Equivalence Between Order and Cell Complex Representations

Alayrangues Sylvie, Lachaud Jacques-Olivier
LaBRI
351, cours de la libération, 33405 TALENCE
telephone and fax number
e-mail: Sylvie.Alayrangues@labri.fr, Jacques-Olivier.Lachaud@labri.fr

Abstract

In order to define consistent models and algorithms for image analysis, many topological representations of images have been proposed. Unfortunately the most generic ones are often not explicitly related, and properties exhibited on one representation are unknown for other representations. The aim of this paper is to show how two different topological representations of images, namely the *order* representation (developped by Bertrand *et al.*) and the *complex representation using strong weak lighting functions*(studied by Ayala *et al.*) may be related in such a way that the results and algorithms proved on one may be applied to the other and conversely.

1 Introduction

The study of the topological properties of discrete spaces is an active field of research. Different approaches have been proposed to represent the support of the images and describe the links between their elements (e.g. pixels in 2D, voxels in 3D). Most of their results and algorithms seem to be model dependent. Nevertheless there are often bridges between these approaches which once discovered could allow the transfer of theoretical results and practical algorithms from one to another. This paper takes an interest in two of these approaches and aims to explicit their relationship. The first one is a work conducted by Bertrand *et al.* [1, 2] and use the *order representation*. From this point of view the elements of the image are represented by the "smallest" elements relative to the order whereas the other elements define the links between them. The second one is based on the approach of Ayala *et al.* [3, 4, 5] which considers the support of an image as a *finite polyhedral complex* equipped with a particular function called *strong weak lighting function*. In this case the elements of the image are represented by the cells of maximal dimension and the cells of lesser dimension connect them. Both models propose formal ways to define several connectednesses between the elements of the image, that is several methods to select only the links of interest (for a given purpose) between the elements of the image. These two approaches are compliant with classical connectednesses (for example those introduced by Rosenfeld) and also create new ones. Our contribution is to show that, one can construct complexes from orders and orders from complexes in such a way that the results found by Bertrand *et al.* and by Ayala *et al.* hold for both models. We first present the main characteristics of both orders and abstract cell complexes and the ways to go from one to the other. We then explicit the relations between their topologies and finally show the correspondence between the construction of connectedness on orders and on complexes equipped with a strong weak lighting function. We will finally list briefly the advantages each model may take of the other.

2 Order and Complexes

2.1 Definitions

This section presents some definitions related to orders and complexes and introduces mappings for building complexes from orders and orders from complexes. For the orders we follow the notations defined by Bertrand *et al.* in [1, 2].

An *order* is a pair $|X| = (X, \alpha)$, where X is a set and α a reflexive, antisymetric, and transitive binary relation. In the sequel we denote β the inverse of α and θ the union of α and β. Moreover we only consider a restricted family of orders, called *CF orders*, that are *countable*, i.e. X is countable, and *locally finite*, i.e. $\forall x \in X, \theta(x)$ is finite.

Some particular sets can be associated to each element x of X. The simplest ones are its α*-adherence*, $\alpha(x) = \{y \in X; (x, y) \in \alpha\}$ and its *strict* α*-adherence*, $\alpha^{\Box}(x) = \alpha(x) \backslash \{x\}$. We also call α*-chain* every fully α-ordered subset of $|X|$. If $|X|$ is CF, every α-chain is finite and its length is equal to the number of its elements less one. If we only consider a subset S of X we denote $\alpha_{|S}$ the reflexive, antisymetric, and transitive binary relation $\alpha \cap S \times S$.

An *abstract cell complex* $C = (E, <, dim)$ is a set E of abstract elements associated with an irreflexive, antisymetric, and transitive binary relation $< \subseteq E \times E$ called *border (or face) relation* and a mapping *dimension* $\dim : E \to I \subseteq \mathbb{Z}^+$ such that $\forall (e, e') \in E \times E$, with $e' < e$, $dim(e') < dim(e)$.

In the sequel, the *star* of a cell e of C will be denoted $st(e)$ and its *combinatorial closure* $cl(e)$.

