Integral based Curvature Estimators in Digital Geometry

David Coeurjolly[1] Jacques-Olivier Lachaud[2] Jérémy Levallois[1,2]

[1]Université de Lyon, CNRS
INSA-Lyon, LIRIS, UMR5205, F-69621, France

[2]Université de Savoie, CNRS
LAMA, UMR5127, F-73776, France

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Context

Differential quantities. . .

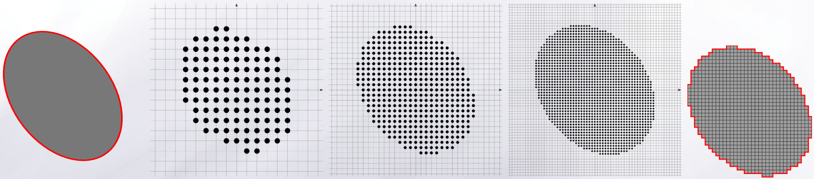
- for shape analysis, shape matching, . . .
- for mathematical modeling of deformable objects (DIGITALSNOW project)

How to make an estimator ?

- Experimental analysis of approximation errors on shapes with known Euclidean values
- Formal proof of convergence
- Computational cost & timing

$\Rightarrow$ Multigrid convergence framework

Multigrid Convergence Framework



Let us consider a **family** $\mathbb{X}$ of smooth and compact subsets of $\mathbb{R}^d$. We denote **shape** X as $X \in \mathbb{X}$, and $\mathtt{D}_h(X)$ the **digitization** of X in a $d-$dimensional grid of resolution h. More precisely, we consider classical Gauss digitization defined as

$$\mathtt{D}_h(X) \stackrel{def}{=} \left(\frac{1}{h} \cdot X\right) \cap \mathbb{Z}^d$$

where $\frac{1}{h} \cdot X$ is the uniform scaling of X by factor $\frac{1}{h}$. Furthermore, the set ∂X denotes the **frontier** of X (i.e. its topological boundary). The h**-boundary** $\partial_h X$ is a $d-1$-dimensional subset of $\mathbb{R}^d$, which is close to ∂X.

Multigrid convergence for local geometric quantities

Definition

A local discrete geometric estimator $\hat{E}$ of some geometric quantity E is ***multigrid convergent*** for the family $\mathbb{X}$ if and only if, for any $X \in \mathbb{X}$, there exists a grid step $h_X > 0$ such that the estimate $\hat{E}(\mathtt{D}_h(X), \hat{x}, h)$ is defined for all $\hat{x} \in \partial_h X$ with $0 < h < h_X$, and for any $x \in \partial X$,

$$\forall \hat{x} \in \partial_h X \text{ with } \|\hat{x} - x\|_\infty \leq h, |\hat{E}(\mathtt{D}_h(X), \hat{x}, h) - E(X, x)| \leq \tau_{X,x}(h),$$

where $\tau_{X,x} : \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ has null limit at 0. This function defines the **speed of convergence** of $\hat{E}$ toward E at point x of X. The convergence is ***uniform*** for X when every $\tau_{X,x}$ is **bounded** from above by a function τ_X independent of $x \in \partial X$ with **null limit at 0**.

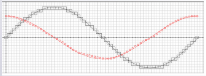
Digital Curvature Estimators

Experimentally convergent in 2D

- MDCA estimator [Roussillon, T. and Lachaud, J.O., 2011]
  Uses the most centered maximal Digital Circular Arc (DCA) to estimate the radius of the osculating circle.

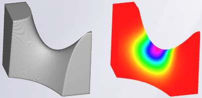
Theoretically & Experimentally convergent in 2D

- BC curvature estimator [Esbelin, H.A. and Malgouyres, R., 2009]
  convergence speed in $O(h^{\frac{4}{9}})$



Non convergent in 3D

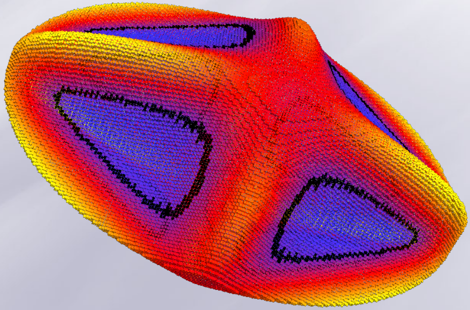
- Curvature estimation for digital surfaces based convolutions [Fourey, S. and Malgouyres, R., 2008]



