Maximal digital straight segments and convergence of discrete geometric estimators

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Abstract

Digital geometric estimators approach geometric quantities on digitized shapes without any knowledge of the continuous shape. A classical yet difficult problem is to show that an estimator asymptotically converges toward the true geometric quantity as the resolution increases. We examine here the possible convergence of a curvature estimator, which is one of the best available at coarse resolutions. Although a convergence theorem for this estimator was published, it was based on a hypothesis related to asymptotic properties of maximal Digital Straight Segments (DSS). We show here that this hypothesis is asymptotically false. The proof involves results from arithmetic properties of digital lines, digital convexity, combinatorics and continued fractions. It exploits a result on convex digital polygons, related to random polytopes.

1 Introduction

Estimating geometric features of shapes or curves solely on their digitization is a classical problem in image analysis and pattern recognition. Some of the geometric features are global: area, perimeter, moments. Others are local: tangents, normals, curvature. Algorithms that performs this task on digitized objects are called *digital geometric estimators*. An interesting property these estimators should have is to converge towards the continuous geometric measure as the digitization resolution increases. However, few estimators have been proved to be convergent. In all works, shapes are generally supposed to have a smooth boundary (at least twice differentiable) and either to be convex or to have a finite number of inflexion points. The shape perimeter estimation has for instance been tackled in [10]. It proved the convergence of a perimeter estimator based on curve segmentation by maximal DSS. The speed of convergence of several length estimators has also been studied in [4]. Klette and Zunic [9] survey results about the convergence (and the speed of convergence) of several global geometric estimators. They show that discrete moments converge toward continuous moments.

As far as we know, there is only one work that deal with the convergence of local geometric estimators [3]. The symmetric tangent estimator appears to be convergent subject to an hypothesis on the growth of DSS as the resolution increases (Hypothesis 4.4). The same hypothesis entails that a curvature estimator is convergent: it is based on DSS recognition and circumscribed circle computation (see Definition 4.3).

In this paper, we relate the number and the lengths of DSS to the number and lengths of edges of convex hulls of digitized shapes. Using arguments related to digital convex polygons, we estimate the asymptotic behaviour of both quantities. We theoretically show that the hypotheses used in [3] are not verified. Experiments confirm our result. The convergence theorem is thus not applicable to digital curves. As a consequence, the existence of convergent digital curvature estimators remains an open problem. The paper — and the proof — is organized as follows. First, we recall some standard notions of digital geometry and combinatoric representation of digital lines, i.e. patterns. The relations between maximal segments and edges of convex digital polygons are then studied to get bounds on maximal segments lengths and number. Finally, the asymptotic behaviour of maximal segments is deduced from the asymptotic behaviour of convex digital polygons, itself linked to random polytopes [1]. Growth of some DSS on the curves is thus proved to be too slow to ensure the convergence of curvature estimation. This theoretical result is further confirmed by experiments.

2 Maximal digital straight segments

We restrict our study to the geometry of 4-connected digital curves. A digital object is a set of pixels and its boundary in $\mathbb{R}^2$ is a collection of vertices and edges. The boundary forms a 4-connected curve in the sense used in the present paper. Our work may easily be adapted to 8-connected curves. In the paper, all the reasoning are made in the first octant, but extends naturally to the whole digital plane. The digital curve is denoted by C. Its points (C_k) are assumed to be indexed. A set of successive points of C ordered increasingly from index i to j will be conveniently denoted by $C_{i,j}$ or $[C_i, C_j]$ when no ambiguities are raised.

2.1 Standard line, digital straight segment, maximal segments

Definition 2.1 *(Réveillès [13]) The set of points (x, y) of the digital plane verifying $\mu \leq ax - by < \mu + |a| + |b|$, with a, b and μ integer numbers, is called the* standard line *with slope a/b and shift μ.*

The *standard lines* are the 4-connected discrete lines. The quantity $ax - by$ is called the *remainder* of the line. The points whose remainder is μ (resp. $|a| + |b| - 1$) are called upper (resp. lower) leaning points. The principal upper and lower leaning points are defined as those with extremal x values. Finite connected portions of digital lines define *digital straight segment*. Since we work with restricted parts of C, we always suppose that indices are totally ordered on this part.

Definition 2.2 *A set of successive points $C_{i,j}$ of C is a* digital straight segment (DSS) *iff there exists a standard line $D(a, b, \mu)$ containing them. The predicate "$C_{i,j}$ is a DSS" is denoted by $S(i, j)$.*

The first index j, $i \leq j$, such that $S(i, j)$ and $\neg S(i, j + 1)$ is called the *front* of i. The map associating any i to its front is denoted by F. Symmetrically, the first index i such that $S(i, j)$ and $\neg S(i - 1, j)$ is called the *back* of j and the corresponding mapping is denoted by B.

Maximal segments form the longest possible DSS in the curve. They are essential when analyzing digital curves: they provide tangent estimations [6, 12], they are used for polygonizing the curve into the minimum number of segments [7].

Definition 2.3 *Any set of points $C_{i,j}$ is called a* maximal segment *iff any of the following equivalent characterizations holds: (1) $S(i, j)$ and $\neg S(i, j + 1)$ and $\neg S(i - 1, j)$, (2) $B(j) = i$ and $F(i) = j$, (3) $\exists k, i = B(k)$ and $j = F(B(k))$, (4) $\exists k', i = B(F(k'))$ and $j = F(k')$.*

From characterizations (3) and (4) of Definition 2.3, any DSS $C_{i,j}$ and hence any point belongs to at least two maximal segments (possibly identical) $C_{B(j),F(B(j))}$ and $C_{B(F(i)),F(i)}$.

2.2 Patterns and DSS

We here recall a few properties about *patterns* composing DSS and their close relations with continued fractions. They constitute a powerful tool to describe discrete lines with rational slopes [2, 8]. Since we are in the first octant, the slopes are between 0 and 1.

