Equivalence between Regular n-G-maps and n-surfaces

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Abstract. Many combinatorial structures have been designed to represent the topology of space subdivisions and images. We focus here on two particular models, namely the n-G-maps used in geometric modeling and computational geometry and the n-surfaces used in discrete imagery. We show that a subclass of n-G-maps is equivalent to n-surfaces. We exhibit a local property characterising this subclass, which is easy to check algorithmatically. Finally, the proofs being constructive, we show how to switch from one representation to another effectively.

1 Introduction

The representation of space subdivisions and the study of their topological properties are significant topics in various fields of research such as geometric modeling, computational geometry and discrete imagery. A lot of combinatorial structures have already been defined to represent such topologies and specific tools have been developed to handle each of them. Although most of them aim at representing manifold-like underlying spaces they have very variable definitions.

Comparing these structures, and highlighting their similarities or specificities are important for several reasons. It can first create bridges between them and offer the possibility to switch from one framework to another according to the needs of a given application. It may also lead to a more general framework which unify most of these structures. Theoretical results and practical algorithms can also be transferred from one to another. However, these structures are most likely not interchangeable. Indeed, there is yet no complete combinatorial characterisation of manifolds. The structures found in the literature generally propose local combinatorial properties that can only approach the properties of space subdivisions. It is therefore extremely important to know precisely what class of objects is associated to each structure. Several studies have already been carried out in this direction. Quad-edge, facet-edge and cell-tuples were compared by

Brisson in [6]. Lienhardt [15] studied their relations with several structures used in geometric modelling like the n-dimensional (generalised or not) map. The relation between a subclass of orders and cell complexes was also studied in [1]. A similar work was done on dual graphs and maps by Brun and Kropatsch in [7].

We focus here mainly on two structures: the n-surface and the n-dimensional generalised map. The n-surface is a specific subclass of orders defined by Bertrand and Couprie in [4] which is similar to the notion previously defined by Evako *et al.* on graphs in [12]. It is essentially an order relation over a set together with a finite recursive property. It is designed to represent the topology of images and objects within. The generalised map introduced by Lienhardt in [15] is an effective tool in geometric modeling and is also used in computational geometry. It is defined by a set of $n+1$ involutions joigning elements dimension by dimension. Although the definitions of these two structures are very different, we show that a subclass of generalised maps, that we call *regular n-G-maps*, is equivalent to n-surfaces. Furthermore, we provide a simple local characterisation of this subclass. This may have various nice consequences. From a theoretical point of view, some proofs may be simplified by expressing them rather on a model than on the other, some notions can also be extended. Moreover the operators defined on each model may be translated onto the other. A possible application would consist in using the tools defined on orders : homotopic thinning, marching chains using frontier orders [8, 9] to obtain n-surfaces. They can then be transformed into n-G-maps which can easily be handled with their associated construction operators: identification, extrusion, split, merge. To prove the equivalence of these models, we use an intermediary structure defined by Brisson in [6]: the augmented incidence graph. This structure is quite similar to orders although its definition does not involve the same local properties as n-surfaces. Moreover Brisson shows a partial link between n-G-maps and such incidence graphs. He effectively proved that an n-G-map may be built from any augmented incidence graph. In [15], Lienhardt gives a necessary condition to build such an augmented incidence graph from an n-G-map. We show here with a counter-example that it is not sufficient.

The main contributions of these papers are: (i) we prove that n-dimensional augmented incidence graphs and n-surfaces are equivalent structures (Theorem 18), (ii) we complete the works of Brisson and Lienhardt with the characterisation of the n-G-map subclass that is equivalent to augmented incidence graphs (Definition 8 of *regular n-G-maps*, Theorem 12 and 14), (iii) we design constructive proofs which allow to effectively switch between the different representation. This result remains very general since any closed n-G-map can be refined into a regular n-G-map with appropriate local subdivisions.

The paper is organized as follows. First, we recall the notions of incidence graphs and orders and show how they are related to each other. We also define precisely the models we wish to compare and give some clues to their equivalence. Then, we give a guideline of the proof before presenting the whole demonstration. We conclude with some perspectives for this work.

2 Models description

We describe below the models we wish to compare, and we list known results about their relationships. We begin with recalling the general notions related to orders and incidence graphs and we characterise then the appropriate submodels.

2.1 Orders and incidence graphs

Orders are used by Bertrand *et al.* [3] to study topological properties of images. The main advantages of this model are its genericity and its simplicity. Orders can be used to represent images of any dimension, whether they are regularly sampled or not.

Definition 1. *An* order *is a pair* $|X| = (X, \alpha)$*, where* X *is a set and* α *a reflexive, antisymmetric, and transitive binary relation. We denote* β *the inverse of* α *and* θ *the union of* α *and* β*.* CF orders *are orders which are* countable*, i.e.* X *is countable, and* locally finite*, i.e.* $\forall x \in X, \theta(x)$ *is finite.*

For any binary relation ρ on a set X, for any element x of X, the set $\rho(x)$ is called the ρ-adherence of x and the set $\rho^{\Box}(x) = \rho(x) \backslash \{x\}$, the strict ρ-adherence of x. A ρ-chain of length n is any sequence $x_0, x_1, \cdots, x_n$ such that $x_{k+1} \in \rho^{\Box}(x_k)$. An implicit dimension, $\mathtt{dim}_\alpha(x)$, may be associated to each element of an order [12, 1], as the length of the longest α-chain beginning at it. We choose here to represent orders as simple directed acyclic graphs (see Fig. 2-a), where each node is associated to an element of the order and only direct α-relations[4] are shown. The remaining relations can be deduced by transitivity.