2.2 Associated orders and complexes

It may be easily seen that there exists a $1-1$ mapping f that builds an order $|X_C| = (X_C, \alpha_C)$ from an abstract cell complex $C = (E, <, dim)$ such that $\forall e \in E$, $\exists! f(e) \in X_C$ and $\forall (e, e') \in E \times E$ with $e < e'$ or $e = e'$, $(f(e), f(e')) \in \alpha_C$. This transformation loses, in general, the notion of dimension. Proving the existence of a mapping that constructs an abstract cell complex from an order and effectively building one is less trivial since orders have no notion of dimension. We first restrain our study to CF-orders since non countable orders cannot be mapped onto a complex. The intuition is to partition X in a finite number of subsets and to attribute to each element a number according to the unique subset it belongs to.

As the order is locally finite, there exists a subset X_0 of X, whose elements have an empty strict α-adherence. Moreover all other elements of X are linked by at least one α-chain to an element of X_0 and the α-chains between an element of X_0 and another element of X have a finite length. There exists then an integer k such that the length of every α-chain on $|X|$ is less than or equal to k. Moreover for each $x \in X$ there exists an integer $i \leq k$ such that i is the maximal length of all the α-chains beginning at x and ending at an element of X_0. X may then be partitionned in X_i, $i = 0..k$ such that $x \in X_i$ if the maximal length of the α-chains from X_0 to x is i. This partition of X is called *α-decomposition* of $|X|$ and is indeed the family $\mathcal{F} = \{X_0, X_1, ..., X_k\}$ such that :
_ $X_0 = \{x \in X,\ \alpha^{\square}(x) = \emptyset\}$,
_ $\forall i \in \{1, \dots, k\}, X_i = \{x \in S, S = X \setminus \bigcup_{j=0}^{j=i-1} X_j, \alpha^{\square}_{|S}(x) = \emptyset\}$,
_ $X_k \neq \emptyset$ and $X = \bigcup_{i=0}^{i=k} X_i$.

For each $x \in X$, there exists then $i \in [0..K]$ such that $x \in X_i$. Let x_a and x_b be respectively elements of X_i and X_j such that $x_b \in \alpha^{\square}(x_a)$ then we deduce from the construction of the partition that $i > j$. This means that an element of $|X|$ "less" than another according to α is associated to a lesser integer number. We call therefore this number the dimension of x and denote it $dim_\alpha(x)$. The dimension of each element of $|X|$ may be recursively computed.

Property 1 : Let x be an element of $|X|$,
_ $dim_\alpha(x) = 0\ iff\ \alpha^{\square}(x) = \emptyset$
_ $dim_\alpha(x) = 1 + \max_{i=1}^{m}(dim_\alpha(y_i))$ with $\alpha^{\square}(x) = \{y_i, i \in [1..m]\}$

We deduce from this how to construct a function that builds an abstract cellar complex from a CF-order.

Theorem (Order and abstract cell complex) : The *abstract cell complex* $C = (E, <, dim)$, associated with the CF-order $|X| = (X, \alpha)$, is defined by the map ψ such that :
_ $\forall x \in X$, $\exists! \psi(x) \in E$ (ψ 1-1 mapping from X to E),
_ $\forall (x, x') \in X \times X$ such that $x' \in \alpha^{\square}(x)$, $\psi(x') < \psi(x)$
_ $\forall x \in X$, $dim(\psi(x)) = dim_\alpha(x)$

In such a complex, the faces of a cell e are precisely the images by ψ of the elements of the strict α-adherence of $\psi^{-1}(e)$. In the sequel we will be interested in another complex that may be built from a CF-order. We call it the *dual abstract cell complex* of the order.