Definition 2.4 *Given a standard line (a, b, μ), we call* pattern *of characteristics (a, b) the succession of Freeman moves between any two consecutive upper leaning points. The Freeman moves defined between any two consecutive lower leaning points is the previous word read from back to front and is called the* reversed pattern.

A pattern (a, b) embedded anywhere in the digital plane is obviously a DSS (a, b, μ) for some μ. Since a DSS contains at least either two upper or two lower leaning points, a DSS (a, b, μ) contains at least one *pattern* or one *reversed pattern* of characteristics (a, b).

Definition 2.5 *We call* simple continued fraction *and we write:*

$$z = a/b = [0, u_1 \ldots, u_i, \ldots, u_n] \quad \text{with} \quad z = 0 + \cfrac{1}{u_1 + \cfrac{1}{\ldots + \cfrac{1}{u_{n-1} + \cfrac{1}{u_n}}}}$$

We call k-th convergent *the simple continued fraction formed of the* $k+1$ *first partial quotients:* $z_k = \frac{p_k}{q_k} = [0, u_1, \ldots, u_k]$.

There exists a recursive transformation for computing the pattern of a standard line from the *simple continued fraction* of its slope [2]. We call E the mapping from the set of positive rationnal number smaller than one onto Freeman-code's words defined as follows. First terms are stated as $E(z_0) = 0$ and $E(z_1) = 0^{u_1}1$ and others are expressed recursively:

$$E(z_{2i+1}) = E(z_{2i})^{u_{2i+1}} E(z_{2i-1}) \qquad (1)$$
$$E(z_{2i}) = E(z_{2i-2}) E(z_{2i-1})^{u_{2i}} \qquad (2)$$

In the following, the *complexity* of a pattern is the depth of its decomposition in simple continued fraction. We recall a few more relations:

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1} \qquad (3)$$
$$(p_k, q_k) = u_k (p_{k-1}, q_{k-1}) + (p_{k-2}, q_{k-2}) \qquad (4)$$

We now focus on computing vector relations between leaning points (upper and lower) inside a pattern. In the following we will consider a DSS$(a, b, 0)$ in the first octant starting at the origin and ending at its second lower leaning point (whose coordinate along the x-axis is positive). We define $a/b = z_n = [0, u_1, \ldots, u_n]$ for some n. Points will be called U_1,L_1, U_2 and L_2 as shown in Fig. 1. We can state $\mathbf{U_1L_1} = \mathbf{U_2L_2}$ and $\mathbf{U_1U_2} = \mathbf{L_1L_2} = (b, a)$. We recall that the Freeman moves of $[U_1, L_1]$ are the same as those of $[U_2, L_2]$. Furthermore Freeman moves between U_1 and U_2 form the *pattern* (a, b) and those between L_1 and L_2 form the *reversed pattern* (a, b).

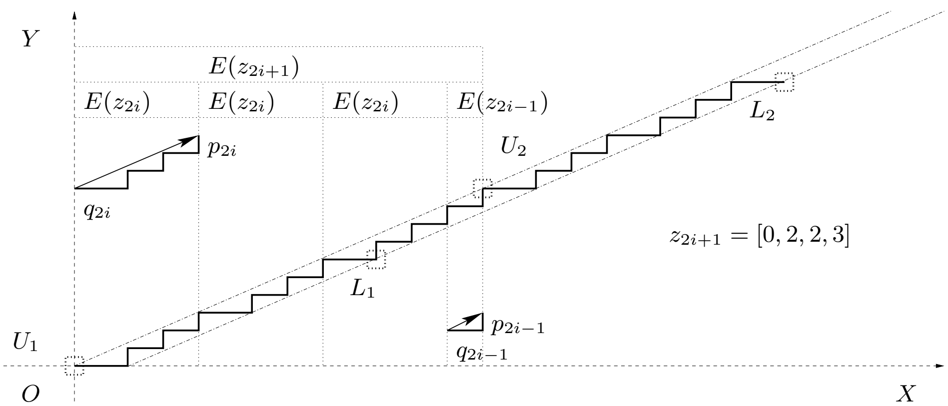

Fig. 1. A DSS$(a, b, 0)$ with an odd complexity of slope, taken between origin and its second lower leaning point.

Proposition 2.6 *A pattern with an odd complexity (say* $n = 2i+1$*) is such that* $\mathbf{U_1L_1} = (u_{2i+1} - 1)(q_{2i}, p_{2i}) + (q_{2i-1}, p_{2i-1}) + (1, -1)$ *and* $\mathbf{L_1U_2} = (q_{2i} - 1, p_{2i} + 1)$*. Moreover the DSS* $[U_1, L_1]$ *has* $E(z_{2i})^{u_{2i+1}-1}$ *as a left factor, and the DSS* $[L_1, U_2]$ *has* $E(z_{2i-1})^{u_{2i}}$ *as a right factor.*

Proof. From Eq. (3) we have: $p_{2i+1}q_{2i} - p_{2i}q_{2i+1} = (-1)^{2i+1+1} = 1$, which can be rewritten as: $aq_{2i} - bp_{2i} = 1$. (q_{2i}, p_{2i}) are clearly the Bézout coefficients of (a, b). The remainder of $(b + 1 - q_{2i}, a - 1 - p_{2i})$ is $a + b - 1$. Since $b + 1 - q_{2i}$ is positive but smaller than b, it is the first positive lower leaning point L_1 and $\mathbf{U_1L_1} = (b + 1 - q_{2i}, a - 1 - p_{2i})$. Using Eq. (4) yields: $\mathbf{U_1L_1} = ((u_{2i+1} - 1)q_{2i} + q_{2i-1} + 1, (u_{2i+1} - 1)p_{2i} + p_{2i-1} - 1)$. From $\mathbf{L_1U_2} = -\mathbf{U_1L_1} + \mathbf{U_1U_2}$, we further get that $\mathbf{L_1U_2} = (q_{2i} - 1, p_{2i} + 1)$. From Eq. (1) $E(z_{2i})^{u_{2i+1}-1}$ is a left factor of $[U_1, U_2]$ but also of $[U_1, L_1]$. Writing $E(z_{2i+1})$ as $E(z_{2i})^{u_{2i+1}-1}E(z_{2i-2})E(z_{2i-1})^{u_{2i}+1}$, and expanding $\mathbf{L_1U_2}$ as $(u_{2i}q_{2i-1} + q_{2i-2} - 1, u_{2i}p_{2i-1} + p_{2i-2} + 1)$ with Eq. (4), we see that $E(z_{2i-1})^{u_{2i}}$ is a right factor of $[L_1, U_2]$. □