Incidence graphs are used to represent subdivisions of topological spaces. They explicitly deal with the different cells of the subdivision and their incidence relations. They have for example been used by Edelsbrunner [10] to design geometric algorithms.

Definition 2. *An* incidence graph *[6] yielded by a cellular partition of dimension* n *is defined as a directed graph whose nodes correspond to the cells of the partition and where each oriented arc connects an* i*-cell to an* $(i-1)$*-cell to which it is incident. With such a graph is associated a labeling of each node given by the dimension of its associated cell.*

Let us denote by I_i *the index set of the* i*-cells of a cellular partition. The associated* n*-dimensional incidence graph is hence denoted by* $IG_C = (C, \prec)$*, where* $C = \bigcup_{i=0}^{i=n} (\bigcup_{\beta \in I_i} c^i_\beta)$ *is the set of cells and* $\prec$ *is the incidence relation between* $(i-1)$*-cells and* i*-cells,* $i \in \{1, \cdots, n\}$*.*

In the sequel we only consider finite incidence graphs. For convenience, it is also sometimes useful to add two more cells c^{-1} and c^{n+1} to incidence graphs, such that c^{-1} is incident to all 0-cells and all n-cells are incident to c^{n+1}. They

[4] an element x is said to be directly related to x' by α if $x' \in \alpha^{\Box}(x)$ and $\alpha^{\Box}(x) \cap \beta^{\Box}(x') = \emptyset$

represent no real cells of the subdivision but make easier the definition of generic operators on incidence graphs. An incidence graph with two such cells is called an *extended incidence graph* and is denoted by $IG_C^* = (C^*, \prec)$. Let IG_C be an incidence graph and c an element of C, if c' is linked to c by a chain of cells (eventually empty) related by $\prec$ then we say that c' is a face of c and we denote it by $c' \leq c$. We write $c' < c$ when $c' \neq c$ and $c' \leq c$. An incidence graph is hence represented by a simple directed acyclic graph, where the nodes respectively representing a cell c^{i+1} and a cell c^i are linked by an arc if and only if $c^i \prec c^{i+1}$ (see Fig. 2-b).

There is an obvious relationship between incidence graphs and orders. The incidence graph $IG_C = (C, \prec)$ can indeed be seen as the order $(C, \leq)$ where the dimension associated to each cell is forgotten. $(C, \leq)$ is the order associated to IG_C. Reciprocally, a relation $\prec_{|X|}$ may be defined on the elements of $|X|$, such that $x' \prec_{|X|} x$ is equivalent to $x' \in \alpha(x)$ and $\mathtt{dim}_\alpha(x') = \mathtt{dim}_\alpha(x) - 1$. The incidence graph $(X, \prec_{|X|})$ where each cell of the graph is labeled by its corresponding α-dimension in $|X|$ is the incidence graph associated to $|X|$. However, in the general case, the relation $\leq$ between the cells of an incidence graph built from an order is different from the order relation α on the set X (see [2]). An order $|X| = (X, \alpha)$ is said to be equivalent to an incidence graph $G = (C, \prec)$ if the order $|G| = (C, \leq)$ is isomorphic to $|X|$.

Theorem 3. *An order* (X, α) *and its associated incidence graph are* equivalent *if and only if :*

1. *each element of* x *belongs to at least one maximal* α*-chain,*
2. $\forall x \in X$, *the elements of* X *directly related to* x *by* α *have dimension* $\mathtt{dim}_\alpha(x) - 1$.

When dealing with incidence graphs we hence use the notations defined on orders. An incidence graph or an order are *connected* if and only if any couple of cells can be joined by a sequence of cells related by $\theta^\square$.

We notice here that orders as well as incidence graphs are not able to represent cellular subdivisions with *multi-incidence*. A simple example is the torus made from a single square with opposite borders glued together. They indeed cannot provide information on how the different cells are glued together. They are for example not able to represent differently the two objects of Fig. 1.

2.2 n-surfaces, augmented incidence graphs and G-maps

Orders and incidence graphs can represent a wide range of objects. We concentrate now on subclasses of these models that are used to represent restricted classes of objects. We aim at comparing these structures with n-dimensional generalised maps (n-G-maps) defined by Lienhardt [14–16].

We begin with a subclass of orders defined by Bertrand *et al.* which is close to the notion of manifold proposed by Kovalevsky [4].

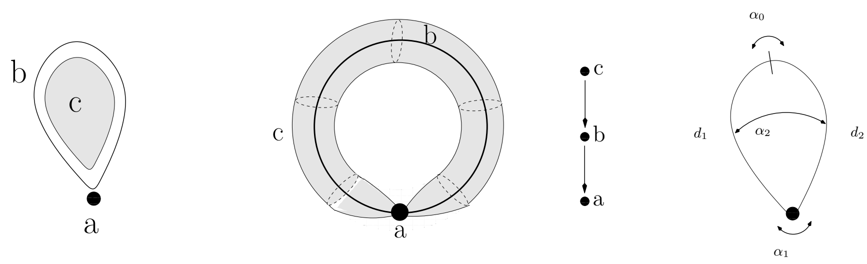

Fig. 1. Both objects have the same cells and the same incidence relations, and hence the same incidence graph. However they are clearly different. 2-G-map with multi-incidence: $\alpha_0(d_1) = \alpha_1(d_1) = \alpha_2(d_1) = \{(d_1, d_2)\}$ corresponding to the first object. The 2-G-map corresponding to the second object is the same except for α_2 which is the identity

Definition 4. *Let $|X| = (X, \alpha)$ be a non-empty CF-order. The order $|X|$ is a* 0-surface *if X is composed exactly of two points x and y such that $y \notin \alpha(x)$ and $x \notin \alpha(y)$ The order $|X|$ is an n*-surface, *$n > 0$, if $|X|$ is connected and if, for each $x \in X$, the order $|\theta^{\square}(x)|$ is an $(n-1)$-surface.*

It can be recursively proved that Theorem 3 holds for any n-surface (see Fig. 2-a), which is then always isomorphic to its associated incidence graph.

