Experimental comparison of continuous and discrete tangent estimators along digital curves

François de Vieilleville[1] and Jacques-Olivier Lachaud[2]

[1] LaBRI, UMR CNRS 5800, Université Bordeaux 1
351 cours de la Libération, 33405 Talence Cedex, France
devieill@labri.fr

[2] Laboratoire de Mathématiques, UMR CNRS 5127
Université de Savoie, 73776 Le-Bourget-du-Lac, France
jacques-olivier.lachaud@univ-savoie.fr

Abstract. Estimating the geometry of a digital shape or contour is an important task in many image analysis applications. This paper proposes an in-depth experimental comparison between various continuous tangent estimators and a representative digital tangent estimator. The continuous estimators belong to two standard approximation methods: least square fitting and gaussian smoothing. The digital estimator is based on the extraction of maximal digital straight segments [9, 10]. The comprehensive comparison takes into account objective criteria such as isotropy and multigrid convergence. Experiments underline that the proposed digital estimator addresses many of the proposed objective criteria and that it is in general as good - if not better - than continuous methods.

1 Introduction

The proper detection of significant features along digital curves often relies on an accurate estimation of the geometry of the underlying curve that has been digitized. Local geometric quantities such as the curvature at given points can lead to corner detection [17], more generally curvature and tangent estimation lead to the detection of dominant points on digital curves [14]. Correct tangent estimation allows length computation by simple integration.

Estimating local geometric quantities on digitized shapes is a difficult task in itself for at least four major reasons:

(1) Given a digitized shape there exists infinitely many continuous Euclidean shapes that have the same digitization.
(2) Given a digital point and a point on the continuous curve, determining the required size of the computation window to achieve a good estimation is tricky.
(3) The digitized curve can be noisy or damaged, worsening the preceding problems.
(4) The time spent on computations may be limited.

The first problem implies that, given a digital shape, additional hypotheses are required to define its reference shape, such as smoothness, compactness, convexity, minimal perimeter or maximal area. For instance, given a digital disk, a reasonnable hypothesis is that the underlying shape is an Euclidean disk, and not some kind of gears. The second problem involves the adaptability of computation windows to the local geometry of the shape, e.g. curves with huge curvature variations require different sizes for the computation windows. Sizes of computation windows have a huge impact on the multi-grid convergence (see [4]). The third problem is a common problem which is efficiently addressed in the continuous world, but lacks proper definitions in the digital world. This entails that continuous methods are generally preferred for the extraction of geometric quantities. The fourth problem arises when the computation windows are too large, and narrowing their sizes has a direct impact of the efficiency or precision of the method. These issues are related to many interesting topics on digital curves such as multi-grid convergence [3, 8], digitization problems and topology issues [12], combinatorial problems [1] and new models for digital straight segments taking into account some distortions [6].

As mentioned earlier usual geometric estimators are based on approximation techniques in the continuous Euclidean spaces, and forget the specificities of subsets of the digital plane. By this way, they address problem (3) considering that it is the main issue. The noise is then handled by tuning some external parameters. In fact the external parameters often reduce to the choice of the size of the computation window, handling problems (2), (3) and (4) at the same time with a trade-off. The continuous methods can be of various type with different aims with respect to the digital curve: interpolation, reconstruction or fit. The choice of the underlying curve in problem (1) is then often made explicitly with the method itself, e.g. using $\mathcal{C}^3$-splines to interpolate points along a digital curve lead to degree three polynomials as the underlying curve. The numerical methods required to extract the chosen solution can be costly and may even require parameters themselves, this is particularly true when the chosen underlying curve is the solution of a non trivial optimisation problem. As a result (1) and (2) have a direct impact on (4).

