Combinatorial view of digital convexity*

S. Brlek, J.-O. Lachaud, and X. Provençal

Laboratoire de Combinatoire et d'Informatique Mathématique,
Université du Québec à Montréal,
CP 8888 Succ. Centre-ville, Montréal (QC) Canada H3C 3P8
`brlek.srecko@uqam.ca, xavierprovencal@gmail.com`

Laboratoire de Mathématiques, UMR 5127 CNRS,
Université de Savoie, 73376 Le Bourget du Lac, France
`jacques-olivier.lachaud@univ-savoie.fr`

Abstract. The notion of convexity translates non-trivially from Euclidean geometry to discrete geometry, and detecting if a discrete region of the plane is convex requires analysis. In this paper we study digital convexity from the combinatorics on words point of view, and provide a fast optimal algorithm checking digital convexity of polyominoes coded by the contour word. The result is based on the Lyndon factorization of the contour word, and the recognition of Christoffel factors that are approximations of digital lines.

Keywords: Digital Convexity, Lyndon words, Christoffel words

1 Introduction

In Euclidean geometry, a given region R is said to be *convex* if and only if for any pair of points p_1, p_2 in R the line segment joining p_1 to p_2 is completely included in R. In discrete geometry on square grids, the notion does not translate trivially, since the only convex (in the Euclidean sense) regions are rectangles. Many attempts have been made to fill the gap, and a first definition of discrete convexity based on discretisation of continuous object came from Sklansky [1] and Minsky and Papert [2]. Later, Kim [3, 4] then Kim and Rosenfeld [5] provided several equivalent characterizations of discrete convex sets, and finally Chaudhuri and Rosenfeld [6] proposed a new definition of digital convexity based this time on the notion of digital line segments (see [7] for a review of digital straightness).

Given a finite subset S of $\mathbb{Z}^2$ its *convex hull* is defined as the intersection of all Euclidean convex sets containing S. Of course all the vertices of the convex hull are points from S. Therefore, throughout this work, a polyomino P (which is the interior of a closed non-intersecting grid path of $\mathbb{Z}^2$) is called *convex* if and only if its convex hull contains no integer point outside P. Debled-Rennesson et al. [8] already provided a linear time algorithm deciding if a given polyomino is convex. Their method uses arithmetical tools to compute series of digital line

* with the support of NSERC (Canada)

segments of decreasing slope: optimal time is achieved with a moving digital straight line recognition algorithm [9, 10]

Recently, Brlek et al. looked at discrete geometry from the combinatorics of words point of view, showing for instance how the discrete Green theorem provides a series of optimal algorithms for diverse statistics on polyominoes [11, 12]. This method is extended to study minimal moment of inertia polyominoes in [13]. This approach also gave an elementary proof and a generalization [14, 15] of a result of Daurat and Nivat [16] relating salient and reentrant points in discrete sets. Some geometric properties of the contour of polyominoes may be found in [17, 18]. Recently it was successfully used to provide an optimal algorithm for recognizing tiles that tile the plane by translation [19, 20]. It is worth noting that the study of these objects goes back to Bernouilli, Markov, Thue and Morse (see Lothaire's books [21–23] for an exhaustive bibliographic account) and as suggested in the recent survey of Klette and Rosenfeld [7] the bridge between discrete geometry and combinatorics on words will benefit to both areas.

Here we study the problem of deciding whether a polyomino coded by its contour word, also called Freeman chain code, is convex or not. To achieve this we use well known tools of combinatorics on words. The first is the existence of a unique Lyndon factorization, and its optimal computation by the linear algorithm of Duval [24]. The second concerns the Christoffel words, a class of finite factors of Sturmian words, that are discrete approximations of straight lines. After recalling the combinatorial background and basic properties, we propose another linear time algorithm deciding convexity of polyominoes. This new purely discrete algorithm is much simpler to implement. Some experiments revealed that it is 10 times faster than previous linear algorithms. Furthermore, one of its main interests lies in the explicit link between combinatorics on words and discrete geometry. Since our method does not rely on geometric and vector computations, it also shows that digital convexity is much more fundamental and abstract property than general convexity.

