	H-convexity	H-convexity $+$ connect.
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A)$ convex	$+$	$-$
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A) = A$ (for A cvx)	$+$	$+$
idempotence	$+$	$+$
fast computation	$+$	$+$
increasing	$+$	$+$

What about a digital convex hull ?

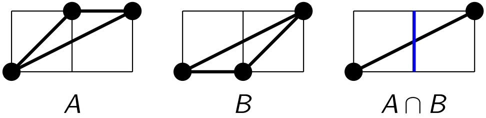
- digital convex hull $\mathrm{Cvxh}_{\mathbb{Z}^d}(A) := \mathrm{Cvxh}(A) \cap \mathbb{Z}^d$

properties	H-convexity	H-convexity + connect.
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A)$ convex	+	−
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A) = A$ (for A cvx)	+	+
idempotence	+	+
fast computation	+	+
increasing	+	+

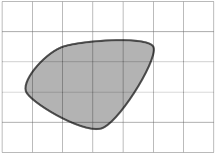
How can we build fully convex sets from arbitrary $A \subset \mathbb{Z}^d$?

Fully convex hull through intersections ?

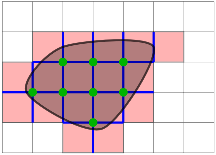
- half-spaces are fully convex
- can we intersect support half-spaces to get fully convex hull ?
- intersections of fully convex sets are not fully convex in general



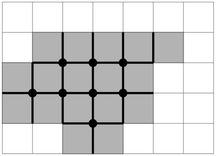
Local operators $\mathrm{Star}(\cdot)$, $\mathrm{Skeleton}(\cdot)$, $\mathrm{Extrema}(\cdot)$



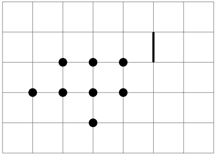
Y



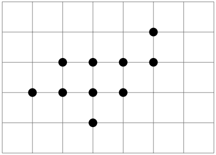
$\mathrm{Star}(Y) = \bar{\mathcal{C}}_0^d[Y] \cup \bar{\mathcal{C}}_1^d[Y] \cup \bar{\mathcal{C}}_2^d[Y]$



K



$K' = \mathrm{Skeleton}(K)$

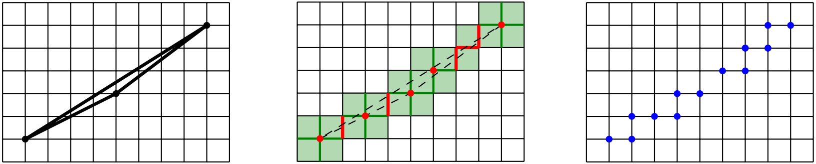


$\mathrm{Extrema}(K')$

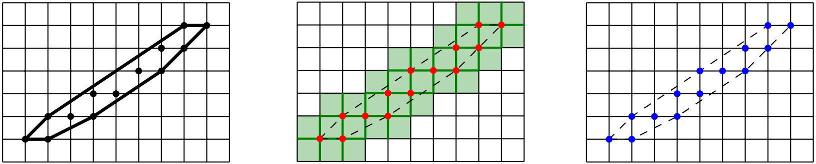
- For any $Y \subset \mathbb{R}^d$, let $\mathrm{Star}(Y) := \bar{\mathcal{C}}^d[Y]$ (coincides with the usual star of combinatorial topology)
- For any complex $K \subset \mathcal{C}^d$, let $\mathrm{Skeleton}(K) := \bigcap_{K' \subset K \subset \mathrm{Star}(K')} K'$
- For any complex $K \subset \mathcal{C}^d$, let $\mathrm{Extrema}(K) := \mathrm{Cl}(K) \cap \mathbb{Z}^d$

Operator $\mathrm{FC}(\cdot)$ and fully convex enveloppe $\mathrm{FC}^*(\cdot)$

- Iterative method for computing a fully convex enveloppe
- Let $\mathrm{FC}(X) := \mathrm{Extrema}\left(\mathrm{Skeleton}\left(\mathrm{Star}\left(\mathrm{Cvxh}\left(X\right)\right)\right)\right)$
- Iterative composition $\mathrm{FC}^n(X) := \underbrace{\mathrm{FC} \circ \cdots \circ \mathrm{FC}}_{n \text{ times}}(X)$
- *Fully convex envelope* of X is $\mathrm{FC}^*(X) := \lim_{n \to \infty} \mathrm{FC}^n(X)$.