On the contrary, standard digital estimators based on digital straight segment recognition estimate local geometric quantities like tangent or curvature with an adaptive computation window and, furthermore, they do not require any external parameters [19, 7, 9]. Recently, an evaluation of digital tangent estimators was performed in [9] and the λ-MST was shown to outperform the others on many criteria like precision, maximal error, isotropy, convergence, convexity. The tangent orientation is determined using digital straight segment recognition, which entails a computation window adapted to the local curve geometry (addressing problem (2)) and without assumptions on the underlying curve (addressing problem (1)). The average size of the computation window is known and is roughly in $\Theta\left(h^{-1/3}\right)$ where h is the grid step (see [4] for technical proofs). As a result, the asymptotic convergence — or multigrid convergence — of the λ-MST estimator is proved for smooth and convex curves [10]. This estimator is also the best among digital ones at rough scale [9, 10]. Its computation on the whole digital curve, i.e. the computation of the tangent orientation field, may be done in time linear with the number of digital points (optimal time, addressing (4)). This estimator is yet to be shown as good as standard continuous methods.

This is precisely the goal of this paper which is achieved by experimental comparison between the λ-MST estimator and two representative classes of continuous estimators. We naturally examine classical criteria like the average absolute error. Furthermore, we propose to use the product precision by computational cost to compare them as objectively as possible. Besides, our aim is not only to compare these estimators but to see if they can benefit from one another. This is the case here where we show how an optimal computation window (problem (2)) can be chosen for the Gaussian derivative technique. The obtained improvements are illustrated experimentally. These experiments indicate that even with the best possible window, the continuous estimators are outperformed by the λ-MST according to the product precision by cost. We stress that we treat only the ideal case where digital contours are perfect digitizations of continuous shapes, without any perturbation or noise. Indeed, a first evaluation must be carried out before in the ideal case, for instance to identify the best precision an estimator may achieve. Secondly, the λ-MST estimator is easily extensible to maximal *blurred* digital straight segments [6], which can accommodate local perturbations in the digital contour. An experimental evaluation of continuous versus discrete estimators in the presence of noise could then be carried out similarly, and would be the object of a future work.

The paper is organized as follow. First we describe continuous tangent estimators methods, more specifically the ones based on least square fitting with polynomials and the ones using convolution with a gaussian derivative. Their main drawbacks are also recalled. In a second time we briefly recall the definition of the λ-MST estimator and its main properties (Section 3). In Section 4, an experimental evaluation between the different estimators is proposed, following some of objective criteria proposed in [9, 10]. We also propose several improvements of the two continuous methods, which are then underlined experimentally. The criterion precision times computational cost shows that, even with these improvments, the digital estimator compete with the best possible continuous methods. Our conclusion is thus that digital straight segments are a powerful tool to analyse the geometry of digital curves.

2 Continuous tangent estimators

This section presents two continuous classes of methods that are used to extract geometric information from curves. Both methods need external parameters to achieve the best possible accuracy. In the remaining of the paper the considered digital curves are digital 4-curves, that is a 4-connected closed sequence of points in $\mathbb{Z}^2$ such that each of them has exactly two 4-neighbors: a predecessor and a follower (given an orientation). Such curves arise naturally from the cellular decomposition of the Gauss digitization of simple Euclidean objects, provided they are well-composed [11]. The obtained digital curve is denoted C and its points are ordered increasingly with a counterclockwise order, C_i denotes the i-th point of the digital curve and $C_{i,j}$ is the digital path from the i-th point to the j-th point.

2.1 Least square methods using polynomials

The aim of these methods is to find a polynomial of finite degree which minimizes a positional squared error from a set of (possibly noisy) samples. More precisely, let us denote by $(s_i = (x_i, y_i))_{1 \leq i \leq M}$ a set of M samples obtained from a planar curve parameterized as $y = f(x)$. We thus seek to minimize the functional $E(a_0, \ldots, a_N) = \sum_{i=1}^{M} \left(y_i - \sum_{j=0}^{j=N} a_j x_i^j \right)^2$.

In the general case, the problem can be reduced to a matrix inversion problem. At least one solution exists and can be efficiently computed using QR factorisation [16]. For small degree polynomials, direct computation is possible as it involves square matrices of order two and three. It is not compulsory that the polynomial be the supposed underlying curve itself. It can also be its local Taylor expansion as explained in [13] for implicit parabola fitting, an approach which is generalized by the *n-jets* of [2].