2 Preliminaries

A word w is a finite sequence of letters $w_1 w_2 \cdots w_n$ on a finite alphabet Σ, that is a function $w : [1..n] \longrightarrow \Sigma$, and $|w| = n$ is its *length*. Consistently its number of a letters, for $a \in \Sigma$, is denoted $|w|_a$. The set of words of length n is denoted Σ^n and the set of all finite words is Σ^*, the free monoid on Σ. The empty word is denoted ε and by convention $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. The k-th power of word w is defined by $w^k = w^{k-1} \cdot w$ with the convention that $w^0 = \varepsilon$. A word is said *primitive* when it is not the power of a nonempty word. A *period* of a word w is a number p such that $w_i = w_{i+p}$, for all $1 \leq i \leq |w| - p$.

Given a total order $<$ on Σ, the *lexicographic ordering* extends this order to words on Σ by using the following rule :

$$w < w' \text{ if either } \begin{array}{l} \text{(i) } w' \in w\Sigma^+, \\ \text{(ii) } w = uav \text{ and } w' = ubv' \text{ with } a < b, a, b \in \Sigma, u \in \Sigma^*. \end{array}$$

Two words w, w' on the alphabet Σ are said to be *conjugate*, written $w \equiv w'$, if there exist u, v such that $w = uv$ and $w' = vu$. The conjugacy class of a word is defined as the set of all its conjugates and is equivalent to the set of all circular permutations of its letters.

Let w be a finite word over the alphabet $\{0, 1\}$. We denote by $\overrightarrow{w}$ the vector $(|w|_0, |w|_1)$. For any word w, the partial function $\phi_w : \mathbb{N} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ associates to any integer j, $0 \leq j \leq |w|$, the vector $\phi_w(j) = \overrightarrow{w_1 w_2 \cdots w_j}$. In other words, this map draws the word as a 4-connected path in the plane starting from the origin, going *right* for a letter 0 and *up* for a letter 1. This extends naturally to

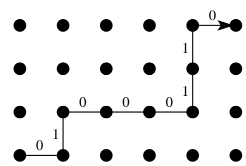


Fig. 1. Path encoded by the word $w = 01000110$.

more general paths by using the four letter alphabet $\Sigma = \{0, 1, \overline{0}, \overline{1}\}$, associating to the letter $\overline{0}$ a *left* step and to $\overline{1}$ a *down* step. This notation allows to code the border of any polyomino by a 4-letter word known as the Freeman chain code.

The lexicographic order $<$ on points of $\mathbb{R}^2$ or $\mathbb{Z}^2$ is such that $(x, y) < (x', y')$ when either $x < x'$ or $x = x'$ and $y < y'$. The *convex hull* of a finite set S of points in $\mathbb{R}^2$ is the smallest convex set containing these points and is denoted by $\mathrm{Conv}(S)$. S being finite, it is clearly a polygon in the plane whose vertices are elements of S. The *upper convex hull* of S, denoted by $\mathrm{Conv}^+(S)$, is the clockwise oriented sequence of consecutive edges of $\mathrm{Conv}(S)$ starting from the lowest vertex and ending on the highest vertex. The *lower convex hull* of S, denoted by $\mathrm{Conv}^-(S)$, is the clockwise oriented sequence of consecutive edges of $\mathrm{Conv}(S)$ starting from the highest vertex and ending on the lowest vertex.

3 Combinatorics on words

Combinatorics on words has imposed itself as a powerful tool for the study of large number of discrete, linear, non-commutative objects. Such objects appears in almost any branches of mathematics and discrete geometry is not an exception. Traditionally, discrete geometry works on characterization and recognition of discrete objects using arithmetic approach or computational geometry. However combinatorics on words provide mathematical tools and efficient algorithms to address this problem as already mentioned. Lothaire's books [21–23] constitute the reference for presenting a unified view on combinatorics on words and many of its applications.