Main contribution

Digital curvature estimators:

- defined in both 2D and 3D
- easy to implement
- multigrid convergence is theoretically proved with an uniform convergence speed in $O(h^{\frac{1}{3}})$
- experimental validation of multigrid convergence



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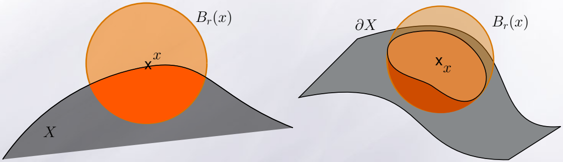
Integration based surface feature



Definition

Given $X \in \mathbb{X}$ and a radius $r \in \mathbb{R}^{+*}$, the volumetric integral $V_r(x)$ at $x \in \partial X$ is given by

$$V_r(x) \stackrel{def}{=} \int_{B_r(x)} \chi(p)dp$$

where $B_r(x)$ is the Euclidean ball (kernel) with radius r and center x and $\chi(p)$ the characteristic function of X. In dimension 2, we simply denote $A_r(x)$ such quantity.

Curvature information with Integration

Lemma [Pottmann2009]

For a sufficiently smooth shape X in $\mathbb{R}^2$ $x \in \partial X$, we have

$$A_r(x) = \frac{\pi}{2}r^2 - \frac{\kappa(X,x)}{3}r^3 + O(r^4)$$

where $\kappa(X,x)$ is the curvature of ∂X at x.
For a sufficiently smooth shape X in $\mathbb{R}^3$ and $x \in \partial X$, we have

$$V_r(x) = \frac{2\pi}{3}r^3 - \frac{\pi H(X,x)}{4}r^4 + O(r^5)$$

where $H(X,x)$ is the mean curvature of ∂X at x.

Local estimators $\tilde{\kappa}_r(x)$ and $\tilde{H}_r(x)$

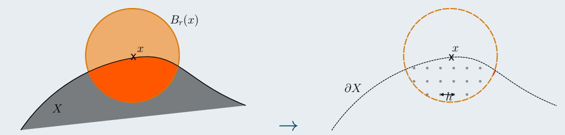
$$\tilde{\kappa}_r(X,x) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}, \quad \tilde{H}_r(X,x) \stackrel{def}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$$

Then:

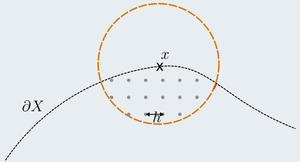
$$\tilde{\kappa}_r(X,x) = \kappa(X,x) + O(r), \quad \tilde{H}_r(X,x) = H(X,x) + O(r)$$

Plan

Proof process

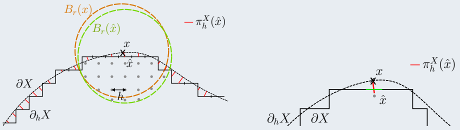


$$A_r(x) \to \widehat{\mathrm{Area}}(\mathtt{D}_h(B_r(x) \cap X), h)$$



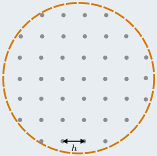
+ [Pottmann2009]

Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), x, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), x, h)$



Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), \hat{x}, h)$

Step 1a - Area/Volume estimation



Given digital shapes $Z \subset \mathbb{Z}^2$, the discrete area estimator by counting at step h are defined:

$$\widehat{\mathrm{Area}}(Z,h) \stackrel{def}{=} h^2 \mathrm{Card}(Z)$$

Given digital shapes $Z' \subset \mathbb{Z}^3$, the discrete area estimator by counting at step h are defined:

$$\widehat{\mathrm{Vol}}(Z',h) \stackrel{def}{=} h^3 \mathrm{Card}(Z')$$

If $Z = \mathtt{D}_h(X)$:

$$\widehat{\mathrm{Area}}(\mathtt{D}_h(X),h) = \mathrm{Area}(X) + O(h^\beta)$$

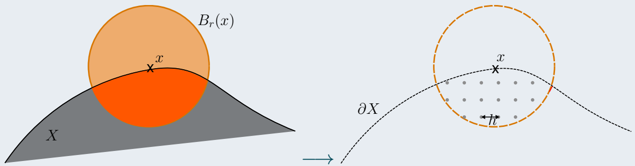
If $Z' = \mathtt{D}_h(X')$:

$$\widehat{\mathrm{Vol}}(\mathtt{D}_h(X'),h) = \mathrm{Vol}(X') + O(h^\gamma)$$

- $\beta = 1$ in general convex case [Gauss]
- $\beta = \frac{15}{11} - \epsilon$ when the shape boundary is C^3 with non-zero curvature [Huxley1990]