Once the optimal polynomial for E is determined, the coefficient associated to its X monomial may be used to estimate the tangent orientation. We naturally focus on low order polynomials. That is the linear regression (LR, Eq. (1)), implicit parabola fitting (IPF, Eq. (2)), and explicit parabola fitting (EPF, Eq. (3)). When used for approaching the tangent orientation at the point of interest C_0, considered as the origin, with a computation window ranging from C_{-q} to C_q, those three methods give very similar results (see Figure 1).

$$E_{LR}(a, b) = E(a, b, 0, \ldots) \tag{1}$$

$$E_{IPF}(a, b) = E(0, a, b, 0, \ldots) \tag{2}$$

$$E_{EPF}(a, b, c) = E(a, b, c, 0, \ldots) \tag{3}$$

A refinement of this method is the *weighted least square fitting*, where each sample have a variable importance in the fitting process: the heavier the weight, the more important the fit. However, it is not easy to find meaningful weights within our context. Another refinement is to use independent coordinates, that is a fit on each coordinates with respect to a given parameterization of the curve. Usually centered windows are considered: C_0 is the point of interest, $M = 2q + 1$ is the size of the computation window going from C_{-q} to C_q. When using independent coordinates, the arc-length from C_0 to C_i, is computed as $\sum_{k=O}^{i-1} d_1(C_k, C_{k+1})$ if $i > 0$ and $-\sum_{k=O}^{i-1} d_1(C_k, C_{k+1})$ otherwise.[3]

2.2 Reconstruction using gaussian smoothing

The use of gaussian filters is a common technique for improving the quality of noisy images. This filter can also be used when trying to analyze a digital curve,

[3] d_1 denotes the distance obtained from the $|| \cdot ||_1$ norm.

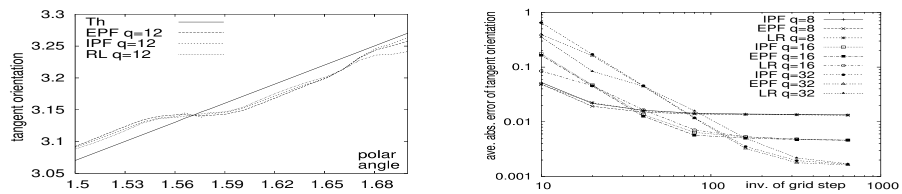

Fig. 1. We represent the tangent orientation estimated with IPF,EPF and LR methods. The test shape is a circle of radius 1. Computation window equals $2q+1$. (Left) Grid step equals 0.01, x-axis represents the polar angle, the y-axis represents the orientation of the tangent. (Right) The plot is in log-space and represent the average absolute error between true tangent and estimated tangent as a function of the grid step. For each grid step 50 experiences are made with a random shift on the center of the shape.

and has been used in the pattern recognition community for almost 30 years. It is essentially a weighted averaging over a finite window. The obtained smoothed continuous curve is considered to be a good approximation of the underlying curve. Its derivatives are easily computed yielding geometric quantities of the first and second order. This reconstruction has one major drawback, which is the choice of the parameter σ. This tuning parameter is often chosen for the whole curve, but it is not satisfying if the curve has huge curvature variations, entailing then over-smoothing for some region and under-smoothing for others. As a result techniques using scale-space were proposed [15, 20] to achieve a better localization of the dominant points across the different σ values. From a discrete point of view we will consider that the estimated derivative at the digital point C_0, say $\hat{\mathbf{C}}'_0$, is obtained as : $\hat{\mathbf{C}}'_0 = \sum_{i=-q}^{q} G'_{\sigma_q}(-i)\mathbf{C}_i$, with $\sigma_q = \frac{2q+1}{3}$ and where $G'_\sigma(t)$ is the first derivative of the Gaussian function $G_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-t^2}{2\sigma^2}\right)$.

2.3 Common drawbacks

In the context of digital geometry, the methods presented above share similar drawbacks, which we try to analyze here. First of all, if we consider the digitization of convex shapes, we see that the analysis of its border with the preceding techniques may lead to false concavity/convexity detection, even in the simplest case of the circle as shown on Figure 2. This is particularly true when the size of the computation window is not large enough.