3.1 Lyndon words

Introduced as *standard lexicographic sequences* by Lyndon in 1954, Lyndon words have several characterizations (see [21]). We shall define them as words being strictly smaller than any of their circular permutations.

Definition 1. *A Lyndon word $l \in \Sigma^+$ is a word such that $l = uv$ with $u, v \in \Sigma^+$ implies that $l < vu$.*

Note that Lyndon words are always primitive. An important result about Lyndon words is that any word w admits a factorization as a sequence of decreasing Lyndon words :

$$w = l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k} \tag{1}$$

where $n_1, n_2, \ldots, n_k \geq 1$ and $l_1 > l_2 > \cdots > l_k$ are Lyndon words (see Lothaire [21] Theorem 5.1.1). Such a factorization is unique and a linear time algorithm to compute it is given in Section 5.

3.2 Christoffel words

Introduced by Christoffel [25] in 1875 and reinvestigated recently by Borel and Laubie [26] who pointed out some of their geometrical properties, Christoffel words reveal an important link between combinatorics on words and discrete geometry.

This first definition of Christoffel word, borrowed from Berstel and de Luca [27], highlights their geometrical properties and helps to understand the main result of this work stated in Proposition 2. Let $\Sigma = \{0, 1\}$. The *slope* of a word is a map

$$\rho : \Sigma^* \to \mathbb{Q} \cup \{\infty\}$$

defined by

$$\rho(\epsilon) = 1, \quad \rho(w) = |w|_1/|w|_0, \quad \text{for} \quad w \neq \epsilon.$$

It is assumed that $1/0 = \infty$. It corresponds to the slope of the straight line joining the first and the last point of the path coded by w. For each k, $1 \leq k \leq |w|$, we define the set

$$\delta_k(w) = \{u \in \Sigma^k | \rho(u) \leq \rho(w)\},$$

of words of length k whose slope is not greater than the slope of w. The quantity

$$\mu_k(w) = \max\{\rho(u) | u \in \delta_k(w)\}$$

is used to define Christoffel words (see Figure 2).

Definition 2. *A word w is a Christoffel word if for any prefix v of w one has $\rho(v) = \mu_{|v|}(w)$.*

A direct consequence of this definition is that given a Christoffel word $u^r = v^s$ for some $r, s \geq 1$, both words u and v are also Christoffel words. From an arithmetical point of view, a Christoffel word is a connected subset of a standard line joining upper leaning points (see Reveillès [28]). The following properties of Christoffel words are taken from Borel and Laubie [26].

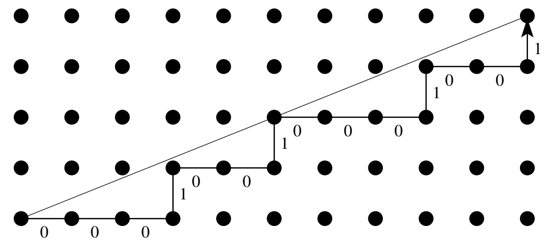


Fig. 2. The path coded by the Christoffel word $w = 00010010001001$ staying right under the straight line of slope $r = \frac{2}{5}$.

Property 1. All primitive Christoffel words are Lyndon words.

Property 2. Given c_1 and c_2 two Christoffel words, $c_1 < c_2$ iff $\rho(c_1) < \rho(c_2)$.

Property 3. Given $r \in \mathbb{Q}^+ \cup \{\infty\}$, let F_r be the set of words w on the alphabet $\{0,1\}$ such that $\rho(v) \leq r$ for all non-empty prefix v of w. F_r correspond to the words coding paths, starting from the origin, that stay below the Euclidean straight line of slope r. Among these paths, those being the closest ones to the line and having their last point located on it are Christoffel words.

Originally Christoffel [25] defined these words as follows. Given $k < n$ two relatively prime numbers, a (primitive) Christoffel word $w = w_1 w_2 \ldots w_n$ is defined by :

$$w_i = \begin{cases} 0 & \text{if } r_{i-1} < r_i, \\ 1 & \text{if } r_{i-1} > r_i, \end{cases}$$

where r_i is the remainder of $(i\,k) \mod n$.