Proposition 2.7 *A pattern with an even complexity (say* $n = 2i$*) is such that* $\mathbf{U_1L_1} = (q_{2i-1} + 1, p_{2i-1} - 1)$ *and* $\mathbf{L_1U_2} = (u_{2i} - 1)(q_{2i-1}, p_{2i-1}) + (q_{2i-2}, p_{2i-2}) + (-1, 1)$*. Moreover the DSS* $[U_1, L_1]$ *has* $E(z_{2i-2})^{u_{2i-1}}$ *as a left factor, and the DSS* $[L_1, U_2]$ *has* $E(z_{2i-1})^{u_{2i}-1}$ *as a right factor.*

3 Properties of maximal segments for convex curves

In this section, we study relations between maximal segments and digital edges of convex shape digitization. The dilation of S by a real factor r is denoted by $r \cdot S$. Let $\mathcal{D}_m$ be the digitization of step $1/m$, i.e. if S is a real shape: $\mathcal{D}_m(S) = (m \cdot S) \cap \mathbb{Z}^2$. The length estimator based on the city-block distance is written as $\mathcal{L}^1$.

3.1 Convex digital polygon (CDP)

Definition 3.1 Γ *is a* convex digital polygon (CDP) *if its vertices* $(V_i)_{i=1..e}$ *form the minimal set of discrete points such that* $\Gamma = \mathcal{D}_1(\mathrm{conv}(V_1, \ldots, V_e))$ *and* Γ *is different from the digitization of the convex hull of any proper subset of the* (V_i)*. The number of vertices or edges of* Γ *is denoted by* $n_e(\Gamma)$ *and its perimeter by* $\mathrm{Per}(\Gamma)$*.*

The points on the boundary of P form a 4-connected contour. A CDP is also called a lattice convex polygon [14]. An *edge* is the Euclidean segment joining two consecutive vertices, and a *digital edge* is the discrete segment joining two consecutive vertices. It is clear that we have as many *edges* as *digital edges* and as vertices. From characterizations of discrete convexity [5], we clearly see that:

Proposition 3.2 *Each digital edge of a CDP is either a pattern or a succession of the same pattern whose slope is the one of the edge. In other words, both vertices are upper leaning points of the digital edge.*

We now recall one theorem concerning the asymptotic number of vertices of CDP that are digitization of continuous shapes. It comes from asymptotic properties of random polytopes.

Theorem 3.3 *(Adapted from Balog, Bárány [1]) If S is a plane convex body with $\mathcal{C}^3$ boundary and positive curvature then*

$$c_1(S)m^{\frac{2}{3}} \leq n_v(\mathrm{conv}(\mathcal{D}_m(S))) \leq c_2(S)m^{\frac{2}{3}}$$

where the constants $c_1(S)$ and $c_2(S)$ depend on extremal bounds of the curvatures along S. Hence for a disc c_1 and c_2 are absolute constants.

3.2 Links between maximal segments and edges of CDP

Maximal segments are DSS: between any two upper (resp. lower) leaning points lays at least a lower (resp. upper) leaning point. The slope of a maximal segment is then defined by two consecutive upper and/or lower leaning points. Digital edges are patterns and their vertices are upper leaning points (from Prop. 3.2). Thus, vertices may be upper leaning points but never lower leaning points of maximal segments. Since a digital edge is a DSS, we get

Lemma 3.4 *A maximal segment cannot be strictly contained into a digital edge.*

We now introduce a special class of digital edge.

Definition 3.5 *We call* supporting edge*, a digital edge whose two vertices define leftmost and rightmost upper leaning points of a maximal segment.*

Relations between maximal DSS and digital edges are given by the following lemmas:

Lemma 3.6 *A supporting edge defines only one maximal segment: it is the only one containing the edge and it has the same slope. If a maximal segment contains two or more upper leaning points then there is a* supporting edge *linking its leftmost and rightmost upper leaning points with the same slope. If a maximal segment contains three or more lower leaning points then it has a* supporting edge.

Lemma 3.7 *If a maximal segment is defined by only two consecutive lower leaning points then it has one upper leaning point which is some vertex of the CDP by convexity.*

Lengths of maximal segments and digital edges are tightly intertwined, as shown by the two next propositions.

Proposition 3.8 *Let $[V_kV_{k+1}]$ be a* supporting edge *of slope $\frac{a}{b}$ made of f patterns (a,b) and let MS be the* maximal segment *associated with it (Lemma 3.6). Their lengths are linked by the inequalities:*

$$\mathcal{L}^1(V_kV_{k+1}) \leq \mathcal{L}^1(MS) \leq \frac{f+2}{f}\mathcal{L}^1(V_kV_{k+1}) - 2 \quad \textit{and} \quad \frac{1}{3}\mathcal{L}^1(MS) \leq \mathcal{L}^1(V_kV_{k+1}) \leq \mathcal{L}^1(MS) \leq 3\mathcal{L}^1(V_kV_{k+1})$$