The false convexity/concavity detection can be alleged to a wrong size of the computation window. Experimentally on digitized circles it seems that if the size of the computation window exceeds some value being a functional of the radius and the grid step, there is no false convexity/concavity points. More precisely, this phenomenon is related to the maximal curvature of the shape under study.

Another fundamental problem related to fixed size computation windows is that one parameter, even if suited for some regions, cannot adapt to the geometry of a digitized shape with huge curvature variations. This statement is

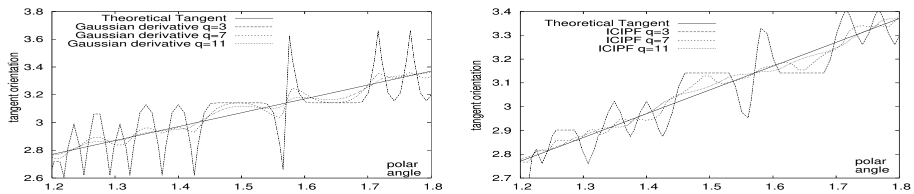

Fig. 2. Test shape is a circle of radius 1, digitized with a grid step equal to 0.01. Tangent orientation is plotted as a function of the polar angle. The x-axis represents the polar angle, the y-axis represents the orientation of the tangent and the size of the computation window equals $2q+1$. (Left) Tangent orientation obtained using convolutions by the gaussian derivative σ_q. (Right) Tangent orientation obtained using implicit parabola fitting with independent coordinates.

particularly underlined on Figure 3. Moreover, a fixed parameter prevents the multigrid convergence of continuous estimators, since it limits the number of data taken into account in the fitting or smoothing process, thus limiting the number of possible local geometries. This is illustrated on Figure 4, where the size on the computation window has a direct impact on the average error.

Last but not least, the computed curvilinear abscissa obtained from the summation of the elementary steps on digital curve is a poor estimation (see [18] for a proof of non convergence for length estimators using fixed-size windows on euclidean segments). Thus the problems of parametrization induce displacements and errors in the continuous proposed methods. A way to solve this problem would be to use an estimation of the elementary steps ds along the curve using the tangent orientation computed with a convergent estimator.

3 Discrete tangent estimators

This section recalls the definitions of elementary objects regarding digital straight segments, we then briefly present the λ-MST estimator and its properties.

3.1 Properties and definitions

Digital straight lines can be simply seen as the digitization of euclidean straight lines. More formally, a *standard line* of characteristics $(a, b, \mu) \in \mathbb{Z}^3$ is the subset of $\mathbb{Z}^2$ $\left\{(x, y) \in \mathbb{Z}^2 \mid \mu \leq ax - by < \mu + |a| + |b|\right\}$. They form 4-connected sequences of digital points. We say that a set of successive points $C_{i,j}$ of the digital curve C is a *digital straight segment (DSS)* iff there exists a standard line (a, b, μ) containing them. The predicate "$C_{i,j}$ is a DSS" is denoted by $S(i, j)$. When $S(i, j)$, the characteristics associated with the digital straight segment (extracted with the **DR95** algorithm [5]) are the characteristics (a, b, μ), which minimize $|a| + |b|$.