Theorem (Order and dual abstract cell complex) : The *dual abstract cell complex* $C^* = (E^*, <^*, dim^*)$, associated with the CF-order $|X| = (X, \alpha)$, is defined by the map ψ^* such that :
_ $\forall x \in X$, $\exists! \psi^*(x) \in E^*$ (ψ^* 1-1 mapping from X to E^*),
_ $\forall (x, x') \in X \times X$ such that $x' \in \beta^{\square}(x)$, $\psi^*(x') <^* \psi^*(x)$ (ψ^* isomorphism from $(X, \beta^{\square})$ to $(E^*, <^*)$),
_ $\forall x \in X$, $dim(\psi^*(x)) = dim^*_\alpha(x)$ with $dim^*_\alpha = max_{x \in X}\{dim_\alpha(x)\} - dim_\alpha$.

The dual abstract cell complex of an order (X, α) is generally *not the same* as the abstract cell complex of the dual order (X, β). The dimension dim^*_α is indeed different from dim_β in most cases. We prefer using the dual abstract cell complex of an order because it is, by construction, a *pure*[1] complex. In such a complex, the faces of a cell e are precisely the images by ψ^* of the elements of the strict β-adherence of ${\psi^*}^{-1}(e)$.

Finally, with an informal notation, we remark that a n-complex C such that $\psi(\psi^{-1}(C)) = C$ (resp. $\psi^*({\psi^*}^{-1}(C)) = C$) must have the following property : every i-cell of C ($i \in [0..n]$) has at least a k-face for each $k \in [0, i-1]$ (resp. is face of a k-cell, $k \in [i+1, n]$). The order built with ψ^{-1} (resp. ${\psi^*}^{-1}$) from such a complex keeps implicitly the information of dimension for its elements : the ψ^{-1}-image (resp ${\psi^*}^{-1}$-image) of each i-cell is an element of X_i (resp X_{n-i}).

2.3 Restriction to particular complexes/orders

In most cases the notion of abstract cell complex is too general to represent the support of images. We deal with particular complexes (such as simplicial and polyhedral complexes). We would like then to know how to characterize orders so that ψ or ψ^* builds a suitable complex. There are ways to characterize *simplicial* orders (i.e. orders such that its abstract cell complex is simplicial). It is not clear whether we can define *polyhedral orders*, because the definition of polyhedral complexes involve geometric constraints. In the sequel, we will be interested by a wider kind of complexes. We will call them *strongly normal* complexes [2].

Strongly normal complex : A complex C is said *strongly normal* if for each cell $e \in C$, the set of cells $e_0, e_1, ..., e_n$, that belong to $st(e)$, is finite and if the intersection of the combinatorial closures of the e_i ($i = 0, 1, \ldots, n$) is either empty or the combinatorial closure of a cell of C.

We define similarly the notion of *strongly normal* order, and will prove that the dual abstract cell complex of such an order is a strongly normal one.

Strongly normal order : An order is said *strongly normal* if it is CF and if the intersection of the β-adherences of the elements of every subset of X_0 is either empty or equal to the β-adherence of an element $x \in X$

Lemma 1 : A *pure* n-complex C is strongly normal if for each cell $e \in C$, the *set of n-cells*, which belong to $st(e)$, is finite and if the intersection of their combinatorial closures is the combinatorial closure of a cell of C.

Theorem (strongly normal order and complex) : The dual abstract cell complex of an order is a strongly normal complex.

Proof : *The dual abstract cell complex $C^*_{|X|}$ of an order $|X|$ is pure. The order is CF and then the star of each cell of $C^*_{|X|}$ is finite. Moreover, let e be a cell of $C^*_{|X|}$, ${\psi^*}^{-1}(e) \in X$, $|X|$*

[1]a pure n-complex is a n-complex whose k-cells, $k < n$, are faces of at least one n-cell

[2]unrestricted simplicial complexes and unrestricted polyhedral complexes are special cases of strongly normal complexes

is strongly normal so $\exists x' \in X$ *such that* $\beta(x') = \bigcap\{\beta(x_0),\ x_0 \in \alpha({\psi^*}^{-1}(e)) \cap X_0\}$. *Considering* $C^*_{|X|}$, *it means that* $\exists e' = \psi^*(x')$ *such that* $cl(\psi^*(x')) = \bigcap\{cl(e_n), e_n \in \psi^*(X_0)\ with\ e\ <\ e_n\}$. *From Lemma 1, we can deduce that* $C^*_{|X|}$ *is strongly normal.* $\square$