Proof. Vertices V_k and V_{k+1} are *leftmost* and *rightmost* upper leaning points of MS. The points $V_k-(b,a)$, $V_{k+1}+(b,a)$ while clearly upper leaning points of the standard line going through $[V_kV_{k+1}]$ cannot belong to the CDP. Hence MS cannot extends further of its supporting edge of more than $|a|+|b|-1$ points on both sides. Consequently $\mathcal{L}^1(MS)\le\mathcal{L}^1(V_kV_{k+1})+2(|a|+|b|-1)$. Using $\mathcal{L}^1(V_kV_{k+1})=f(|a|+|b|)$ brings: $\mathcal{L}^1(V_kV_{k+1})\le\mathcal{L}^1(MS))\le\frac{f+2}{f}\mathcal{L}^1(V_kV_{k+1})-2$. Worst cases bring $\mathcal{L}^1(V_kV_{k+1})\le\mathcal{L}^1(MS)\le 3\mathcal{L}^1(V_kV_{k+1})$ □

Proposition 3.9 *Let $MS_{k'}$ be a maximal segment in the configuration of Lemma 3.7, and so let V_k be its upper leaning point. The length of the maximal segment is upper bounded by:*

$$\mathcal{L}^1(MS_{k'})\le 4\left(\mathcal{L}^1(V_{k-1}V_k)+\mathcal{L}^1(V_kV_{k+1})\right)$$

Proof. We call L_1, L_2 the leftmost and rightmost lower leaning points and $U_2\equiv V_k$ the upper leaning point (see Fig. 1). Suppose that $MS_{k'}$ has a slope with an odd complexity (say $2i+1$). Proposition 2.6 implies $\mathcal{L}^1(\mathbf{L_1U_2})=q_{2i}+p_{2i}$. There is clearly a right part of $[L_1U_2]$ (i.e. $[L_1V_k]$) that is contained in $[V_{k-1}V_k]$ and touches V_k. The pattern $E(z_{2i-1})^{u_{2i}}$ is a right factor of $[L_1U_2]$ (Proposition 2.6 again). It is indeed a right factor of $[V_{k-1}V_k]$ too, since it cannot extends further than V_{k-1} to the left without defining a longer digital edge. We get $[V_{k-1}V_k]\supseteq E(z_{2i-1})^{u_{2i}}$ and the length inequality $\mathcal{L}^1(V_{k-1}V_k)\ge u_{2i}(q_{2i-1}+p_{2i-1})$.

From Eq. (4), we have: $q_{2i}+p_{2i}=u_{2i}(q_{2i-1}+p_{2i-1})+q_{2i-2}+p_{2i-2}$ and $q_{2i-2}+p_{2i-2}\le q_{2i-1}+p_{2i-1}$. We obtain immediately $\mathcal{L}^1(\mathbf{L_1U_2})=q_{2i}+p_{2i}\le(u_{2i}+1)(q_{2i-1}+p_{2i-1})$. By comparing this length to the length of the digital edge $[V_{k-1}V_k]$, we get $\mathcal{L}^1(\mathbf{L_1U_2})\le\frac{u_{2i}+1}{u_{2i}}\mathcal{L}^1(V_{k-1}V_k)$.

Proposition 2.6 and similar arguments on $[V_kV_{k+1}]$ brings $\mathcal{L}^1(\mathbf{U_2L_2})\le\frac{u_{2i+1}}{u_{2i+1}-1}\mathcal{L}^1(V_{k-1}V_k)$. Worst cases are then $\mathcal{L}^1(\mathbf{L_1U_2})\le 2\mathcal{L}^1(V_{k-1}V_k)$ and $\mathcal{L}^1(\mathbf{U_2L_2})\le 2\mathcal{L}^1(V_kV_{k+1})$. The case where $MS_{k'}$ has a slope with an even complexity (say $2i$) uses Prop. 2.7 and is treated similarly.

Since MS has only one upper leaning point, it cannot be extended further than $\mathcal{L}^1(\mathbf{U_2L_2})$ on the left and $\mathcal{L}^1(\mathbf{L_1U_2})$ on the right (Lemma 3.6). We thus get $\mathcal{L}^1(MS_{k'})\le 4(\mathcal{L}^1(V_{k-1}V_k)+\mathcal{L}^1(V_kV_{k+1}))$. □

The proof of the following theorem is given in Appendix B for limited space reasons.

Theorem 3.10 *Let E be a supporting edge whose slope has a complexity n, $n\ge 2$, then the maximal segment containing E includes at most n other edges on each side of E.*

Corollary 3.11 *The shortest pattern of a supporting edge for which a maximal segment contains $2n+1$ digital edge is $z_n=[0,2,\dots,2]$. If the DCP is enclosed in a $m\times m$ grid, then the maximal number n of digital edges included in one maximal segment is upper bounded as: $n\le\frac{\log\frac{4m}{\sqrt{2}}}{\log(1+\sqrt{2})}-1$.*

Proof. The number $L=[0,2,\dots,2,\dots]$ is a quadratic number equal to $-1+\sqrt{2}$. Its recursive characterization is $U_n=2U_{n-1}+U_{n-2}$ with $U_0=0$ and $U_1=1$. Solving it leads to $U_n=\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^n-(1-\sqrt{2})^n\right)$. Hence asymptotically, $U_n\approx\frac{\sqrt{2}}{4}(1+\sqrt{2})^n$ and $\lim\frac{U_n}{U_{n+1}}{}_{n\to\infty}=L$.

The shortest edge (whose slope is an n-th convergent of L) that fits into an $m\times m$ grid is such that $U_{n+1}\le m$. We thus obtain that $n\le\frac{\log\frac{4m}{\sqrt{2}}}{\log(1+\sqrt{2})}-1$. □

3.3 Asymptotic number and size of maximal segments

We assume in this section that the digital convex polygon Γ is enclosed in a $m\times m$ grid. We wish to compute a lower bound for the number of edges related to at least one maximal segment. We show in Theorem 3.12 that this number is significant and increases at least as fast as the number of edges of the DCP divided by $\log m$. From this lower bound, we are able to find an upper bound for the length of the smallest maximal segment of a DCP (Theorem 3.13). We first label each vertex of the DCP as follows: (i) a *2-vertex* is an upper leaning point of a supporting edge, (ii) a *1-vertex* is an upper leaning point of some maximal segment but is not a 2-vertex, (iii) *0-vertices* are all the remaining vertices. The number of i-vertices is denoted by n_i. Given an orientation on the digital contour, the number of edges going from an i-vertex to a j-vertex is denoted by n_{ij}.