input X, $Y := \mathrm{Cvxh}\,(X)$ — $\mathrm{Star}\,(Y)$, $\mathrm{Skeleton}\,(\mathrm{Star}\,(Y))$ — $X' = \mathrm{FC}(X)$



input X', $Y' := \mathrm{Cvxh}\,(X')$ — $\mathrm{Star}\,(Y')$, $\mathrm{Skeleton}\,(\mathrm{Star}\,(Y'))$ — $X'' = \mathrm{FC}(X') = \mathrm{FC}^2(X)$

The fully convex enveloppe is well defined

Lemma
For any $X \subset \mathbb{Z}^d$*,* $X \subset \mathrm{FC}(X)$*.*

Lemma
For any finite $X \subset \mathbb{Z}^d$*,* X *and* $\mathrm{FC}(X)$ *have the same bounding box.*

Theorem
For any finite digital set $X \subset \mathbb{Z}^d$*, there exists a finite* n *such that* $\mathrm{FC}^n(X) = \mathrm{FC}^{n+1}(X)$*, hence* $\mathrm{FC}^*(X)$ *exists and is equal to* $\mathrm{FC}^n(X)$*.*

Envelope $\mathrm{FC}^*(\cdot)$ acts as a fully convex hull operator

Lemma

If $X \subset \mathbb{Z}^d$ is fully convex, then $\mathrm{FC}(X) = X$. So $\mathrm{FC}^*(X) = X$.

Proof.

$$
\begin{aligned}
\mathrm{FC}(X) &= \mathrm{Extrema}\left(\mathrm{Skeleton}\left(\mathrm{Star}\left(\mathrm{Cvxh}\left(X\right)\right)\right)\right) \\
&= \mathrm{Extrema}\left(\mathrm{Skeleton}\left(\mathrm{Star}\left(X\right)\right)\right) && \text{(X is assumed fully convex)} \\
&= \mathrm{Extrema}\left(X\right) && \text{(Skeleton inverse of star)} \\
&= X && (X \subset \mathbb{Z}^d)
\end{aligned}
$$

$\square$

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□

Lemma

If $X \subset \mathbb{Z}^d$ is not fully convex, then $X \subsetneq \mathrm{FC}(X)$

Envelope $\mathrm{FC}^*(\cdot)$ acts as a fully convex hull operator

Lemma

If $X \subset \mathbb{Z}^d$ is fully convex, then $\mathrm{FC}(X) = X$. So $\mathrm{FC}^*(X) = X$.

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Lemma

If $X \subset \mathbb{Z}^d$ is not fully convex, then $X \subsetneq \mathrm{FC}(X)$

Theorem

$X \subset \mathbb{Z}^d$ is fully convex if and only if $X = FC(X)$.

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\end{aligned}
$$

□

Lemma

If $X \subset \mathbb{Z}^d$ is not fully convex, then $X \subsetneq \mathrm{FC}(X)$

Theorem

$X \subset \mathbb{Z}^d$ is fully convex if and only if $X = FC(X)$.

Theorem

For any finite $X \subset \mathbb{Z}^d$, $\mathrm{FC}^*(X)$ is fully convex.

Envelope $\mathrm{FC}^*(\cdot)$ acts as a fully convex hull operator

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If $X \subset \mathbb{Z}^d$ is fully convex, then $\mathrm{FC}(X) = X$. So $\mathrm{FC}^*(X) = X$.

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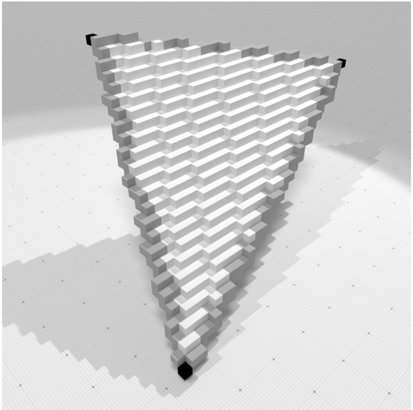
Theorem

For any finite $X \subset \mathbb{Z}^d$, $\mathrm{FC}^*(X)$ is fully convex.

Theorem

Computation of $\mathrm{FC}(\cdot)$ is bounded by $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$, with $n = \#(X)$.