- $\gamma = 1$ in general convex case [Kratzel1988]
- $\gamma = \frac{243}{158}$ for smoother boundary [Guo2010]

Step 1b - Convergence of volumetric integral estimation



Lemma

$$|\widehat{\text{Area}}(\mathtt{D}_h(B_r(x) \cap X), h) - A_r(x)| \leq K_1'(r)\, h^\beta$$

$$\widehat{\text{Area}}(\mathtt{D}_h(B_r(x) \cap X), h) = r^2 \widehat{\text{Area}}(\mathtt{D}_{h/r}(B_1(\frac{1}{r} \cdot x) \cap \frac{1}{r} \cdot X), h/r)$$

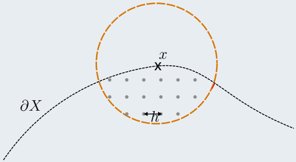
$$|\widehat{\text{Area}}(\mathtt{D}_h(B_r(x) \cap X), h) - A_r(x)| \leq K_1 h^\beta r^{2-\beta}$$

with $1 \leq \beta < 2$.

Proof hints

- Rescale shapes Z to only a unit ball B_1
- True for any point of $\mathbb{R}^2$

Step 2a - Integral digital curvature estimators



+ [Pottmann2009]

Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), x, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), x, h)$

Reminder:

$$\tilde{\kappa}_r(X,x) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}, \quad \tilde{H}_r(X,x) \stackrel{def}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$$

Then, we can define:
Integral digital curvature estimator $\hat{\kappa}_r$ of a digital shape Z at point $x \in \mathbb{R}^2$ and step h:

$$\forall 0 < h < r, \hat{\kappa}_r(Z,x,h) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3\widehat{\mathrm{Area}}(B_{r/h}(\frac{1}{h} \cdot x) \cap Z, h)}{r^3}.$$

Integral digital curvature estimator $\hat{H}_r$ of a digital shape Z' at point $x \in \mathbb{R}^3$ and step h:

$$\forall 0 < h < r, \hat{H}_r(Z',x,h) \stackrel{def}{=} \frac{8}{3r} - \frac{4\widehat{\mathrm{Vol}}(B_{r/h}(\frac{1}{h} \cdot x) \cap Z', h)}{\pi r^4}.$$

Step 2b - Convergence when $x \in \partial_h X$

Rem: $\tilde{\kappa}_r(X,x) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}$ and $\tilde{\kappa}_r(X,x) = \kappa(X,x) + O(r)$

$$|\hat{\kappa}_r(\mathtt{D}_h(X),x,h) - \kappa(X,x)| \leq O(r) + 3K_1 \frac{h^\beta}{r^{1+\beta}}$$

Let us set $\boldsymbol{r = kh^\alpha}$, then

$$|\hat{\kappa}_r(\mathtt{D}_h(X),x,h) - \kappa(X,x)| \leq K_2 k h^{\boldsymbol{\alpha}} + \frac{3K_1}{k^{1+\beta}} h^{\boldsymbol{\beta - \alpha(1+\beta)}}.$$

Theorem (Convergence of digital curvature estimator $\hat{\kappa}_r$ along ∂X)

Let X be some convex shape of $\mathbb{R}^2$, with at least C^2-boundary and bounded curvature. Then $\exists h_0, K_1, K_2$, such that

$$\forall h < h_0, r = k_m h^{\alpha_m}, |\hat{\kappa}_r(\mathtt{D}_h(X),x,h) - \kappa(X,x)| \leq K h^{\alpha_m},$$

where $\alpha_m = \frac{\beta}{2+\beta}$, $k_m = ((1+\beta)K_1/K_2)^{\frac{1}{2+\beta}}$, $K = K_2 k_m + 3K_1/k_m^{1+\beta}$.

$\alpha_m = \frac{15}{37} - \epsilon \approx 0.405$ when the boundary of X is C^3 without null curvature points,
$\alpha_m = \frac{1}{3}$ otherwise.