Theorem 3.12 *The number of supporting edges and of 1-vertices of Γ are related to its number of edges with*

$$\frac{n_e(P)}{\Omega(\log m)}\le n_1+2n_{22}. \tag{5}$$

An immediate corollary is that there are at least $n_e(\Gamma)/\Omega(\log m)$ maximal segments.

Proof. From Theorem 3.10 and its Corollary 3.11, we know that a DSS hence a maximal segment cannot include more than $\Omega(\log m)$ edges. Hence there cannot be more than $\Omega(\log m)$ 0-vertices for one 1-vertex or for one 2-vertex. We get $n_{00} \leq (n_1 + n_2)\Omega(\log m)$. We develop the number of edges with each possible label: $n_e(\Gamma) = n_{22} + n_{02} + n_{12} + n_{20} + n_{21} + n_{00} + n_{01} + n_{10} + n_{11}$. Since, $n_{02} + n_{12} \leq n_{22}$, $n_{20} + n_{21} \leq n_{22}$ and $n_{01} + n_{10} + n_{11} \leq 3n_1$, we get $n_e(\Gamma) \leq 3n_{22} + n_{00} + 3n_1$. Noting that a 2-vertex cannot be isolated by definition of supporting edges (Definition 3.5) gives $n_2 \leq 2n_{22}$. Once inserted in $n_{00} \leq (n_1 + n_2)\Omega(\log m)$ and compared with $n_e(\Gamma)$, we get the expected result. □

We now relate the DCP perimeter to the length of maximal segments.

Theorem 3.13 *The length of the smallest maximal segment of the DCP Γ is upper bounded:*

$$\min_l \mathcal{L}^1(MS_l) \leq \Omega(\log m)\frac{\mathrm{Per}(\Gamma)}{n_e(\Gamma)}. \tag{6}$$

Proof. We have $Per(\Gamma) = \sum_{n_e} \mathcal{L}^1(E_i)$. We now may expand the sum on supporting edges (22-edges), on edges touching a 1-vertex, and on others. Edges touching 1-vertices may be counted twice, therefore we divide by 2 their contribution to the total length.

$$\sum_{n_e} \mathcal{L}^1(E_i) \quad \geq \quad \sum_{n_{22}} \mathcal{L}^1(E_j^{22}) + \frac{1}{2}\sum_{n_1} \mathcal{L}^1(E_{k-1}^{?1}) + \mathcal{L}^1(E_k^{1?}) \tag{7}$$

For the first term, each supporting edge indexed by j (a 22-edge) has an associated maximal segment, say indexed by j'. From Proposition 3.8, we know that $\mathcal{L}^1(E_j^{22}) \geq \frac{1}{3}\mathcal{L}^1(MS_{j'})$.

For the second term, each 1-vertex indexed by k is an upper leaning point of some maximal segment indexed by k'. Proposition 3.9 holds and $\mathcal{L}^1(E_{k-1}^{?1}) + \mathcal{L}^1(E_k^{1?}) \geq \frac{1}{4}\mathcal{L}^1(MS_{k'})$.

Putting everything together in Eq. (7), we get:

$$\sum_{n_e} \mathcal{L}^1(E_i) \geq \frac{1}{3}\sum_{n_{22}} \mathcal{L}^1(MS_{j'}) + \frac{1}{8}\sum_{n_1} \mathcal{L}^1(MS_{k'}) \geq (\frac{1}{3}n_{22} + \frac{1}{8}n_1)\min_l \mathcal{L}^1(MS_l) \geq \frac{1}{8}(n_1 + 2n_{22})\min_l \mathcal{L}^1(MS_l)$$

Inserting the lower bound of Theorem 3.12 into the last inequality concludes. □

4 Asymptotic properties of shapes digitized at increasing resolutions

We may now turn to the main interest of the paper: studying the asymptotic properties of discrete geometric estimators on digitized shapes. We therefore consider a plane convex body S which is contained the square $[0, 1] \times [0, 1]$ (w.l.o.g.). Furthermore, we assume that its boundary $\gamma = \partial S$ is $\mathcal{C}^3$ with everywhere strictly positive curvature. This assumption is not very restrictive since people are mostly interested in regular shapes. Furthermore, the results of this section remains valid if the shape can be divided into a *finite* number of convex and concave parts; each one is then treated separately. The digitization of S with step $1/m$ defines a digital convex polygon $\Gamma(m)$ inscribed in a $m \times m$ grid. We first examine the asymptotic behavior of the maximal segments of $\Gamma(m)$, both theoretically and experimentally. We then study the *asymptotic convergence* of a discrete curvature estimator. We show that contrary to what was thought its convergence is still an open problem. Experimental evaluation confirms this result.

4.1 Asymptotic behavior of maximal segments

The next theorem summarizes the asymptotic size of the smallest maximal segment wrt the grid size m.