A 3D digital triangle

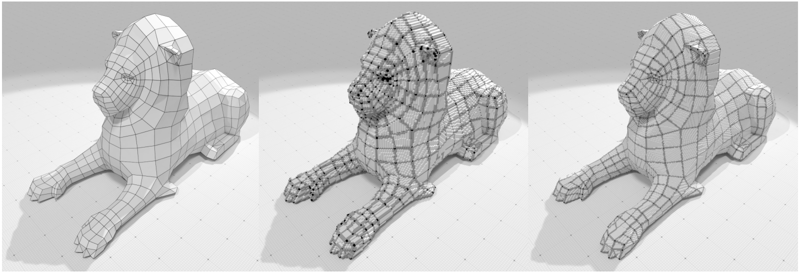


vertices $A = (8, 4, 18)$, $B = (-22, -2, 4)$, $C = (18, -20, -8)$ (black),

edges $\mathrm{FC}^*(\{A, B\})$, $\mathrm{FC}^*(\{A, C\})$, $\mathrm{FC}^*(\{B, C\})$ (grey+black)

triangle $\mathrm{FC}^*(\{A, B, C\})$ (white+grey+black)

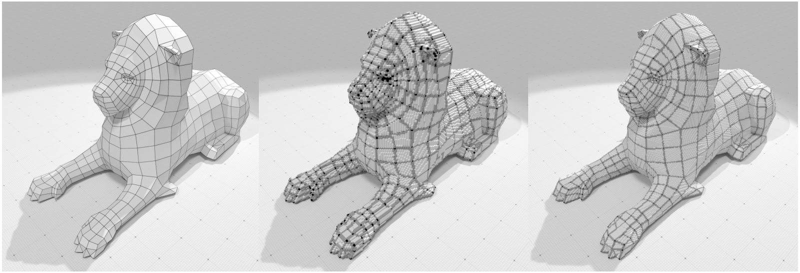
A generic digital polyhedral model



Quad-mesh $\mathcal{Q}$, non planar faces | $\sharp\mathcal{Q}^* = 81044$ | $\sharp\mathcal{Q}^* = 373225$

- combinatorial polyhedron $\mathcal{P}$ made of k-cells (facets, edges, vertices), with incidence relations
- vertices have integer coordinates
- a digital k-cell σ with vertices V_σ is $\mathrm{FC}^*(V_\sigma)$

A generic digital polyhedral model



Quad-mesh $\mathcal{Q}$, non planar faces | $\sharp\mathcal{Q}^* = 81044$ | $\sharp\mathcal{Q}^* = 373225$

- combinatorial polyhedron $\mathcal{P}$ made of k-cells (facets, edges, vertices), with incidence relations
- vertices have integer coordinates
- a digital k-cell σ with vertices V_σ is $\mathrm{FC}^*(V_\sigma)$

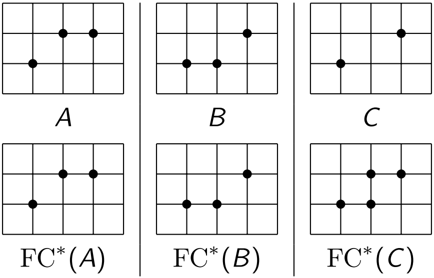
But no control on the thickness of digital facets.

Is the fully convex enveloppe a hull operator ?

properties	fully convex enveloppe
$\mathrm{FC}^*(A)$ convex	$+$
$\mathrm{FC}^*(A) = A$ (for A fully cvx)	$+$
idempotence	$+$
fast computation	$\approx$ (# iterations)
increasing	$-$

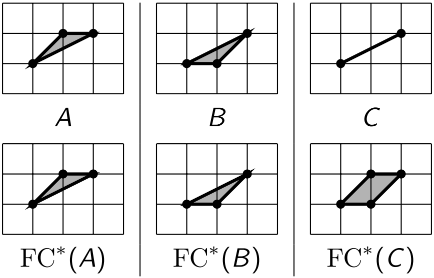
Is the fully convex enveloppe a hull operator ?

properties	fully convex enveloppe
$\mathrm{FC}^*(A)$ convex	$+$
$\mathrm{FC}^*(A) = A$ (for A fully cvx)	$+$
idempotence	$+$
fast computation	$\approx$ ($\#$ iterations)
increasing	$-$



Is the fully convex enveloppe a hull operator ?

properties	fully convex enveloppe
$\mathrm{FC}^*(A)$ convex	$+$
$\mathrm{FC}^*(A) = A$ (for A fully cvx)	$+$
idempotence	$+$
fast computation	$\approx$ ($\#$ iterations)
increasing	$-$