2.4 Links between their topologies

The notions defined on orders and complexes are linked through the mapping ψ.
We first describe those that allow to define a topology on an order and its associated complexes. We consider an order $|X|$ and its abstract cell complex $C_{|X|}$. The image of the α-*adherence* of an element x (resp. a subset S) of X is the *combinatorial closure* of $\psi(x)$ (resp. $\psi(S)$). The image of the *strict* α*-adherence* of an element x (resp. a subset S) of X is the *combinatorial frontier* of $\psi(x)$ (resp. $\psi(S)$). The α-interior of a subset S of $|X|$ is the set : $\star\alpha(S) = \overline{\alpha(\overline{S})} = \{x \in S/\beta(x) \subseteq S\}$. The image of the α-interior of a subset S of $|X|$ is the set of cells $\psi(x) \in \psi(S)$ whose star in $C_{|X|}$ belongs to $\psi(S)$. A subset S of $|X|$ is α*-closed* if $S = \alpha(S)$, and α*-open* if $S = \star\alpha(S)$. The notions of open and closed subcomplex are the ones usually used (cf [6]). With these definitions, a *discrete topology in the sense of Alexandroff*[3] may be defined on both orders and complexes. Equipped with this topology, orders and complexes become *Alexandroff spaces* [4]. Moreover the image of an α-closed (resp. α-open) subset of $|X|$ is a closed (resp. open) subcomplex of $C_{|X|}$. The inverse is also true. ψ is hence an *homeomorphism* between $|X|$ and $C_{|X|}$.
We examine then an order $|X|$ and its dual abstract cell complex $C^*_{|X|}$. The image of the β-*adherence* of an element x (resp. a subset S) of X is the *combinatorial closure* of $\psi^*(x)$ (resp. $\psi^*(S)$). The image of the *strict* β*-adherence* of an element x (resp. a subset S) of X is the *combinatorial frontier* of $\psi^*(x)$ (resp. $\psi^*(S)$). The image of the β-interior of a subset S of $|X|$ is the combinatorial closure of $\psi^*(S)$. Moreover the image of a β-closed (resp. β-open) subset of $|X|$ is an open (resp. closed) subcomplex of $C^*_{|X|}$. ψ^* is not an homeomorphism between $|X|$ and $C^*_{|X|}$.
In the sequel we call θ*-path* on $|X|$ the image by ψ^{-1} or ${\psi^*}^{-1}$ of a path (defined by the border relation) on $C_{|X|}$ or $C^*_{|X|}$, i.e. a sequence of elements such that every couple of consecutive elements is in $\alpha \cup \beta$.

3 Connectedness between image elements

We consider now the notions and properties that are linked with the connectedness of orders and complexes. It may first be proved that *connectedness* and *path-connectedness* are equivalent on both orders and complexes. In the sequel we determine which elements of an order $|X|$ (resp. a complex C equipped with a strong weak lighting function) must be kept to grant a given connectivity between the α-terminals of $|X|$(resp. its n-cells of C). We will call "inessential"

[3] A topology is said discrete in the sense of Alexandroff iff the intersection of every family of open sets (finite or infinite) is an open set

[4] An Alexandroff space is a $\mathcal{T}_0$-separable space with a discrete topology in the sense of Alexandroff

elements of either $|X|$ or K, those that are not needed to characterize the chosen connectedness between the main elements. For each notion introduced in one of the models we give an interpretation into the other. Finally we prove the equivalence between the definition of inessential elements in the two models when the considered order and the associated pure n-complex are *strongly normal*.