Theorem 4.1 *The length of the smallest maximal segment of $\Gamma(m)$ has the following asymptotic upper bound:*

$$\min_i \mathcal{L}^1(MS_i(\Gamma(m))) \leq \Omega(m^{1/3}\log m) \tag{8}$$

Proof. Theorem 3.13 gives for the DCP $\Gamma(m)$ the inequality $\min_i \mathcal{L}^1(MS_i(\Gamma(m))) \leq \Omega(\log m)\frac{\mathrm{Per}(\Gamma(m))}{n_e(\Gamma(m))}$. Since $\Gamma(m)$ is convex included in the subset $m \times m$ of the digital plane, its perimeter $\mathrm{Per}(\Gamma(m))$ is upper bounded by $4m$. On the other hand, Theorem 3.3 indicates that its number of edges $n_e(\Gamma(m))$ is lower bounded by

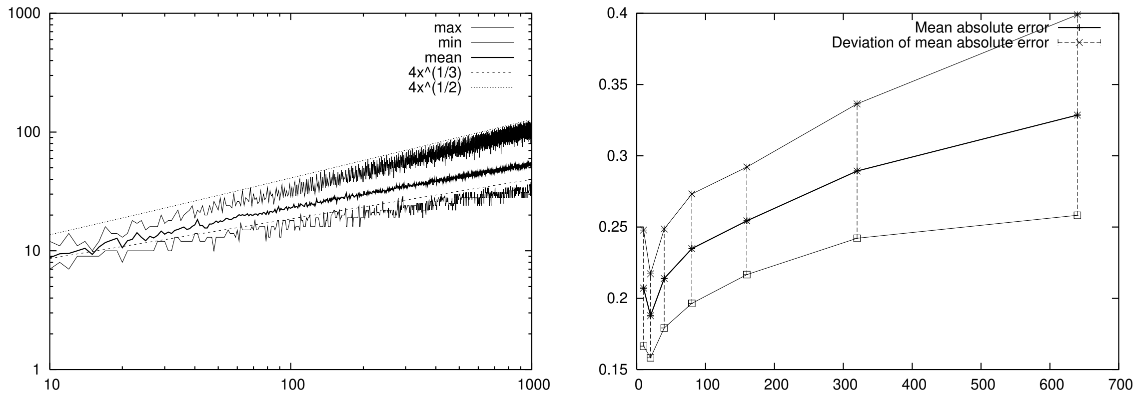

Fig. 2. For both curves, the digitized shape is a disk of radius 1 and the abscissa is the digitization resolution. Left: plot in log-space of the $\mathcal{L}^1$-size of maximal segments. Right: plot of the mean and standard deviation of the absolute error of curvature estimation, $|\hat{\kappa} - 1|$ (expected curvature is 1).

$c_1(S)m^{2/3}$. Putting everything together gives $\min_i \mathcal{L}^1(MS_i(\Gamma(m))) \leq \Omega(\log m)\frac{4m}{c_1(S)m^{2/3}}$ which is once reduced what we wanted to show. $\square$

Although there are points on a shape boundary around which maximal segments grow as fast as $O(m^{1/2})$ (the critical points in [11]), some of them do not grow as fast. A closer look at the proofs of Theorem 3.13 shows that a significant part of the maximal segments (at least $\Omega(1/(\log m))$) has an average length that grows no faster than $\Omega(m^{1/3} \log m)$. This fact is confirmed with experiments. Fig. 2, left, plots the size of maximal segments for a disk of radius 1 digitized with increasing resolution. The average size is clearly closer to $m^{1/3}$ than to $\sqrt{m}$.

4.2 Asymptotic convergence of discrete geometric estimators

A useful property that a discrete geometric estimator may have is to converge toward the geometric quantity of the continuous shape boundary when the digitization grid gets finer [3, 4, 9]. It may be expressed as follows,

Definition 4.2 *Let $\mathcal{F}$ be any geometric descriptor on the shape S with boundary γ and digitizations $\Gamma(m)$. The discrete geometric estimator $\mathcal{E}$* asymptotically converges *toward the descriptor $\mathcal{F}$ for γ iff*

$$|\mathcal{E}(\Gamma(m)) - \mathcal{F}(\gamma)| \leq \epsilon(m) \text{ with } \lim_{m \to +\infty} \epsilon(m) = 0. \tag{9}$$

Of course, interesting discrete geometric estimator should converge for a large class of curves. We now recall the definition of a discrete curvature estimator based on the estimation of an osculating circle [3].

Definition 4.3 *Let P be any point on a discrete contour, $Q = B(P)$ and $R = F(P)$ are the extremities of the longest DSS starting from P (called* half-tangents*). Then the* curvature estimator by circumcircle *$\hat{\kappa}(P)$ is the inverse of the radius of the circle circumscribed to P, Q and R.*

Experiments show that this estimator rather correctly estimates the curvature of discrete circles *on average* ($\approx 10\%$ error). It is indeed better than any other curvature estimators proposed in the litterature. Theorem B.4 of [3] demonstrates the *asymptotic convergence* of this curvature estimator, subject to the hypothesis:

Hypothesis 4.4 *Half-tangents on digitized boundaries grow at a rate of $\Omega(\sqrt{m})$ with the resolution m.*

However, with our study of maximal segments, we can state that

Claim 4.5 *Hypothesis 4.4 is* not *verified for digitizations of C^3-curves with strictly positive curvature. We cannot conclude on the asymptotic convergence of the curvature estimator by circumcircle.*

Proof. It is enough to note that half-tangents, being DSS, are included in maximal segments and may not be longer. Furthermore, since maximal segments cover the whole digital contour, some half-tangents will be included in the smallest maximal segments. Since the smallest maximal segments have a length upper bounded by $\Omega(m^{1/3} \log m)$ (Theorem 4.1), the length of some half-tangents has the same upper bound, which is smaller than $\Omega(\sqrt{m})$. □

The asymptotic convergence of a curvature estimator is thus still an open problem. Furthermore, precise experimental evaluation of this estimator indicates that it is most certainly not asymptotically convergent, although it is actually on average one of the most stable discrete curvature estimator (see Fig. 2, right). Former experimental evaluations of this estimator were averaging the curvature estimates on all contour points. The convergence of the average of all curvatures does not induce the convergence of the curvature at one point.

5 Conclusion

We show in this paper the relations between edges of convex hulls and maximal segments in terms of number and sizes. We provide an asymptotical analysis of the worst cases of both measures. A consequence of the study is the refutation of an hypothesis related to the asymptotic growth of maximal segments and which was essential in proving the convergence of a curvature estimator based on DSS and circumcircles [3]. Our work also applied to digital tangents since their convergence relies on the same hypothesis. The existence of a convergent discrete estimator of curvature based on DSS is thus still a challenging problem and we are currently investigating it.