In [27] Berstel and de Luca provided an alternative characterization of primitive Christoffel words. Let CP be the set of primitive Christoffel words, PAL the set of palindromes and PER the set of words w having two periods p and q such that $|w| = p + q - 2$. The following relations hold :

$$CP = \big(\{0,1\} \cup 0 \cdot PER \cdot 1\big) \subset \big(\{0,1\} \cup 0 \cdot PAL \cdot 1\big).$$

These properties of Christoffel words are essential for deciding if a given word is Christoffel or not.

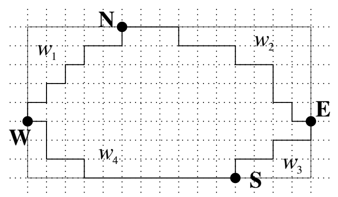
4 Digital convexity

There are several (more or less) equivalent definitions of digital convexity, depending on whether or not one asks the digital set to be connected. We say that a finite 4-connected subset S of $\mathbb{Z}^2$ is *digitally convex* if it is the Gauss digitization of a convex subset X of the plane, i.e. $S = \mathrm{Conv}(X) \cap \mathbb{Z}^2$.

The *border* $\mathrm{Bd}(S)$ of S is the 4-connected path that follows clockwise the pixels of S that are 8-adjacent to some pixel not in S. This path is a word of $\{0, 1, \overline{0}, \overline{1}\}^*$, starting by convention from the lowest point and in clockwise order.

Definition 3. *A word w is said to be* digitally convex *if it is conjugate to the word coding the border of some finite 4-connected digitally convex subset of $\mathbb{Z}^2$.*

Note that implicitely, a digitally convex word is necessarily closed. Now, every closed path coding the boundary of a region is contained in a smallest rectangle such that its contour word w may be factorized as follows. Four extremal points are defined by their coordinates:



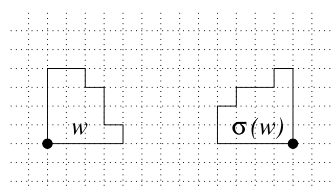
W is the lowest on the Left side;
N is the leftmost on the Top side;
E is the highest on the Right side;
S is the rightmost on the Bottom side;
So that $w \equiv w_1 w_2 w_3 w_4$.

This factorization is called the *standard decomposition*. We say that a word w_1 in $\{0,1\}^*$ is *NW-convex* iff there are no integer points between the upper convex hull of the points $\{\phi_w(j)\}_{j=1\ldots|w|}$ and the path w.

Define the counterclockwise $\pi/2$ circular rotation by

$$\sigma : (0, 1, \overline{0}, \overline{1}) \longmapsto (1, \overline{0}, \overline{1}, 0).$$

Then we have w_2 in $\{0,\overline{1}\}^*$ is *NE-convex* iff $\sigma(w_2)$ is NW-convex, and more



$$w = 111100\overline{1}0\overline{11}0\overline{10000},$$
$$\sigma(w) = \overline{0000}11010010\overline{1111}.$$

generally, in the factorization above

$$w_i \quad \text{is convex} \iff \sigma^{i-1}(w_i) \quad \text{is NW-convex.}$$

Clearly, the convexity of w requires the convexity of each w_i for $i = 1, 2, 3, 4$, and we have the following obvious property.

Proposition 1. *Let $w \equiv w_1 w_2 w_3 w_4$ be the standard decomposition of a polyomino. Then w is digitally convex iff $\sigma^{i-1}(w_i)$ is NW-convex, for all i.*

Let $\mathrm{Alph}(w)$ be the set of letters of w. Observe that if for some i, w_i contains more than 2 letters, that is if $\mathrm{Alph}(\sigma^{i-1}(w_i)) \not\subseteq \{0,1\}$, then w is not digitally convex.

We are now in position to state the main result which is used in Section 5 to design an efficient algorithm for deciding if a word is convex.

Proposition 2. *A word v is NW-convex iff its unique Lyndon factorization $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is such that all l_i are primitive Christoffel words.*

In order to prove Proposition 2, we first need the following lemma.