3.1 Connectedness between α-terminals in an order

As said in introduction, the order approach consider the elements in X_0, i.e. the α-terminals of $|X|$, as the points of the image and the elements of the X_i,$i > 0$, as the connections between them. The images of the α-terminals in $C_{|X|}$ are the cells without faces, i.e. by construction of $C_{|X|}$, the 0-cells, and their images in $C^*_{|X|}$ are the cells with an empty strict star, i.e. by construction of $C^*_{|X|}$, the n-cells where n is the maximal dimension of $C^*_{|X|}$. Determining whether an element of $|X|$ is inessential or not consists in a local observation.
We are then interested in the α-*closeness* of each element x of $|X|$, $\alpha^\bullet(x)$. Formally, $\alpha^\bullet(x) = \{y \in X / y \in \alpha^\Box(x), \alpha^\Box(x) \cap \beta^\Box(y) = \emptyset\}$, i.e. it is the set of elements of $|X|$ that are linked to x by a single α-chain of length 1. It may be proved that

$$\alpha^\Box(x) = \bigcup_{x_i \in \alpha^\bullet(x)} \alpha(x_i).$$

If we consider $C_{|X|}$ (resp. $C^*_{|X|}$), the image of $\alpha^\bullet(x)$ by ψ is the smallest subset S of $cl(\psi(x))\backslash\{\psi(x)\}$ (resp. $st(\psi^*(x))\backslash\{\psi^*(x)\}$) such that $cl(S)$ (resp. $st(S)$) is equal to $cl(\psi(x))\backslash\{\psi(x)\}$ (resp. $st(\psi^*(x))\backslash\{\psi^*(x)\}$). The elements whose α-closeness contains one and only one element are called α-*unipolar* (we note that an α-terminal cannot be α-unipolar). These points are the simplest inessential points in the order, if x is α-unipolar then $\exists x' \in \alpha^\Box(x), \alpha^\Box(x) = \alpha(x')$. The other inessential points of the orders are called k-α-*free* and are α-unipolar for the order obtained after the recursive deletion of a sequence of α-unipolar points (or 0-α-free points) and i-α-free points ($i \in \{1, \ldots, k-1\}$). Their deletion do not disconnect any couple of α-terminals of their common adherences :

Lemma 1 : Let $|X|$ be a CF-order, if $x \in |X|$ is α-free then there exists $x' \in \alpha^\Box(x)$ such that $\alpha(x) \cap X_0 = \alpha(x') \cap X_0$.

Proof : *x^k is$k$-α-free, x^k α_{Y_k}-unipolar in $Y_k \subseteq X$ with $Y_k \cap X_0 = X_0$, then $\exists! x'^k \in Y_k, \alpha^\bullet_{|Y_k}(x^k) = \{x'^k\}$. That is to say that $\alpha^\bullet_{|X}(x^k)$ contains some i-α-free elements ($i \in \{0, \ldots, k-1\}$) and eventually x'^k. It may be seen (with an appropriate recursion) that every strict α-adherence of every i-α-free element belonging to $\alpha^\bullet_{|X}(x^k)$ is contained in $\alpha(x'^k)$. The α-adherence of x^k in X can then be expressed as the union of the singletons containing the i-α-free elements of $\alpha^\bullet_{|X}(x^k)$ and the α-adherence of x'^k. It implies then $\alpha(x^k) \cap X_0 = \alpha(x'^k) \cap X_0$.* $\Box$

The deletion of the images of α-free elements in $C_{|X|}$ do not remove the shortest paths whose elements have the smaller dimension between any two 0-cells of their neighborhood. The suppression of their images in $C^*_{|X|}$ do not delete the shortest paths whose elements have the higher dimension between any two n-cells of their neighborhood.
Finally an element of $|X|$ that is not α-free is called *α-link* and the set consisting of all the α-links of an order is called the *α-kernel* of this order.

3.2 Strong Weak Lighting Functions

The weak lighting functions introduced by Ayala *et al.* are usually defined on a homogeneously n-dimensional locally finite polyhedral complex K, but the formal definition is valid on a wider range of complexes. These functions allow to "light" the cells required to define a chosen connexity on the complex.
In the sequel we denote by O a subset of n-cells of K, by $st_n(e, O)$ the set of n-cells in O belonging to $st(e)$.