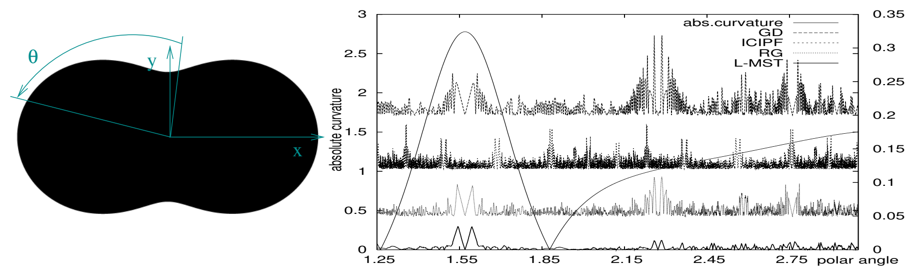

Fig. 3. (Left) The digitized shape is the Gauss digitization at grid step 0.001 of a flower with two extremities, maximum radius 1.4 and minimum radius 0.6. (Right) We plot the absolute error between theoretical tangent orientation and estimations obtained using window size of 21 points, except for the λ-MST. To identify difficult points, we also superposed the theoretical curvature with a dash-dotted plot, but in an other scale. The different estimators are: gaussian derivative (GD - dashed plot), parabola fitting with independent coordinates (ICIPF - dotted plot), linear regression (RG - small dotted plot), λ-MST (L-MST - solid thick plot). We see that the three methods are less precise on the part of the border of the shape which has the fastest curvature variation because of the size of the computation window which is not adapted to the local geometry of the shape. On the contrary, the λ-MST which has an adaptive window size behave much better.

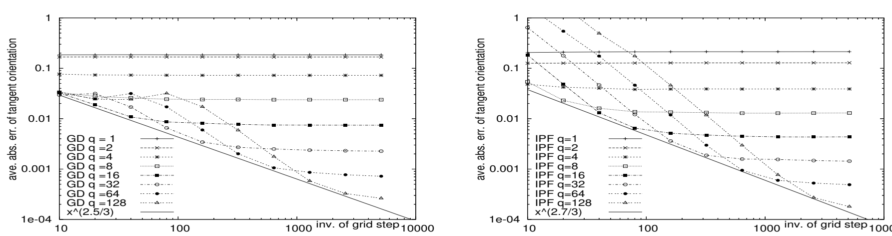

Fig. 4. Experimental multigrid convergence analysis drawn in log-space: x-axis is the inverse of the grid step, y-axis is the average of the absolute error between theoretical tangent and estimated tangent, shape of reference is a circle of radius one. At each grid step 50 experiences are made and center is shifted randomly. (Left) Gaussian derivative (GD) with various window size. (Right) Implicit parabola fitting (IPF). In both cases, fixed parameters cannot achieve convergence.

The slope a/b of a DSS provides a coarse estimation of the slope of the underlying tangent. Upon the many existing classes of DSS, we choose to focus on a particular class, the one that contains all the other DSS:

Definition 1. *We say that a portion $C_{i,j}$ of C is a* maximal digital straight segment *(MS) iff $S(i,j) \wedge \neg S(i-1,j) \wedge \neg S(i,j+1)$.*

Maximal segments can be numbered with increasing indexes on the digital curve, $M^i = C_{b_i,f_i}$ denoting the i-th maximal segment. With an incremental version of the **DR95** algorithm (see [7, 10, 9]), the set of all the maximal segments on a finite digital curve can be extracted in linear time with respect to the number of points of the curve. As maximal segments generally overlap, we introduce the set of all the maximal segments traversing a point.

Definition 2. *The* pencil of maximal segments *of C_k, denoted $\mathcal{P}(k)$ is the set of MS containing C_k.*

Since every DSS can be extended to form a MS, the pencil of any point is never empty. We also define the *eccentricity* of a point C_k with respect to a maximal segment M^i in $\mathcal{P}(k)$ as: $e_i(k) = \frac{\|C_k - C_{b_i}\|_1}{L_i} = \frac{k-b_i}{L_i}$ with $L_i = \|C_{f_i} - C_{b_i}\|_1$. This value indicates if a digital point is centered within a maximal segment: it is perfectly centered if the value equals 1/2, limit values are 0 and 1 for extremal points of a maximal segment.

3.2 The λ-MST tangent estimator

The λ-MST tangent estimator at one point is designed to take into account the various orientations of the MS in the pencil weighted by a functional of their respective eccentricity with respect to the point of interest:

Definition 3. *The* λ-maximal segment tangent direction at point C_k (λ-MST) *is defined as $\hat{\theta}(k) = \frac{\sum_{i \in \mathcal{P}(k)} \lambda(e_i(k))\theta_i}{\sum_{i \in \mathcal{P}(k)} \lambda(e_i(k))}$, where θ_i is the angle of the slope of the i-th MS with the x-abscissa.*

Considering the properties of the eccentricity and the non-emptyness of pencils, this value is always defined and may be computed locally. For particular λ functions the λ-MST estimator satisfies the convexity/concavity property[4] (see Theorem 8 of [10]).