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A Proof of Proposition 2.7

PROOF: (ex prop 2.8) From Eq. (3) we have: $p_{2i}q_{2i-1} - p_{2i-1}q_{2i} = (-1)^{2i+1} = -1$, which can be rewritten as: $a(-q_{2i-1}) - b(-p_{2i-1}) = 1$ and eventually $a(q_{2i} - q_{2i-1}) - b(p_{2i} - p_{2i-1}) = 1$. We clearly ontain the Bézout coefficients. From its remainder we get the relatives coordinates of L_1, as: $\mathbf{U_1L_1} = (q_{2i-1}+1, p_{2i-1}-1)$. From $\mathbf{L_1U_2} = -\mathbf{U_1L_1} + \mathbf{U_1U_2}$ we get : $\mathbf{L_1U_2} = ((u_{2i}-1)q_{2i-1} + q_{2i-2} - 1, (u_{2i}-1)p_{2i-1} + p_{2i-2} + 1)$. Using $E(z_{2i}) = E(z_{2i-2})^{u_{2i-1}+1}E(z_{2i-3})E(z_{2i-1})^{u_{2i}-1}$ and $\mathbf{U_1L_1} = (u_{2i-1}q_{2i-2} + q_{2i-3} + 1, u_{2i-1}p_{2i-2} + p_{2i-3} - 1)$, it is clear that $E(z_{2i-2})^{u_{2i-1}}$ is a left factor of the DSS between U_1 and L_1. From Eq. (2) and $\mathbf{L_1U_2}$ we clearly see that $E(z_{2i-1})^{u_{2i}-1}$ is a right factor of the DSS between L_1 and U_2. □

B Proof of Theorem 3.10

Lemma B.1 *We call P_n a pattern of complexity n whose Freeman code is $E(z_n)$. One can build strict right and left factors (called respectively R and L) of P_n such that:*

(i) $[RP_n]$, $[P_nL]$ *and* $[RP_nL]$ *are DSS of slope* z_n,

(ii) R *and* L *are patterns (or successions of the same pattern) ,*

(iii) RP_n, P_nL *and* RP_nL *are not patterns,*

(iv) the slope of R *is greater than that of* P_n *and the slope of* P_n *is greater than that of* L,

(v) maximal complexity of slope of R *and* L *depends on parity of* n:

Complexity of P_n	*maximal complexity of* R	*maximal complexity of* L
$2i+1$	$2i+1$	$2i$
$2i$	$2i-1$	$2i$

(vi) Complexity of factors obtained by substracting R *or* L *from* P_n *depends on parity of* n:

Complexity of P_n	*complexity of* $P_n \setminus R$	*complexity of* $P_n \setminus L$
$2i+1$	$2i$	$2i+1$
$2i$	$2i$	$2i-1$

Proof. Since R and L are strict factors of P_n, their Freeman moves are compatible with those of $E(z_n)$, giving same slope when R,P_n and L are put together. Thus $[RP_nL]$ is a DSS of slope z_n. From digital straightness we clearly have digital convexity (see [5]). Upper leaning points of this DSS are located at extremities of P_n.

We simply choose among strict factors R and L those that are patterns so that they fit descriptions given in Eq. (1) and Eq. (2). We may now describe them given the parity of n.

Consider the case where n is odd (say $n = 2i+1$), from Eq. (1) we get: $R = E(z_{2i})^{u_{2i+1}-r}E(z_{2i-1})$ and $L = E(z_{2i})^{u_{2i+1}-l}$. If R and L are longer patterns, they are no longer strict factors of P_{2i+1}. We see that R is a pattern of complexity $2i+1$ and that L is a succession of the pattern $E(z_{2i})$, with a complexity of $2i$. Factor obtained by substracting R from P_{2i+1} equals $E(z_{2i})^r$ and substracting L from P_{2i+1} gives $E(z_{2i})^lE(z_{2i-1})$. The slope of R equals $z'_{2i+1} = [0, u_1, \ldots, u_{2i}, u_{2i+1} - r] = \frac{p'_{2i+1}}{q'_{2i+1}}$. From Eq. (4) we get that $\frac{p_{2i+1}}{q_{2i+1}} = \frac{p'_{2i+1}+rp_{2i}}{q'_{2i+1}+rq_{2i}}$. The sign of $z'_{2i+1} - z_{2i+1}$ is that of $p'_{2i+1}q_{2i} - q'_{2i+1}p_{2i}$, and is positive (see Eq. (3)). Thus the slope of R is greater than that of P_{2i+1}. Same reasoning applied to $z_{2i+1} - z_{2i}$ brings that the slope of P_{2i+1} is greater than that of L.

Consider now that n is even (say $n = 2i$), from Eq. (2) we get: $R = E(z_{2i-1})^{u_{2i}-r}$ and $L = E(z_{2i-2})E(z_{2i-1})^{u_{2i}-l}$. If R and L are longer patterns, they are no longer strict factors of P_{2i}. Clearly, R has a complexity of $2i-1$ and that of L equals $2i$. Factor obtained by substracting R from P_{2i} equals $E(z_{2i-2})E(z_{2i-1})^r$ and substracting L from P_{2i} gives $E(z_{2i-1})^l$. The slope of L equals $z'_{2i} = [0, u_1, \ldots, u_{2i-1}, u_{2i} - l] = \frac{p'_{2i}}{q'_{2i}}$. From Eq. (4) we get that $\frac{p_{2i}}{q_{2i}} = \frac{p'_{2i}+lp_{2i-1}}{q'_{2i}+lq_{2i-1}}$. The sign of $z_{2i} - z'_{2i}$ is that of $q'_{2i}p_{2i-1} - p'_{2i}q_{2i-1}$, and is positive (see Eq. (3)). Thus the slope of P_n is greater than that of L. Same reasoning applied to $z_{2i-1} - z_{2i}$ brings that the slope of R is greater than that of P_n.