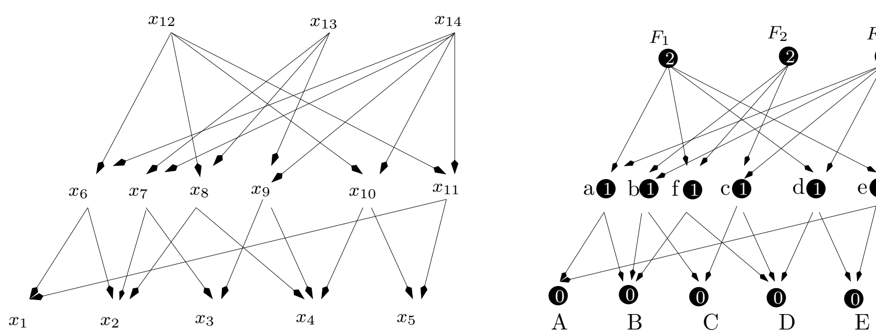

Fig. 2. Example of an order (Left) which is 2-surface and of an incidence graph (Right) which is augmented, and represents the subdivision of Fig. 4-a

We present now a subclass of incidence graphs defined by Brisson [6], to represent CW-complexes whose underlying spaces are d-manifolds. It will form the bridge between n-surfaces and n-G-maps.

Definition 5. *An extended incidence graph IG^*_C of dimension n is said to be an* augmented incidence graph *when it is connected and :*

1. *each i-cell of C belongs to at least one $(n+1)$-tuple of cells $(c^0, \cdots, c^n)$*
2. $\forall (c^{i-1}, c^i, c^{i+1}) \in C^* \times C \times C^*$, $c^{i-1} \prec c^i \prec c^{i+1}$, $\exists ! c^{'i} \in C, c^{'i} \neq c^i$, $c^{i-1} \prec c'^i \prec c^{i+1}$ (c^{i-1}, c^i, c^{i+1}) *and* $(c^{i-1}, c^{'i}, c^{i+1})$ *are called* switch IG^*_C-triplets. *The operator* `switch` *is hence defined by* $\texttt{switch}(c^{i-1}, c^i, c^{i+1}) = c'^i$. *The cell* $c^{'i}$ *is then called the* (c^{i-1}, c^{i+1})-twin of c^i in IG^*_C.

The n-G-maps defined by Lienhardt are used to represent the topology of subdivisions of topological spaces. They can only represent quasi-manifolds (see [16]), orientable or not, with or without boundary.

Definition 6. *Let $n \geq 0$, an n-G-*map *is an $(n+2)$-tuple $G = (D, \alpha_0, \cdots, \alpha_n)$ such that :*

- *D is a finite set of darts*
- *α_i, $i \in \{0 \cdots, n\}$ are permutations on D such that :*
  - *$\forall i \in \{0, \cdots, n\}$, α_i is an involution*[5].
  - *$\forall i$, j such that $0 \leq i < i+2 \leq j \leq n$, $\alpha_i \alpha_j$ is an involution.*

The i-cells of the cellular subdivision represented by an n-G-map are determined by the orbits of the darts of D (see Figs. 4-b, c and d). In the following an orbit of a dart d is denoted by $<\!\alpha\!>$ (d) indexed by the indices of the involved permutations.

Definition 7. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be an n-G-map. Each i-cell of the corresponding subdivision is given by $<\!\alpha\!>_{N-\{i\}}$ (d) where d is a dart incident to this i-cell.*

The set of i-cells is a partition of the darts of the n-G-map, for each i between 0 and n. The incidence relations between the cells is defined by: $c^j = <\!\alpha\!>_{N-\{j\}}$ (d) is a face of a cell $c^i = <\!\alpha\!>_{N-\{i\}}$ (d) when $j \leq i$ and $<\!\alpha\!>_{N-\{j\}}$ $(d) \cap <\!\alpha\!>_{N-\{i\}}$ $(d) \neq \emptyset$ [15]. Two such cells are said to be consecutive when $j = i+1$.

We list below three classical properties attached to n-G-maps.

1. *closeness* : $\forall i \in N = \{0, \cdots, n\}$, α_i is without fixed point : $\forall d \in D, \alpha_i(d) \neq d$
2. *without multi-incidence* : $\forall d \in D, \bigcap_{i=0}^{i=n} <\!\alpha\!>_{N-\{i\}} (d) = \{d\}$
3. *connectedness* : $\forall d \in D,\ <\!\alpha\!>_N (d) = D$

We note that a subdivision represented by a closed n-G-map has no boundary. A subdivision with multi-incidencee is displayed in Fig. 1.