This implies that the border of digitally convex shapes analysed with the λ-MST estimator under the conditions of the preceding theorem does not contain any false concavity. In practice the triangle function is used as the λ function: it matches the preceding conditions and brings good results.[5] Other nice properties are a good isotropic behaviour, multigrid convergence and computation of the tangent field in time linear with respect to the number of curve points (see [10, 9] and Figure 5).

[4] Estimated tangent directions are monotone for digitization of convex shapes.

[5] The triangle function is defined as $x \to x$ if $x \in [0, \frac{1}{2}]$ and $x \to 1 - x$ if $x \in [\frac{1}{2}, 1]$.

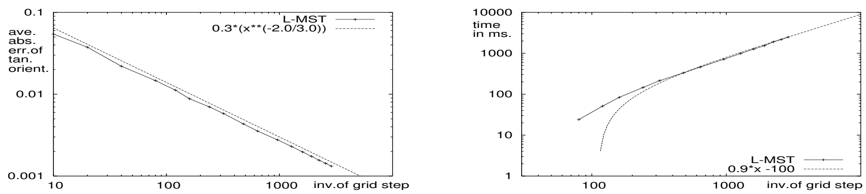

Fig. 5. The test shape is a circle of radius 1, the x-axis represents the inverse of the grid step. For each grid step, fifty experiences were launched with uniform random shift of the center of the shape. Plots are drawn in log-space. (Left) Average absolute error between true tangent and estimated tangent with the λ-MST, the law seems to be in $\mathcal{O}(h^{-2/3})$. (Right) Time spent on computing the tangent orientation field with the λ-MST, the law follows $\mathcal{O}(1/h)$, the same magnitude as the number of points constituting the border of the digitized shape.

4 Experimental evaluation

The multigrid convergence of the λ-MST estimator is shown on Figure 5 and its good behaviour with respect to huge curvature variations is exemplified on Figure 3. On the contrary the non multigrid convergence of the proposed estimators using fixed size computation window is shown on Figure 4 with the measure of the average absolute error as a function of the grid step. Though the precision of an estimator is important the time spent on the computation has also to be taken into account, a criterion measuring these two parameters at the same time is proposed in the next subsection, yielding the same conclusion.

4.1 A new criterion balancing precision and computation time

This subsection introduces a new criterion to compare local tangent estimators, called AAEBT: we measure the product of the average absolute error of tangent direction estimation by the computation time for the whole curve. The lower the quantity as the grid step decreases, the better. As problem (2) penalizes estimators using fixed size windows on curves with huge curvature variations we ran the experiments on digitizations of a disk. The experiments on Figure 6 clearly show that criterion AAEBT for the GD estimator of fixed size window becomes linear with the inverse of the grid step after some rank. For each window size, there is thus a bound to the maximum reachable precision (Figure 4 also illustrates this matter). However, judging from the experiments, the λ-MST estimator has a much better AAEBT which seems to be in $\mathcal{O}((1/h)^{1/3})$. This behaviour is consistent with the average absolute error of the tangent orientation in $\mathcal{O}(h^{2/3})$ and the computational cost in $\mathcal{O}(1/h)$ (see Figure 5).

4.2 Improving continuous estimators using fixed-size windows

The Figure 4 clearly suggests that there is a somewhat best window size to pick for each grid step. Judging from experiments on the circle for the GD estimator,

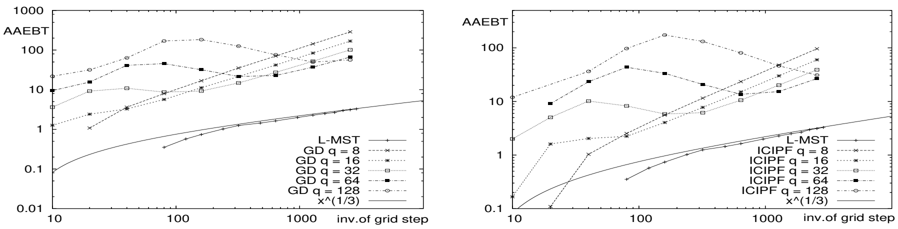

Fig. 6. Test shape is a circle of radius 1, the x-axis is the inverse of the grid step. We represent the time spent on computing the tangent field multiplied by the average absolute error between true tangent orientation and estimation with particular estimators. Plots are drawn in the log-space. (Left) Comparison between gaussian derivative with various window sizes and the λ-MST estimator. For the GD estimator, curves tend to be linear once the maximum precision is reached. (Right) Comparison between implicit parabola fitting with independent coordinates with various window sizes and the λ-MST estimator. For the λ-MST estimator, the law seem to be in $\mathcal{O}((1/h)^{1/3})$.