Lemma 1. *Let $v \in \{0,1\}^*$ be a word coding an NW-convex path and let e be one of the edges of its convex hull. The factor u of v corresponding to the segment of the path determined by e is a Christoffel word.*

This is a direct consequence of Property 3 since both the starting and ending points of an edge of the convex hull of a discrete figure are necessarily part of its border. We may now proceed to the proof of Proposition 2.

Proof. Let v be a word coding an NW-convex path and let the ordered sequence of edges $(e_1, e_2, \ldots, e_k)$ be the border of its convex hull. For each i from 1 to k, let u_i be the factor of v determined by the edge e_i so that $v = u_1 u_2 \cdots u_k$. Let l_i be the unique primitive word such that $u_i = l_i^{n_i}$. By definition of NW-convexity and Lemma 1, u_i is a Christoffel word, which implies that l_i is a primitive Christoffel word. By Property 1, l_i is also a Lyndon word. Now, since $(e_1, e_2, \ldots, e_k)$ is the convex hull of w, it follows that the slope s_i of the edge e_i is greater than the slope s_{i+1} of the edge e_{i+1} leading to the following inequality :

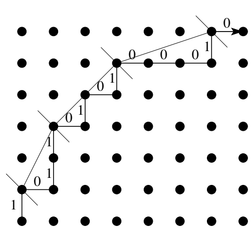
$$\rho(l_i) = \rho(u_i) = s_i > s_{i+1} = \rho(u_{i+1}) = \rho(l_{i+1}).$$

By Property 2 we conclude that $l_i > l_{i+1}$. Thus $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ is the unique factorization w as a decreasing sequence of Lyndon words.

Conversely, let $v \in \{0,1\}^+$ be such that its Lyndon factorization $l_1^{n_1} l_2^{n_2} \cdots l_k^{n_k}$ consists of primitive Christoffel words. For each i from 1 to k, let e_i be the segment joining the starting point of the path coded by $l_i^{n_i}$ to its ending point. We shall show that $(e_1, e_2, \ldots, e_k)$ is the upper convex hull of ϕ_v. Since $l_i^{n_i}$ is a Christoffel word, Property 3 ensures that no integer point is located between the path coded by $l_i^{n_i}$ and the segment e_i and, moreover, the path always stays below the segment. By hypothesis, $l_i > l_{i+1}$. Using the same argument as before we have that the slope of e_i is strictly greater than the slope of e_{i+1}.

We have just built a sequence of edges which is above the path ϕ_v, such that no integer points lies in-between, and with decreasing slopes. $(e_1, e_2, \ldots, e_k)$ is thus the upper convex hull of ϕ_v and v is NW-convex. □

For example, consider the following NW-convex path $v = 1011010100010$.



The Lyndon factorization of v is

$$v = (1)^1 \cdot (011)^1 \cdot (01)^2 \cdot (0001)^1 \cdot (0)^1,$$

where 0, 011, 01, 0001 and 0 are all Christoffel words.

5 Algorithm to check word convexity

We give now a linear time algorithm checking digital NW-convexity for paths encoded on $\{0,1\}$. This is achieved in two steps: first we compute the prefix $l_1^{n_1}$ of the word w using the Fredricksen and Maiorana algorithm [29] (rediscovered by Duval [24]), and then Algorithm 2 below checks that the Lyndon factor $l_1 \in CP$. Iterating this process on all Lyndon factors of w provides the answer whether all l_i are primitive Christoffel words.

Given a word $w \in \Sigma^*$ whose Lyndon factorization is $w = l_1^{n_1} l_2^{n_2} \dots l_k^{n_k}$, the following algorithm, taken from Lothaire's book [23], computes the pair (l_1, n_1).