It is now clear that slopes are strictly decreasing from R to P_n and from P_n to L whatever the parity of n. □

Proof. [of Theorem 3.10] We construct $2n$ digital edges around E:

- $(R_i)_{1 \le i \le n}$ at left of E,

- $(L_i)_{1\leq i\leq n}$ at right of E.

These edges are such that $[R_n \dots R_i \dots R_1 E L_1 \dots L_j \dots L_n]$ is a DSS of slope $z_n = a/b$ and has no upper leaning points but those located on E. E may contain several time the pattern $E(z_n)$. It is clear that $R_n \dots R_i \dots R_1$ (resp. $L_1 \dots L_j \dots L_n$) has to be a right (resp. left) strict factor of $E(z_n)$. Moreover R_i is a right strict factor of $E(z_n) \setminus R_1 \dots R_{i-1}$ and L_i is a left strict factor of $E(z_n) \setminus L_1 \dots L_{i-1}$. From Proposition 3.2 any of the digital edges $(R_i)_{1\leq i\leq n}$ and $(L_i)_{1\leq i\leq n}$ is a pattern or a succession of the same pattern. From Eq. (1) and Eq. (2) two successive digital edges with same complexity (say n) cannot form a right or left strict factor of a pattern with same complexity. Thus complexities of $(R_i)_{1\leq i\leq n}$ and $(L_i)_{1\leq i\leq n}$ are decreasing when i increases. Moreover to fullfil convexity properties, slopes of edges are decreasing from R_n to L_n.

We now build $(R_i)_{1\leq i\leq n}$ when n is odd (say $n = 2i+1$). From Lemma B.1, R_1 has a complexity that equals $2i+1$ and R_2 is a right strict factor of a pattern whose complexity equals $2i$. Applying Lemma B.1 brings R_2 with a complexity of $2i-1$. Applying the same reasoning recursively brings other edges as shown on Table 1. Lemma B.1 also bring requirements with decreasing slopes and maximality in complexity of factors.

Constructions for the three other cases are given in Tables 2, 3, 4 and follow the same reasoning. To satisfy full decomposition each $(u_k)_{1\leq n}$ has to be equal or greater than 2. If this condition is not meet for some k, than steps associated with it (e.g. any factors containing $u_k - r_j$ or $u_k - lj$ as powers of some pattern) are skipped.

Table 1. Constructions of $(R_i)_{1\leq i\leq n}$ when $n = 2i+1$.

R_1	$E(z_{2i})^{u_{2i+1}-r_1}E(z_{2i-1})$
R_2	$E(z_{2i-1})^{u_{2i}-r_2}$
R_3	$E(z_{2i-2})^{u_{2i-1}-r_3}E(z_{2i-3})$
R_4	$E(z_{2i-3})^{u_{2i-2}-r_4}$
$\vdots$	$\vdots$
R_{2j}	$E(z_{2i+1-2j})^{u_{2i+2-2j}-r_{2j}}$
R_{2j+1}	$E(z_{2i-2j})^{u_{2i+1-2j}-r_{2j+1}}E(z_{2i-1-2j})$
$\vdots$	$\vdots$
R_{2i+1}	$0^{u_1-r_{2i+1}}1$

Table 2. Constructions of $(L_i)_{1\leq i\leq n}$ when $n = 2i+1$.

L_1	$E(z_{2i})^{u_{2i+1}-l_1}$
L_2	$E(z_{2i-2})E(z_{2i-1})^{u_{2i}-l_2}$
L_3	$E(z_{2i-2})^{u_{2i-1}-l_3}$
L_4	$E(z_{2i-4})E(z_{2i-3})^{u_{2i-2}-l_4}$
$\vdots$	$\vdots$
L_{2j}	$E(z_{2i-2j})E(z_{2i+1-2j})^{u_{2i+2-2j}-l_{2j}}$
L_{2j+1}	$E(z_{2i-2j})^{u_{2i+1-2j}-l_{2j+1}}$
$\vdots$	$\vdots$
L_{2i+1}	$0^{u_1-l_{2i+1}}$

Table 3. Constructions of $(R_i)_{1\le i\le n}$ when $n = 2i$.

R_1	$E(z_{2i-1})^{u_{2i}-r_1}$
R_2	$E(z_{2i-2})^{u_{2i-1}-r_2}E(z_{2i-3})$
R_3	$E(z_{2i-3})^{u_{2i-2}-r_3}$
R_4	$E(z_{2i-4})^{u_{2i-3}-r_4}E(z_{2i-5})$
$\vdots$	$\vdots$
R_{2j}	$E(z_{2i-2j})^{u_{2i+1-2j}-r_{2j}}E(z_{2i-1-2j})$
R_{2j+1}	$E(z_{2i-1-2j})^{u_{2i-2j}-r_{2j+1}}$
$\vdots$	$\vdots$
R_{2i}	$0^{u_1-r_{2i}}1$

Table 4. Constructions of $(L_i)_{1\le i\le n}$ when $n = 2i$.

L_1	$E(z_{2i-2})E(z_{2i-1})^{u_{2i}-l_1}$
L_2	$E(z_{2i-2})^{u_{2i-1}-l_2}$
L_3	$E(z_{2i-4})E(z_{2i-3})^{u_{2i-2}-l_3}$
L_4	$E(z_{2i-4})^{u_{2i-3}-l_4}$
$\vdots$	$\vdots$
L_{2j}	$E(z_{2i-2j})^{u_{2i+1-2j}-l_{2j}}$
L_{2j+1}	$E(z_{2i-2-2j})E(z_{2i-1-2j})^{u_{2i-2j}-l_{2j+1}}$
$\vdots$	$\vdots$
L_{2i}	$0^{u_1-l_{2i}}$