The associated incidence graph of an n-G-map is the extended incidence graph corresponding to the cellular subdivision it represents. There is an immediate link between the darts of an n-G-map and the maximal chains, called $(n+1)$ cell-tuples, of its associated incidence graph. A dart d actually defines a unique $(n+1)$ cell-tuple $(c^0, \cdots, c^n)$ with all cells having at least the dart d in common (see Definition 7). $(c^0, \cdots, c^n)$ is called the $(n+1)$ cell-tuple associated

[5] A permutation π on the domain set D is an involution if and only if $\pi \circ \pi =$ *identity of D*

to d. The condition of non multi-incidence is needed to reciprocally associate a unique dart to each $(n+1)$ cell-tuple. There exists hence a bijection between the set of darts of an n-G-map without multi-incidence and the set of $(n+1)$ cell-tuples in the associated incidence graph. For instance, on Fig. 4-a, the dart 1 is uniquely associated to (A, a, F_1).

However, despite what has been written in [15], the property of non multi-incidence of a generalised map is not sufficient to guarantee that its associated incidence graph is augmented. A counterexample is given in Fig. 3. We introduce hence a more accurate subclass of n-G-maps.

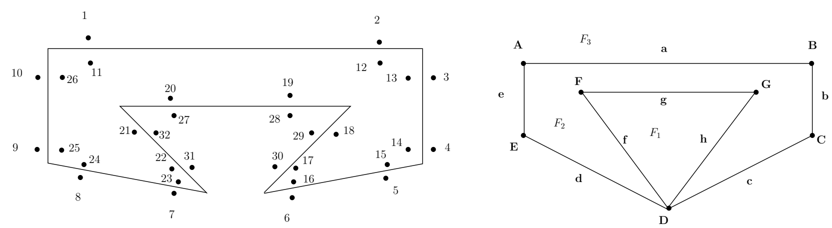

Fig. 3. Example of a closed n-G-map without multi-incidence (Left) and its associated space subdivision (Right) whose associated incidence graph is not augmented: there are four 1-cells (c,d,f,h) between D and F_2

2.3 Regular n-G-maps

The insufficiency of the non multi-incidence property comes from a subtler kind of multi-incidence. The classical non multi-incidence condition guarantees that there are no multi-incidence on the cellular subdivision associated to the n-G-map. As many other models, each n-G-map may be associated to a "set of simplices" but it has not to be a simplicial complex. It is namely a numbered simplicial set[6] (see Fig. 4-e) in which other kinds of multi-incidence may appear. Such a simplicial set is related to the barycentric triangulation of the corresponding cellular partition. The set of vertices is exactly the set of cells of the incidence graph, each vertex being labeled by the dimension of the corresponding cell. The classical non multi-incidence property implies that two different maximal simplices of the associated numbered simplicial set cannot have the same set of vertices. But it does not force lower dimensional simplices to fullfill the same requirement. We consider hence a restricted subclass of n-G-maps, the *regular n-G-maps*, which avoids more configurations of simplicial multi-incidence.

[6] A numbered simplicial set is a simplicial set in which a positive integer is associated to each 0-simplex, see [16, 17].

Definition 8. *A* regular n-G-map *is a connected closed n-G-map without multi-incidence with the additional property, $\forall i \in \{1, \cdots, n-1\}$ and $\forall d \in D$:*

$$<\alpha>_{N-\{i-1\}} (d) \cap <\alpha>_{N-\{i+1\}} (d) = <\alpha>_{N-\{i-1,i+1\}} (d) \ (\textit{simplicity})$$

The simplicity condition of such an n-G-map impose that the cells of the associated subdivision are more similar to topological disks. It implies that the numbered simplicial set must have a single edge between every two vertices when there is a difference of two between their associated numbers.

This limitation is not too restrictive because of the following property: any closed n-G-map may be refined into a regular n-G-map by appropriate barycentric subdivisions. There is indeed always possible to obtain a simplicial set without multi-incidence from any simplicial set by refining it [13]. Moreover as this process involves barycentric subdivisions [13], we are sure that the resulting simplicial set can be numbered [16]. We also note that the refinement process has not to be done on the whole map but only locally where some multi-incidence appears.

3 Equivalence of regular n-G-maps, augmented incidence graph and n-surfaces

We give first the main ideas and the organisation of the proof. We detail then the whole demonstration.

3.1 Guideline of the proof

Incidence graphs are used as a bridge between regular n-G-maps and n-surfaces.

Generalised map and incidence graph An example of a regular n-G-map and an equivalent augmented incidence graph is given in Figs. 4 and 2-b.

We first prove that the incidence graph associated to any regular n-G-map is augmented. It already fullfills a part of the definition since each cell of such an incidence graph belongs to at least one $(n+1)$ cell-tuple. We must then show that the $(n+1)$ involutions of the map induce a switch operator on the incidence graph, which makes it augmented. These involutions are indeed involutions on the darts of the map and thus induce $(n+1)$ involutions on the $(n+1)$ cell-tuples of the associated incidence graph. The regularity of the map allows to prove that these involutions induce a switch property on the $(n+1)$ cell-tuples.

The converse has already partially been proved by Brisson [6]. We begin with proving that the switch operator on an augmented incidence graph of dimension n induces $n+1$ involutions without fixed point on the $(n+1)$ cell-tuples of this graph. We show then that they commute when there is a difference of two between their indices. The $(n+2)$-tuple made of the set containing all $(n+1)$ cell-tuples of the incidence graph and the $(n+1)$ involutions is hence a closed n-G-map without multi-incidence. The switch property allows then to prove that it also verifies the simplicity property.