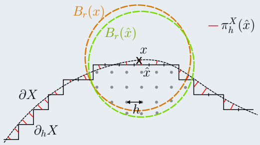
Step 3a - Convergence of $\hat{x} \in \partial_h X$

$\hat{x}$ lies on the normal direction to ∂X at x, at a distance $\delta \stackrel{def}{=} \|x - \hat{x}\|_2 \stackrel{def}{=} h^{\alpha'}$

$$|A_r(\hat{x}) - A_r(x)| = 2r\delta(1 + O(r^2) + O(\delta)) \text{ [Pottmann2009]}$$

$$|\widehat{\mathrm{Area}}(\mathtt{D}_h(B_r(\hat{x}) \cap X), h) - A_r(x)| \leq K_1 h^\beta r^{2-\beta} + 2r\delta(1 + O(r^2) + O(\delta))$$

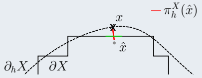
$$|\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h) - \kappa(X, x)| \leq O(r) + 3K_1 \frac{h^\beta}{r^{1+\beta}} + \frac{6\delta}{r^2}(1 + O(r^2) + O(\delta))$$



Back-projection π_h^X [Lachaud2006]

Let $\hat{x} \in \partial_h X$ and set $x_0 = \pi_h^X(\hat{x})$.

$$\|\hat{x} - x_0\|_\infty \leq \frac{\sqrt{2}}{2} h < h$$



Step 3b - Convergence when $\hat{x} \in \partial_h X$

Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), \hat{x}, h)$

Theorem (Uniform convergence of curvature estimator $\hat{\kappa}_r$ along $\partial_h X$)

Let X be some convex shape of $\mathbb{R}^2$, with at least C^3-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0$, setting $r = kh^{\alpha}$, $\delta = O(h^{\alpha'})$ where $\alpha \geq 1$, we have

$$\forall x \in \partial X, \forall \hat{x} \in \partial_h X, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$

$$\begin{aligned} |\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h) - \kappa(X, x)| \leq \; & \boldsymbol{O(h^{\alpha})} \\ & + \boldsymbol{O(h^{\beta - \alpha(1+\beta)})} \\ & + O(h^{\alpha' - 2\alpha}) + O(h^{\alpha'}) + O(h^{2\alpha' - 2\alpha}) \end{aligned}$$

Finding the best possible parameter $\alpha_m = \frac{\beta}{1+\beta}$ if $\alpha' \geq \frac{3\beta}{1+\beta}$, otherwise $\alpha_m = \frac{\alpha'}{3}$

$$\left.\begin{array}{l} \text{[Gauss]} \Rightarrow \beta = 1 \\ \text{[Lachaud2006]} \Rightarrow \alpha' = 1 \end{array}\right\} \Rightarrow \alpha_m = \frac{1}{3} \Rightarrow |\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h) - \kappa(X, x)| \leq K h^{\frac{1}{3}}$$

Mean curvature in 3D

In the same way, we have in 3D :

Theorem (Uniform convergence of $\hat{H}_r$ along $\partial_h X$)

Let X' be some convex shape of $\mathbb{R}^3$, with at least C^2-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0$ $\forall x \in \partial X', \forall \hat{x} \in \partial_h X', \|\hat{x} - x\|_\infty \leq h$

$$\forall 0 < h < r, \hat{H}_r(\partial_h X', \hat{x}, h) \stackrel{def}{=} \frac{8}{3r} - \frac{4\widehat{\mathrm{Vol}}(B_{r/h}(\hat{x}) \cap \partial_h X', h)}{\pi r^4}.$$

Setting $r = k' h^{\frac{1}{3}}$, we have

$$|\hat{H}_r(\mathtt{D}_h(X'), \hat{x}, h) - H(X', x)| \leq K' h^{\frac{1}{3}}.$$