Full convexity for polyhedral models in digital spaces

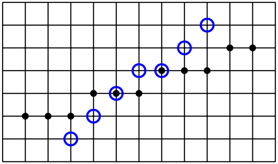
A relative fully convex enveloppe

- For $X \subset Y$, let $\mathrm{FC}_{|Y}(X) := \mathrm{FC}(X) \cap Y$
- $\mathrm{FC}^n_{|Y}(X) := \mathrm{FC}_{|Y} \circ \cdots \circ \mathrm{FC}_{|Y}(X)$, composed n times
- *Fully convex envelope of X relative to Y* is $\mathrm{FC}^*_{|Y}(X) := \lim_{n\to\infty} \mathrm{FC}^n_{|Y}(X)$
- we have $\mathrm{FC}^*(X) = \mathrm{FC}^*_{|\mathbb{Z}^d}(X)$

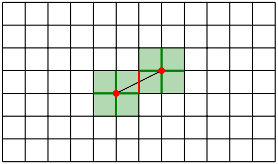
Theorem

Let $X \subset \mathbb{Z}^d$ and $Y \subset \mathbb{Z}^d$ fully convex.
Then $\mathrm{FC}^*_{|Y}(X \cap Y)$ is fully convex and is included in Y.

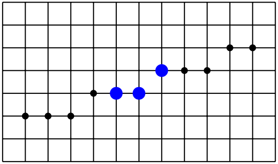
Intersections of fully convex sets



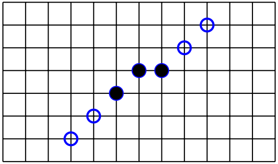
X, Y



$\mathrm{Skeleton}(\mathrm{Star}(\mathrm{Cvxh}\,(X \cap Y)))$



$\mathrm{FC}^*_{|Y}\,(X \cap Y)$



$\mathrm{FC}^*_{|X}\,(X \cap Y)$

Full convexity for polyhedral models in digital spaces

Polyhedral models (here 3D)

- combinatorial polyhedron $\mathcal{P}$ made of k-cells (facets, edges, vertices), vertices are simply digital points.
- thick enough arithmetic planes are fully convex
- use relative full convexity for facets
- $T \subset \mathbb{Z}^3$ made of coplanar points,
  $P_1(T)$ (resp. $P_\infty(T)$) is its median standard (resp. naive) plane.

Definition (standard digital polyhedron)

$\mathcal{P}_1^*$ is the collection of digital cells that are subsets of $\mathbb{Z}^d$:

- if σ is a facet of $\mathcal{P}$ with vertices $V(\sigma)$, then σ_1^* is a cell of $\mathcal{P}_1^*$ with $\sigma_1^* := \mathrm{FC}^*_{|P_1(V(\sigma))}(V(\sigma))$.
- if τ is an edge, then it has as many geometric realizations as incident facets σ: $(\tau, \sigma)_1^* := \mathrm{FC}^*_{|\sigma_1^*}(V(\tau))$.

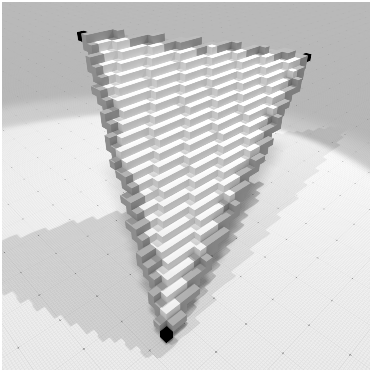
Definition (naive digital polyhedron)

$\mathcal{P}_\infty^*$ defined similarly by replacing 1 with ∞ above.

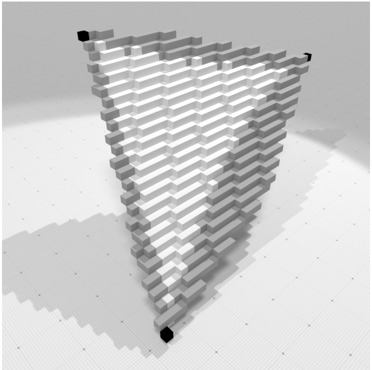
Standard and naive 3D triangle

Theorem

All digital cells are fully convex.