Incidence graph and n-surface An example of an augmented incidence graph and an equivalent n-surface is displayed on Fig. 2.

The proof is made with an induction over the dimension n. The equivalence is clear for $n = 0$. For $n > 0$, we prove that each subgraph built on the strict θ-adherence of any element of an augmented incidence graph is itself an augmented incidence graph. We also show that an extended incidence graph which is locally everywhere an augmented incidence graph is globally an augmented incidence graph. This means that an n-dimensional augmented incidence graph, $n > 0$, can be recursively defined. It is simply an extended incidence graph such that each subgraph built on the strict θ-adherence of any of its elements is an $(n-1)$-dimensional augmented incidence graph. Now n-dimensional augmented incidence graphs and n-surfaces are equivalent for $n = 0$. Given that they are built with the same recurrence for all $n > 0$, they are hence equivalent for all n.

Organisation of the proof

$$\begin{array}{c} \text{regular} \\ n\text{-}G\text{-map} \end{array} \begin{array}{c} \overset{\substack{nGIG-\text{conversion} \\ \text{(Theorem 12 p. 10)}}}{\Longrightarrow} \\ \overset{\substack{IGnG-\text{conversion} \\ \text{(Theorem 14 p. 11)}}}{\Longleftarrow} \end{array} \begin{array}{c} \text{augmented} \\ \text{incidence} \\ \text{graph} \end{array} \begin{array}{c} \overset{\substack{IGnS-\text{conversion} \\ \text{(Theorem 18 p. 12)}}}{\Longrightarrow} \\ \overset{\substack{nSIG-\text{conversion} \\ \text{(Theorem 18 p. 12)}}}{\Longleftarrow} \end{array} n\text{-surface}$$

3.2 Proof

We first prove the equivalence between regular n-G-maps and augmented incidence graph. We show then that augmented incidence graph and n-surfaces are equivalent. Finally we deduce the link between regular n-G-maps and n-surfaces.

Equivalence between regular n-G-maps and augmented incidence graphs We first show how to define an $nGIG$-conversion which builds an augmented incidence graph from a regular n-G-map. We then define the $IGnG$-conversion which is the inverse of $nGIG$-conversion up to isomorphism.

$nGIG$**-conversion**

As previously said, there is an n-dimensional incidence graph associated with any regular n-G-map. We are going to prove that this incidence graph is augmented. We need first to state some properties of particular orbits of n-G-maps. The first two lemmas have an interesting interpretation on the numbered simplicial set associated to the n-G-map (Fig. 4-e). The last one is better related to the cellular subdivision. The proofs of the three following lemmas can be found in [2].

The first lemma states that, for any n-G-map without multi-incidence, three 0-simplices with consecutive numbers belong to exactly one 2-simplex.

Lemma 9. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be a closed n-G-map without multi-incidence. Let d be any dart of D and $i \in \{1, \cdots, n-1\}$,*

$$<\!\alpha\!>_{N-\{i-1\}}(d) \cap <\!\alpha\!>_{N-\{i\}}(d) \cap <\!\alpha\!>_{N-\{i+1\}}(d) = <\!\alpha\!>_{N-\{i-1,i,i+1\}}(d)$$

This second lemma says that, for any n-G-map, a 1-simplex between two 0-simplices numbered $i-1$ and $i+1$ belongs to at most two 2-simplexes.

Lemma 10. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be an n-G-map. Let d be any dart of D and $i \in \{1, \cdots, n-1\}$,*

$$<\alpha>_{N-\{i-1,i+1\}} (d) =<\alpha>_{N-\{i-1,i,i+1\}} (d) \cup <\alpha>_{N-\{i-1,i,i+1\}} (d\alpha_i)$$

This third lemma states that every 1-cell of the cellular subdivision associated to an n-G-map has at most two 0-faces and that any $(n-1)$-cell is face of at most two n-cells.

Lemma 11. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be an n-G-map and d,d' two darts of D.*

1. $<\alpha>_{N-\{1\}} (d) = (<\alpha>_{N-\{1\}} (d) \cap <\alpha>_{N-\{0\}} (d)) \cup (<\alpha>_{N-\{1\}} (d) \cap <\alpha>_{N-\{0\}} (d\alpha_0))$
2. $<\alpha>_{N-\{n-1\}} (d) = (<\alpha>_{N-\{n-1\}} (d) \cap <\alpha>_{N-\{n\}} (d)) \cup (<\alpha>_{N-\{n-1\}} (d) \cap <\alpha>_{N-\{n\}} (d\alpha_n))$

Theorem 12. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be a regular n-G-map. Its associated incidence graph is then augmented. The construction of an augmented incidence graph from a regular n-G-map is called an $nGIG-conversion$.*

Proof. We must prove that the incidence graph associated to the n-G-map has the property needed to build a switch operator. This property can be equivalently expressed on the $(n+1)$ cell-tuples of the graph with the two additional fictive cells c^{-1} and c^{n+1} [6]. For all couple of cells (c^{i-1}, c^{i+1}), there must exist exactly two different cells c^i and c'^i such that all $(n+1)$ cell-tuples containing c^{i-1} and c^{i+1} contains either c^i or c'^i. Moreover since there exists a bijection between the set of darts of a closed n-G-map without multi-incidence and the set of $(n+1)$ cell-tuples of the associated incidence graph, we can equivalently achieve the demonstration with darts or cell-tuples.