Gaussian Curvature on Digital Surface

Main idea

- Instead of computing the volume of $Y = B_r(x) \cap X$, we compute its covariance matrix

$$J(Y) \stackrel{def}{=} \int_Y (p - \overline{Y})(p - \overline{Y})^T dp = \int_Y pp^T dp - \mathrm{Vol}(Y)\overline{Y}\overline{Y}^T,$$

  where $\overline{Y}$ denotes the centroid of Y.
- Principal curvatures k^1 and k^2 at x are related to eigenvalues of $J(Y)$

$$\lambda_1 = \frac{2\pi}{15}R^5 - \frac{\pi}{48}(3\kappa^1(X,x) + \kappa^2(X,x))R^6 + O(R^7)$$
$$\lambda_2 = \frac{2\pi}{15}R^5 - \frac{\pi}{48}(\kappa^1(X,x) + 3\kappa^2(X,x))R^6 + O(R^7)$$
$$\lambda_3 = \frac{19\pi}{480}R^5 - \frac{9\pi}{512}(\kappa^1(X,x) + \kappa^2(X,x))R^6 + O(R^7)$$

$\Rightarrow$ From convergence of high order moment estimator and specific error propagation analysis, convergence proofs can be designed for κ^1 and κ^2

Plan

Experimentation

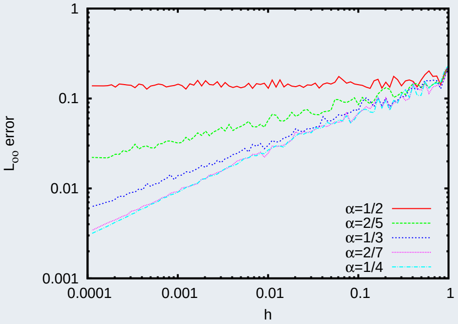
Experimental Settings

- Family of Euclidean shapes (implicit, parametric) with *exact* curvature information
- Digitization process at resolution h
- Error metrics
  - Worst-case l_∞ error: maximum of absolute difference value $\max_{\hat{x} \in \partial_h X, x \in \partial X} (|\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h) - \kappa(X, x)|)$
  - Quadratic l_2 error

Validation of α parameter

Convolution kernel radius

$r = kh^{\alpha}$

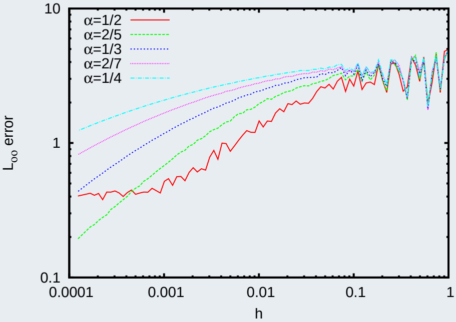


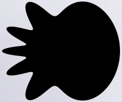
α	Observed convergence speed
$1/2$	$0(h^{0.024})$
$2/5$	$0(h^{0.24})$
$1/3$	$0(h^{0.38})$
$2/7$	$0(h^{0.41})$
$1/4$	$0(h^{0.44})$

Validation of α parameter

Convolution kernel radius

$r = kh^{\alpha}$

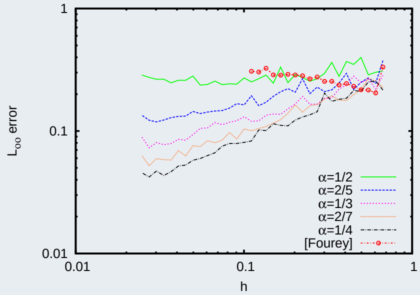


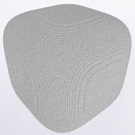


Validation of α parameter

Convolution kernel radius

$r = kh^{\alpha}$

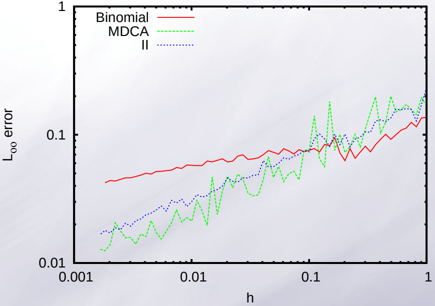


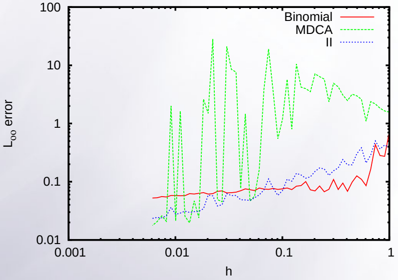


α	Observed convergence speed
$1/2$	$0(h^{0.088})$
$2/5$	$0(h^{0.29})$
$1/3$	$0(h^{0.42})$
$2/7$	$0(h^{0.49})$
$1/4$	$0(h^{0.57})$