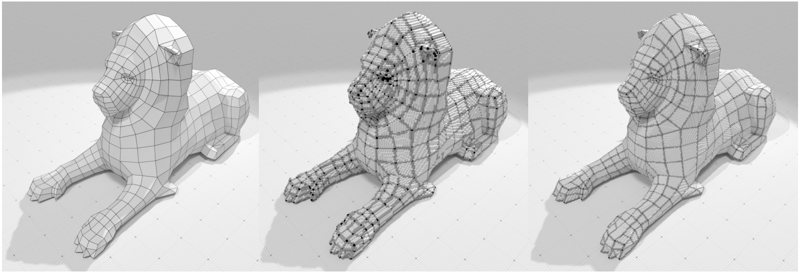
standard triangle $\mathcal{T}_1^*$
985 points



naive triangle $\mathcal{T}_\infty^*$
567 points

Polyhedron $\mathcal{T}$ with vertices $A = (8, 4, 18)$, $B = (-22, -2, 4)$, $C = (18, -20, -8)$, edges $\{(A, B), (A, C), (B, C)\}$ and one facet $\{(A, B, C)\}$.

Generic/standard/naive digital polyhedron

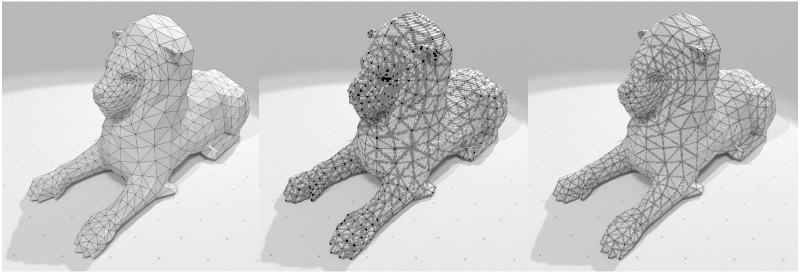


Quad-mesh $\mathcal{Q}$, non planar faces

$\sharp\mathcal{Q}^* = 81044$

$\sharp\mathcal{Q}^* = 373225$

Generic/standard/naive digital polyhedron

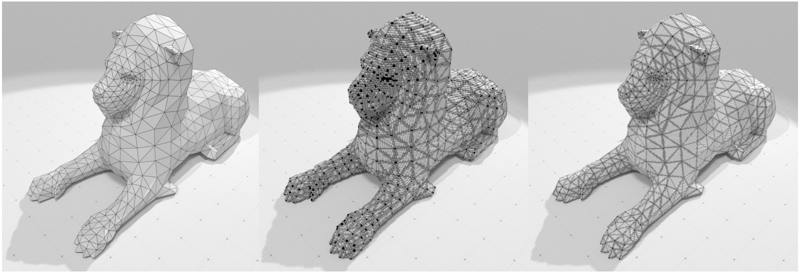


Tri-mesh $\mathcal{T}$, planar faces

$\sharp\mathcal{T}_1^* = 68603$

$\sharp\mathcal{T}_1^* = 275931$

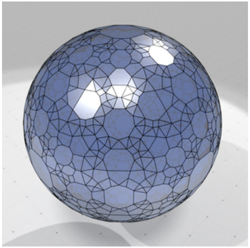
Generic/standard/naive digital polyhedron



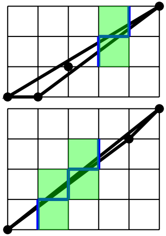
Tri-mesh $\mathcal{T}$, planar faces	$\sharp\mathcal{T}^*_\infty = 46639$	$\sharp\mathcal{T}^*_\infty = 182451$

Full convexity packages in DGtal

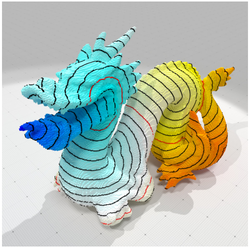
dgtal.org



dD convex hull and Delaunay triangulation



Full convexity in dD, tangency, envelope



Local shape analysis, geodesics

- most of full convexity and applications implemented in DGtal
- open source library, efficient efficient generic C++
- a nice tutorial yesterday !

Conclusion

- an envelope operator for building fully convex set
- a new characterization of full convexity $X = \mathrm{FC}^*(X)$
- a relative envelope operator
  - induces asymmetric intersections of fully convex sets
  - allows fully convex sets within planes
- polyhedral models with facets that are pieces of planes

Future works

Theoretical side

- increasingness of enveloppe still under study
- redefine intersection of fully convex digital sets
- new characterization of full convexity
- arithmetic planes without arithmetic ?

Algorithmic and implementation side

- fast cell and lattice point enumeration within polytopes
- faster full convexity tests
- bound number of iterations of $\mathrm{FC}^*(\cdot)$

Explore its natural applications

- we know how to pass from a polyhedron to a digital polyhedron
- how can we do the other way around ? Optimal decomposition into fully convex facets ?