Given two cells c^{i-1} and c^{i+1}, let us choose one of the $(n+1)$ cell-tuples containing them and let d be its associated dart. By Definition 7, the dart $d\alpha_i$ also corresponds to an $(n+1)$ cell-tuple containing c^{i-1} and c^{i+1}. But as the map is closed and without multi-incidence, the i-cell associated to d, $<\alpha>_{N-\{i\}} (d)$, is different from the i-cell associated to $d\alpha_i$, $<\alpha>_{N-\{i\}} (d\alpha_i)$. We have then at least two distinct i-cells between c^{i-1} and c^{i+1}. We must prove that there is no other. We translate this condition in terms of orbits of darts.

- If $i \in \{1, \cdots, n\}$, $\forall d' \in D$ such that $d' \in <\alpha>_{N-\{i-1\}}(d)$ and $d' \in <\alpha>_{N-\{i+1\}}(d) \Rightarrow$ either $d' \in <\alpha>_{N-\{i\}}(d)$ or $d' \in <\alpha>_{N-\{i\}} (d\alpha_i)$
- $\forall d' \in D, d' \in <\alpha>_{N-\{1\}} (d) \Rightarrow d' \in <\alpha>_{N-\{0\}} (d)$ or $d' \in <\alpha>_{N-\{0\}} (d\alpha_0)$,
- $\forall d' \in D$, $d' \in <\alpha>_{N-\{n-1\}} (d) \Rightarrow d' \in <\alpha>_{N-\{n\}} (d)$ or $d' \in <\alpha>_{N-\{n\}} (d\alpha_n)$

The last two points comes directly from Lemma 11. We prove the first point.

$$
\begin{aligned}
& d' \in <\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i+1\}}(d) \\
\overset{simplicity}{\Longleftrightarrow}\; & d' \in <\alpha>_{N-\{i-1,i+1\}}(d) \\
\overset{Lemma\ 10}{\Longleftrightarrow}\; & d' \in <\alpha>_{N-\{i-1,i,i+1\}}(d) \cup <\alpha>_{N-\{i-1,i,i+1\}}(d\alpha_i) \\
\overset{Lemma\ 9}{\Longleftrightarrow}\; & d' \in (<\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i\}}(d) \cap <\alpha>_{N-\{i+1\}}(d)) \\
& \cup (<\alpha>_{N-\{i-1\}}(d\alpha_i) \cap <\alpha>_{N-\{i\}}(d\alpha_i) \cap <\alpha>_{N-\{i+1\}}(d\alpha_i))
\end{aligned}
$$

Otherwise said $d' \in <\alpha>_{N-\{i\}}(d)$ or $d' \in <\alpha>_{N-\{i\}}(d\alpha_i)$. □

IGnG-**conversion**

We here show how to build an n-G-map from an augmented incidence graph. The first lemma says that the operator `switch` induces $(n+1)$ involutions on the set of $(n+1)$ cell-tuples of the incidence graph.

Lemma 13. *Let IG^*_C be an augmented incidence graph. Its switch operator induces $(n+1)$ involutions without fixed point α_i, $i \in \{0, \cdots, n\}$ on the set of the $(n+1)$ cell-tuples of IG^*_C, $(c^0, \cdots, c^i, \cdots, c^n)$, defined by :*

$$
\alpha_i((c^0, \cdots, c^{i-1}, c^i, c^{i+1}, \cdots, c^n)) = (c^0, \cdots, c^{i-1}, {c'}^i, c^{i+1}, \cdots, c^n)
$$
$$
\text{where } {c'}^i = \mathtt{switch}(c^{i-1}, c^i, c^{i+1})
$$

Theorem 14. *Let IG^*_C be an augmented incidence graph. Let us define*

- $D = \{(c^0_{\beta_0}, \cdots, c^n_{\beta_n}), c^{-1} \prec c^0_{\beta_0} \prec c^1_{\beta_1} \prec \cdots \prec c^n_{\beta_n} \prec c^{n+1}\}$
- α_i, $i \in \{0, \cdots, n\}$ *such that*

$$
(c^0_{\beta_0}, \cdots, c^{i-1}_{\beta_{i-1}}, c^i_{\beta_i}, c^{i+1}_{\beta_{i+1}}, \cdots, c^n_{\beta_n}) \overset{\alpha_i}{\mapsto} (c^0_{\beta_0}, \cdots, c^{i-1}_{\beta_{i-1}}, c^i_{\beta'_i}, c^{i+1}_{\beta_{i+1}}, \cdots, c^n_{\beta_n})
$$

with $c^i_{\beta'_i} = \mathtt{switch}(c^{i-1}_{\beta_{i-1}}, c^i_{\beta_i}, c^{i+1}_{\beta_{i+1}})$

Then $(D, \alpha_0, \cdots, \alpha_n)$ is a regular n-G-map.