the best possible accuracy is in $\mathcal{O}(h^{5/6})$ provided the size of the computation window follow $\mathcal{O}((1/h)^{1/2})$ as shown on Figure 7. The parameter σ of GD is set to one third of the computation window size. It is clear that the size of the computation window should increase with the inverse of the grid step.

Let us use an adaptive window defined as the maximum distance between the point of interest and the ends of its pencil. The defined size of the computation window increase with the inverse of the grid step, and even though on average it only grows in $\mathcal{O}((1/h)^{1/3})$ this size brings multigrid convergence for both fits and gaussian derivative, as exemplified on Figure 8 (H-GD and H-ICIPF). The size can also be set globally using the average size of the maximal segments, again multigrid convergence is observed, see Figure 8 (HG-GD and HG-ICIPF).

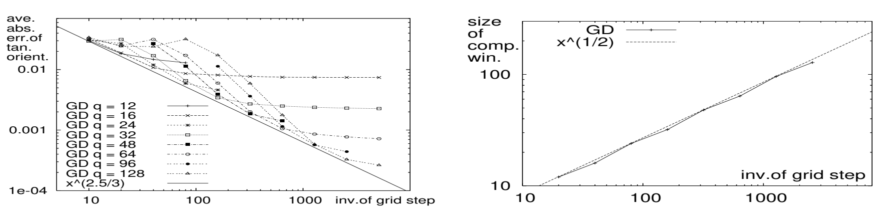

Fig. 7. (Left) Suggested best possible average absolute error with GD as being some $\mathcal{O}((1/h)^{5/6})$, with parameter $\sigma = (2q+1)/3$. (Right) The suggested size of the computation window to achieve best possible accuracy is in $\mathcal{O}((1/h)^{1/2})$.

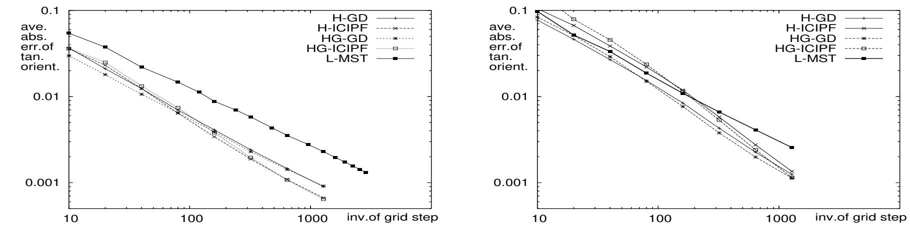

Fig. 8. Average absolute error between true tangent and estimated tangent. The plotted estimators use computation window whose size is determined with maximal segments. Hybrid estimators (H-GD and H-ICIPF) use the maximal distance between the point of interest and the ends of its pencil as q parameter. Hybrid global estimators (HG-GD and HG-ICIPF) use the average size in terms of number of points of the maximal segments as q parameter. (Left) The test shape is a circle of radius one. (Right) Test shape is a flower with two extremities, maximal radius 1.4 and minimal radius 0.6.

5 Conclusion

The presented experiments have shown how digital tangent estimators compare to classic continuous methods in the ideal digitization case: they are as precise and they are faster. This is clearly underlined when using the criterion precision multiplied by cost. Furthermore, we have shown how to introduce the adaptive window of digital estimators into continuous estimators to get an optimal window size. Future works will consider noise in the evaluation. Although defining noise in the discrete world is tricky, we plan to use maximal blurred digital straight segments to take into account distortion in the digital curve.

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