Algorithm 1 (FirstLyndonFactor)
Input $w \in \Sigma^n$; **Output** (l_1, n_1)

```
(i, j) ← (1, 2)
  while j ≤ n and w_i ≤ w_j do
    If w_i < w_j then
      i ← 1
    else
      i ← i + 1
    end if
    j ← j + 1
  end while
return (w_1 w_2 ··· w_{j−i}, ⌊(j − 1)/(j − i)⌋)
```

Clearly this algorithm is linear in $n_1|l_1|$, and hence the Lyndon factorization of w is computed in linear time with respect to $|w|$. On the other hand, given a primitive word $w \in \{0,1\}^*$, checking whether it is a Christoffel word is also achieved in linear time using the definition from [25]: first, compute $k = |w|_1$ and $n = |w|$; then compute successively $r_1, r_2, \dots, r_{\lceil n/2 \rceil}$ where $r_i = (i\,k) \mod n$ and verify that w_i satisfies the definition. Note that since $CP \setminus \{0,1\} \subset 0PAL1$ the second half of w is checked at the same time by verifying that $w_i = w_{n-i+1}$ for $2 \leq i \leq \lceil n/2 \rceil$. This yields the following algorithm.

Algorithm 2 (IsChristoffelPrimitive)
Input $w \in \Sigma^n$

```
k ← |w|_1; i ← 1; r ← k ;
rejected := not(w_1 = 0 and w_n = 1)
while not(rejected) and i < ⌈n/2⌉ do
  i ← i + 1 ; r' ← r + k  mod n
  If r < r' then
    rejected ← not(0 = w_i and 0 = w_{n−i−1})
  else
    rejected ← not(1 = w_i and 1 = w_{n−i−1})
  end if
  r ← r'
end while
return not(rejected)
```

Combining these two algorithms provides this following algorithm that checks NW-convexity of a given word $w \in \Sigma^*$.

Algorithm 3 (IsNW-Convex)
Input $w \in \Sigma^n$

$index \leftarrow 1$; $rejected \leftarrow false$
while not$(rejected)$ **and** $index \leq n$ **do**
  $(l_1, n_1) \leftarrow$ **FirstLyndonFactor**$(w_{index} w_{index+1} \cdots w_n)$
  $rejected \leftarrow$ **not**(**IsChristoffelPrimitive**(l_1))
  $index \leftarrow index + n_1 |l_1|$
end while
return not$(rejected)$

Equation (1) ensures that $\sum_i |l_i| \leq |w|$ so that this algorithm is linear in the length of the word w.

5.1 The final algorithm

According to Proposition 1, we have to check convexity for each term in the standard decomposition $w \equiv w_1 w_2 w_3 w_4$. Instead of applying the morphism σ to each w_i, which requires a linear pre-procesing, it suffices to implement a more general version of Algorithm 1 and Algorithm 2, with the alphabet and its order relation as a parameter. For that purpose, ordered alhabets are denoted as lists $Alphabet = [\alpha, \beta]$ with $\alpha < \beta$.

The resulting algorithm is the following where we assume that w is the contour of a non-empty polyomino.

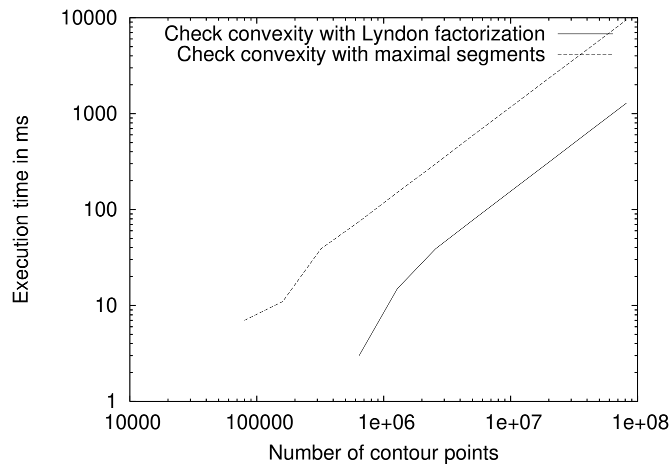
Algorithm 4 (IsConvex)
Input $w \in \Sigma^n$