This process is called a IGnG-conversion

Proof. The proof is decomposed in four parts. The *closeness*, *commutativity* and *without multi-incidence* properties have already been proved by Brisson [6] and may also be found in [2]. We just prove here the *simplicity* property. The switch property and the definition of α_i guarantees that if $d' \in <\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i+1\}}(d)$ then either $d' \in <\alpha>_{N-\{i\}}(d)$ or $d' \in <\alpha>_{N-\{i\}}(d\alpha_i)$. Otherwise said,

$$
\begin{aligned}
& <\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i+1\}}(d) \\
\overset{\text{(by Switch prop.)}}{=}\; & (<\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i+1\}}(d) \cap <\alpha>_{N-\{i\}}(d)) \\
& \cup (<\alpha>_{N-\{i-1\}}(d) \cap <\alpha>_{N-\{i+1\}}(d) \cap <\alpha>_{N-\{i\}}(d\alpha_i))) \\
\overset{\text{(by Lemma 9)}}{=}\; & <\alpha>_{N-\{i-1,i,i+1\}}(d) \cup <\alpha>_{N-\{i-1,i,i+1\}}(d\alpha_i) \\
\overset{\text{(by Lemma 10)}}{=}\; & <\alpha>_{N-\{i-1,i+1\}}(d)
\end{aligned}
$$

□

Equivalence between augmented incidence graphs and n-surfaces We state below two lemmas which together provide a recursive definition of n-dimensional augmented incidence graphs. Their proofs can be found in [2].

The first lemma expresses that given an augmented incidence graph IG_C^* all subgraphs of the form $\theta^{\square}(c)$ with $c \in C$ are augmented incidence graphs too.

Lemma 15. *Let IG_C^* be an augmented incidence graph of dimension $n \geq 1$, then $\forall c \in C, \theta^{\square}(c)$ is an augmented incidence graph of dimension $n-1$.*

The next lemma shows that an extended incidence graph IG_C^* with dimension at least 1, which is locally everywhere an augmented indicence graph, is also itself an augmented incidence graph.

Lemma 16. *Let IG_C^* be an extended incidence graph of dimension at least 1, such that $\forall c \in C$, $\theta^{\square}(c)$ is an augmented incidence graph then IG_C^* is also an augmented incidence graph.*

These two lemmas lead to the following theorem which gives a recursive characterisation of augmented incidence graphs of dimension n.

Theorem 17. *Let $IG_C^* = (C^*, \prec)$ be an extended incidence graph of dimension n, the two following propositions are equivalent :*

1. *IG_C^* is a non empty augmented incidence graph*
2. *IG_C^* is such that:*
   - *if $n = 0$, C contains exactly two 0-cells c^0 and $c^{'0}$, such that c^0 and $c^{'0}$ are (c^{-1}, c^1)-twins in IG_C^*.*
   - *if $n > 0$, C is such that for all $c \in C$, $\theta^{\square}(c)$ is an augmented incidence graph of dimension $n-1$.*

Proof. We are going to prove that (1) $\Leftrightarrow$ (2) for all n. The proof is quite immediate for $n = 0$ [2] (Both models consist in two disconnected points). We show it for $n > 0$:

- $\Rightarrow$ Let IG_C^* be an augmented incidence graph of dimension n. $\forall c \in C_n$, $\theta^{\square}(c)$ is by Lemma 15 an augmented incidence graph of dimension $n-1$.
- $\Leftarrow$ Let IG_C^* be an extended incidence graph of dimension n fullfilling the conditions of (2). For all $c \in C_n$, $\theta^{\square}(c)$ is an $(n-1)$-dimensional augmented incidence graph. IG_C^* has dimension strictly greater than 0. It is then by Lemma 16 an augmented incidence graph.

$\square$

This recursive characterisation identical to the definition of n-surfaces leads immediately to the following theorem :

Theorem 18. *Let $IG_C = (C, \prec)$ be an incidence graph and $|X| = (X, \alpha)$ an order*

1. *$IG_C^* = (C \cup \{c^{-1}, c^{n+1}\}, \prec)$ is augmented $\Rightarrow$ its associated order is an n-surface*
2. *$|X| = (X, \alpha)$ is an n-surface $\Rightarrow$ its associated incidence graph is augmented*

Equivalence between regular n-G-maps and n-surfaces The two preceding results leads to the following equivalence between regular n-G-maps and n-surfaces.

Theorem 19. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be an n-G-map and $|X| = (X, \alpha)$ an order such that there exists an isomorphism between their associated incidence graphs. Then the following propositions are equivalent :*

1. *G is a regular n-G-map*
2. *$|X|$ is an n-surface*

If C_G is the set of cells of the subdivision represented by a regular n-G-map G and $\leq_G$ the incidence relation between these cells then $(C_G, \leq_G)$ is the n-surface associated to G by $nGnS$-conversion where the dimension of the cells of C_G are forgotten. Reciprocally, if $D_{|X|}$ is the set of $(n+1)$ α-chains of an n-surface $|X|$, then $(D_{|X|}, \alpha_0, \cdots, \alpha_n)$ is the regular n-G-map associated to $|X|$ by $nSnG$-conversion, where for each $d = (x^0, \cdots, x^{i-1}, x^i, x^{i+1}, \cdots, x^n) \in D_{|X|}$, $d\alpha_i = (x^0, \cdots, x^{i-1}, x'^i, x^{i+1}, \cdots, x^n)$ with $x'^i = (\alpha^\square(x^{i+1}) \cap \beta^\square(x^{i-1})) \setminus \{x^i\}$ if $i \in \{1, \cdots, n-1\}$, $x'^0 = \alpha^\square(x^1) \setminus \{x^0\}$ and $x'^n = \beta^\square(x^{n-1}) \setminus \{x^n\}$.

Theorem 20. *Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be an n-G-map and $|X| = (X, \alpha)$ an order :*

1. *G is a regular n-G-map $\Rightarrow$ its associated order is an n-surface*
2. *$|X|$ is an n-surface $\Rightarrow$ its associated n-G-map is regular*

With the previous construction processes, any n-surface may be built from some regular n-G-map, and any regular n-G-map may be built from some n-surface. We also prove, in [2], that these conversions are inverse to each other up to isomorphism which prove the equivalence of both structures.