Implicit surface is $x^2 + y^2 + z^2 - 25 = 0$

Comparison

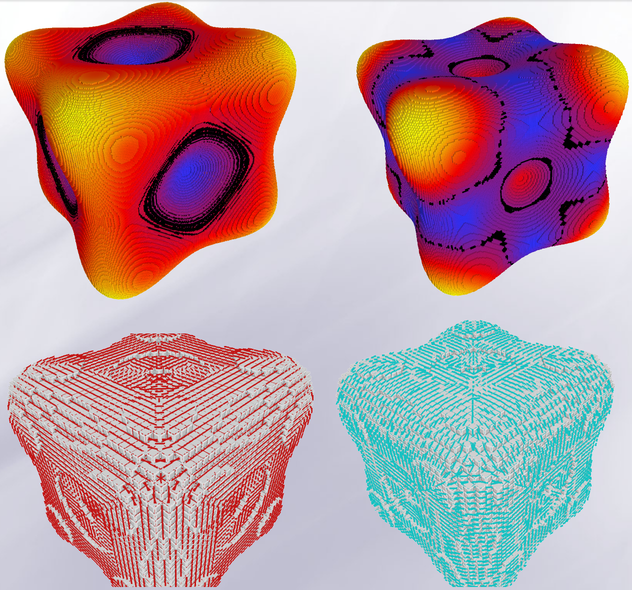




Estimator	Observed convergence speed
II	$O(h^{0.38})$
MDCA	$O(h^{0.42})$
BC	$O(h^{0.154})$

- Kanungo noise
- noise parameter: 0.5 ($\in]0,1[$)

3D curvature estimation

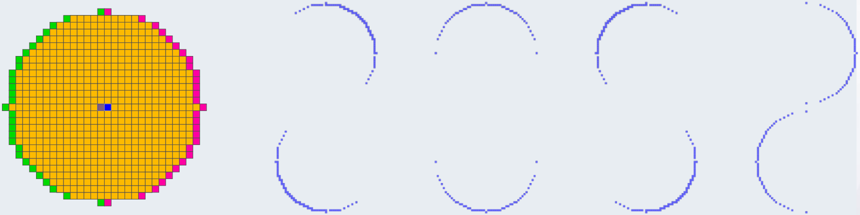


Optimizations with convolution

DGtal

- Open-source C++ library
- Geometry structures, algorithm & tools for digital data
- `http://libdgtal.org`

Optimization with displacement masks



Complexity:

- without optimization: $O((r/h)^d)$
- with optimization: $O((r/h)^{d-1})$

Plan

Conclusion & Future work

- Integral Invariant is perfect for digital geometry
- A unique estimator for both 2D and 3D
- Easy to implement
- Fast computation with masks
- Convergent with a least a uniform convergence speed in $O(h^{\frac{1}{3}})$
- Needs a parameter (r for the kernel radius), as BC .. but we have better results

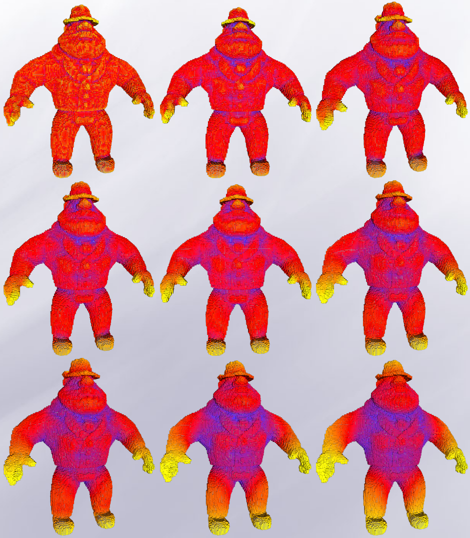
For an Ellipse, and the same L_∞ error between BC and our estimator (**0.0461726**), we have :

	BC	II
h	0.00302755	0.0301974
time (in ms.)	421945	350
mask size	**91336**	**8349**
mask size (optim.)		**8*145**

Future work

- Theoretical demonstration of Gaussian curvature convergence
- Comportment with noised data
- Scale-Space analysis

Choice of radius



More information

For more information (l_2 graphs, high-res images, scripts, etc.):

http://liris.cnrs.fr/jeremy.levallois/Papers/DGCI2013/