Compute the standard decomposition $w \equiv w_1 w_2 w_3 w_4$;
$rejected \leftarrow false; i \leftarrow 1; Alphabet \leftarrow [0, 1]$;
while not$(rejected)$ **and** $i \leq 4$ **do**
  $u \leftarrow w_i; k \leftarrow |u|$;
  if $\mathrm{Alph}(u) \subseteq Alphabet$ **then**
    $index \leftarrow 1$;
    while not$(rejected)$ **and** $index \leq k$ **do**
      $(l_1, n_1) \leftarrow$ **FirstLyndonFactor**$([u_{index} u_{index+1} \cdots u_k], Alphabet)$;
      $rejected \leftarrow$ **not**(**IsChristoffelPrimitive**$(l_1), Alphabet)$;
      $index \leftarrow index + n_1 |l_1|$;
    end while
  else
    $rejected \leftarrow true$;
  end if
  $i \leftarrow i + 1; Alphabet \leftarrow [\sigma^{i-1}(0), \sigma^{i-1}(1)]$;
end while
return not$(rejected)$

Remark. For more efficiency, testing that the letters of w_i belong to $\sigma^{i-1}(\{0,1\}^*)$ (Line 4) can be embedded within the algorithm **FirstLyndonFactor** or in the computation of the standard decomposition (Line 0) and returning an exception.

6 Concluding remarks

The implementation of our algorithm was compared to an implementation of that of Debled-Rennesson et al. [8]. The results (see figure below) showed that our technique was 10 times faster than the technique of maximal segments.



This speedup is partially due to the fact that computing maximal segments provides more geometrical informations while testing convexity is simpler. Nevertheless, our algorithm is much simpler conceptually and suggests that the notion of digital convexity might be a more fundamental concept than what is usually perceived. The fact that the combinatorial approach delivers such an elegant algorithm begs for a systematic study of the link between combinatorics on words and discrete geometry. In particular, there exist another characterization of Christoffel words that involve their palindromic structure.

Among the many problems that can be addressed with this new approach we mention the computation of the convex hull. It is also possible to improve algorithm **IsConvex** by merging some computations in one pass instead of calling independent routines. The resulting algorithm is more tricky, but providing a still faster implementation, and its description will appear in third author's PhD dissertation [30].

Acknowledgements. We are grateful to Christophe Reutenauer for his beautiful lectures on Sturmian words during the School on Combinatorics on words held in Montreal in march 2007. Our fruitful discussions led to a better understanding of Christoffel words and inspired the present work.