4 Conclusion

We have shown that two topological models namely regular n-G-maps and n-surfaces are equivalent structures. This result is important because these models come from various research fields, and are defined very differently. Moreover we have given an explicit way to switch from one representation to another. The equivalence between both models gives us more information on them. It implies for example that the neighbourhood of any cell of a regular n-G-map is a generalised map too.

Future works will be lead into three main directions. It will first be interesting to take advantage of this equivalence by transfering tools, namely operators and properties from one to another or by integrating them in a chain of operations. Besides such models can only represent quasi-manifolds, it would be useful to go on with more general structures such as chains of maps [11] which represent more general subdivisions that are not necessarily quasi-manifolds but such that each cell is a quasi-manifold. We could also focus on subclasses of these models. Finally it could be useful to study more precisely the class of regular n-G-maps, we have introduced here.

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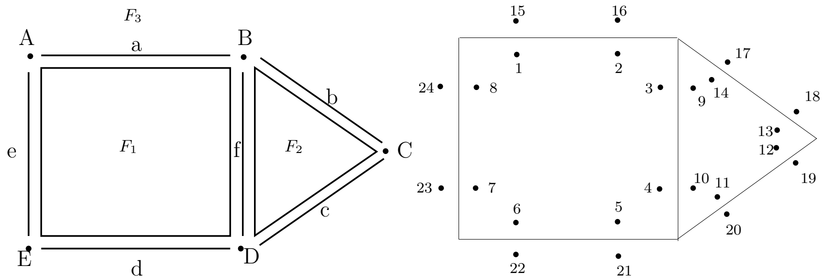
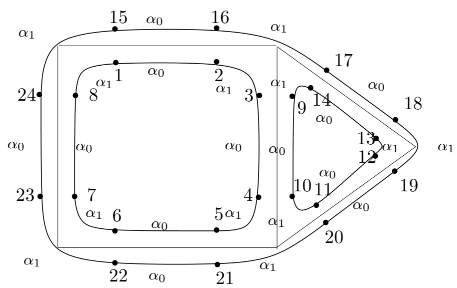
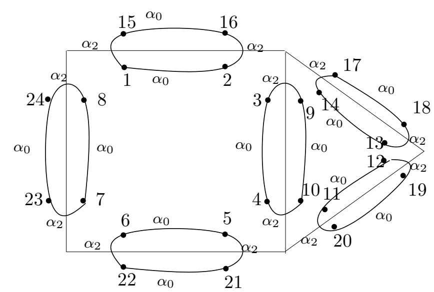


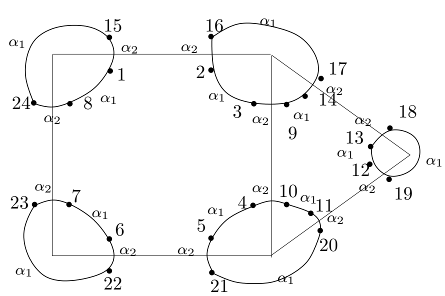
(a) Subdivision of $\mathbb{R}^2$ and its corresponding 2-G-map, $G = (D, \alpha_0, \alpha_1, \alpha_2)$, with $D = \{1, \cdots, 24\}$



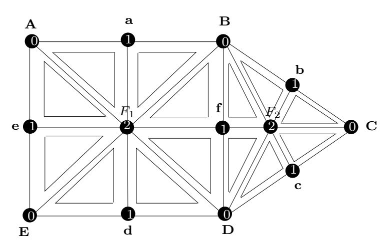
(b) $< \alpha_0\alpha_1 >$ orbits / 2-cells on a 2-G-map
$< \alpha_0\alpha_1 > (15) \Leftrightarrow F_3$
$< \alpha_0\alpha_1 > (9) \Leftrightarrow F_2$
$< \alpha_0\alpha_1 > (1) \Leftrightarrow F_1$



(c) $< \alpha_0\alpha_2 >$ orbits / 1-cells on a 2-G-map
$< \alpha_0\alpha_2 > (1) \Leftrightarrow a$,
$< \alpha_0\alpha_2 > (13) \Leftrightarrow b$
$< \alpha_0\alpha_2 > (11) \Leftrightarrow c$,
$< \alpha_0\alpha_2 > (5) \Leftrightarrow d$
$< \alpha_0\alpha_2 > (7) \Leftrightarrow e$,
$< \alpha_0\alpha_2 > (3) \Leftrightarrow f$



(d) $< \alpha_1\alpha_2 >$ orbits / 0-cells on a 2-G-map
$< \alpha_1\alpha_2 > (1) \Leftrightarrow A$,
$< \alpha_1\alpha_2 > (2) \Leftrightarrow B$
$< \alpha_1\alpha_2 > (12) \Leftrightarrow C$,
$< \alpha_1\alpha_2 > (4) \Leftrightarrow D$
$< \alpha_1\alpha_2 > (6) \Leftrightarrow E$



(e) numbered simplicial set associated to $G' = (D', \alpha_{0_{|D'}}, \alpha_{1_{|D'}}, \alpha_{2_{|D'}})$ with $D' = \{1, \cdots, 14\}$

Fig. 4. A 2-G-map with its associated cellular decomposition and the numbered simplicial set associated to one of its submaps