References

1. J. Sklansky, Recognition of convex blobs, Pattern Recognition 2 (1) (1970) 3–10.
2. M. Minsky, S. Papert, Perceptrons, 2nd Edition, MIT Press, 1988.
3. C. Kim, On the cellular convexity of complexes, Pattern Analysis and Machine Intelligence 3 (6) (1981) 617–625.
4. C. Kim, Digital convexity, straightness, and convex polygons, Pattern Analysis and Machine Intelligence 4 (6) (1982) 618–626.
5. C. Kim, A. Rosenfeld, Digital straight lines and convexity of digital regions, Pattern Analysis and Machine Intelligence 4 (2) (1982) 149–153.
6. B. Chaudhuri, A. Rosenfeld, On the computation of the digital convex hull and circular hull of a digital region, Pattern Recognition 31 (12) (1998) 2007–2016.
7. R. Klette, A. Rosenfeld, Digital straightness—a review, Discrete Appl. Math. 139 (1-3) (2004) 197–230.
8. I. Debled-Rennesson, J.-L. Rémy, J. Rouyer-Degli, Detection of the discrete convexity of polyominoes, Discrete Appl. Math. 125 (1) (2003) 115–133, 9th International Conference on Discrete Geometry for Computer Imagery (DGCI 2000) (Uppsala).
9. F. Feschet, L. Tougne, Optimal time computation of the tangent of a discrete curve: Application to the curvature, in: Proc 8th Int. Conf. Discrete Geometry for Computer Imagery (DGCI'99), no. 1568 in Lecture Notes in Computer Science, Springer Verlag, 1999, pp. 31–40.
10. J.-O. Lachaud, A. Vialard, F. de Vieilleville, Fast, accurate and convergent tangent estimation on digital contours, Image and Vision Computing 25 (2007) 1572–1587.
11. S. Brlek, G. Labelle, A. Lacasse, Incremental algorithms based on discrete Green theorem, in: G. S. d. B. I. Nystrom, S. Svensson (Eds.), Proc. DGCI 2003, 11-th International Conference on Discrete Geometry for Computer Imagery, no. 2886 in LNCS, Springer-Verlag, Napoli, Italia, 2003, pp. 277–287.
12. S. Brlek, G. Labelle, A. Lacasse, Algorithms for polyominoes based on the discrete Green theorem, Discrete Applied Math. 147 (3) (2005) 187–205.
13. S. Brlek, G. Labelle, A. Lacasse, On minimal moment of inertia polyominoes, in: DGCI 2008, 14-th International Conference on Discrete Geometry for Computer Imagery, Lyon, France, 2007, this volume.
14. S. Brlek, G. Labelle, A. Lacasse, A note on a result of Daurat and Nivat, in: C. de Felice, A. Restivo (Eds.), Proc. DLT 2005, 9-th International Conference on Developments in Language Theory, no. 3572 in LNCS, Springer-Verlag, Palermo, Italia, 2005, pp. 189–198.
15. S. Brlek, G. Labelle, A. Lacasse, Properties of the contour path of discrete sets, Int. J. Found. Comput. Sci. 17 (3) (2006) 543–556.
16. A. Daurat, M. Nivat, Salient and reentrant points of discrete sets, in: A. del Lungo, V. di Gesu, A. Kuba (Eds.), Proc. IWCIA'03, International Workshop on Combinatorial Image Analysis, Electronic Notes in Discrete Mathematics, Elsevier Science, Palermo, Italia, 2003.
17. S. Brlek, G. Labelle, A. Lacasse, Shuffle operations on lattice paths, in: M.Rigo (Ed.), Proc. CANT2006, International school and conference on Combinatorics, Automata and Number theory, University of Liège, Liège, Belgium, 2006.

18. S. Brlek, G. Labelle, A. Lacasse, Shuffle operations on discrete paths, Theoret. Comput. Sci. In press.
19. S. Brlek, X. Provençal, An optimal algorithm for detecting pseudo-squares., in: A. Kuba, L. G. Nyúl, K. Palágyi (Eds.), Discrete Geometry for Computer Imagery, 13th International Conference, DGCI 2006, Szeged, Hungary, October 25-27, 2006, Proceedings, Vol. 4245 of Lecture Notes in Computer Science, Springer, 2006, pp. 403–412.
20. S. Brlek, X. Provençal, On the tiling by translation problem, Discrete Applied Math. To appear.
21. M. Lothaire, Combinatorics on words, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1997.
22. M. Lothaire, Algebraic combinatorics on words, Vol. 90 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2002.
23. M. Lothaire, Applied combinatorics on words, Vol. 105 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2005.
24. J.-P. Duval, Factorizing words over an ordered alphabet, J. Algorithms 4 (4) (1983) 363–381.
25. E. B. Christoffel, Observatio arithmetica, Annali di Mathematica 6 (1875) 145–152.
26. J.-P. Borel, F. Laubie, Quelques mots sur la droite projective réelle, J. Théor. Nombres Bordeaux 5 (1) (1993) 23–51.
27. J. Berstel, A. de Luca, Sturmian words, Lyndon words and trees, Theoret. Comput. Sci. 178 (1-2) (1997) 171–203.
28. J.-P. Reveillès, Géométrie discrète, calcul en nombres entiers et algorithmique, Ph.D. thesis, Université Louis Pasteur, Strasbourg (December 1991).
29. H. Fredricksen, J. Maiorana, Necklaces of beads in k colors and k-ary de Bruijn sequences, Discrete Math. 23 (3) (1978) 207–210.
30. X. Provençal, Combinatoire des mots, pavages et géométrie discrète, Ph.D. thesis, Université du Québec à Montréal, Montréal (2008).