We denote by $supp(O)$ the support of O that is the set of cells e whose combinatorial closure is equal to the intersection of the combinatorial closure of the elements of $st_n(e, O)$.
We define on a complex K a *Strong Weak Lighting Function* (s.w.l.f.) that is a map, f : $\mathcal{P}(cell_n(K)) \times K \rightarrow \{0, 1\}$ with $\forall O \in \mathcal{P}(cell_n(K))$ and $e \in K$:
1. if $e \in O$ then $f(O, e) = 1$;
2. if $e \notin supp(O)$ then $f(O, e) = 0$;
3. $f(O, e) \leq f(cell_n(K), e)$;
4. $f(O, e) = f(st_n(e, O), e)$;

The first property of the s.w.l.f. specifies that all the n-cells of an objet are lighted, the second one expresses that cells that are not intersection of combinatorial closures of subsets of O are not useful to connect it, the third one imposes that a cell lighted for an object is also lighted for the whole image, and the fourth one induces that for a given object, the lighting of a cell is a local property of the objet. Many s.w.l.f. may be constructed on a complex depending on which connexity we want to associate to the complex. These functions are equal to 0 on inessential elements of the complex and to 1 on the others.

3.3 Connectedness in strongly normal orders/complexes

We are going to prove that it is possible to build a fonction on a strongly normal order $|X|$ that will define a s.w.l.f. on its dual abstract cell complex $C_{|X|}$. To do so, we first express some useful lemma (their proofs will not be given because of the lack of space).

The first one indicates that the strong normality property is hereditary for some subcomplex of a strongly normal complex.

Lemma 2 : Let C be a pure strongly normal complex, every subcomplex of C built without removing any cell belonging to the support of at least one set of n-cells, is also pure and strongly normal.

The next lemma means that if two different cells are faces of exactly the same n-cells and if one of them has a lower dimension then this latter cell is inessential.

Lemma 3 : Let X be a CF-order and $C_{|X|}$ its dual abstract cell complex, if x and x' are such that $\alpha(x) \cap X_0 = \alpha(x') \cap X_0 = S_0$ and $x' \in \alpha^{\square}(x)$ then $\psi^*(x) \notin supp(S_0)$

The following lemma is a corollary of Lemma 3 , it shows that the images of the α-unipolar elements of $|X|$ in $C^*_{|X|}$ are inessential.

Lemma 4 : Let X be a CF-order and $C_{|X|}$ its dual abstract cell complex, if x is α-unipolar then $\psi^*(x) \notin supp(\psi^*(\alpha(x) \cap X_0))$

The next lemma concerns only the orders, it says that an element linked to only one α-terminal is α-free and hence inessential in the order.

Lemma 5 : Let $|X|$ be a CF-order, $\forall x \in X$, such that $x \notin X_0$ and $\alpha(x) \cap X_0$ only contains one element, then x is α-free in $|X|$.

The following lemma shows that every inessential cell linked to more than one n-cell is face of an element of the support of this group of n-cells.

Lemma 6 : Let $|X|$ be a strongly normal order, $\forall S_0 \subseteq X_0$, every cell of $C^*_{|X|}$ that does not belong to $supp(\psi^*(S_0))$ and whose star contains at least two n-cells of $\psi^*(S_0)$ is face of at least one k-cell, $k < n$, of $supp(\psi^*(S_0))$.

The next lemma indicates that an inessential cell may be the maximal face of at most one cell of the support of an object O.

Lemma 7 : Let $|X|$ be a strongly normal order, $\forall S_0 \subseteq X_0$, a cell e of $C^*_{|X|}$ such that $e \notin supp(\psi^*(S_0))$ cannot be the maximal face of more than one cell $e' \in supp(S_0)$ (that is $e < e'$ and $\nexists e'' \in \psi^*(\beta(S_0))$ such that $e < e'' < e'$).

The last lemma shows that the cells not belonging to the support of an object and whose dimension are maximal are some of the first inessential cells that can be removed (it may be proved thanks to Lemma 7).

Lemma 8 : Let $|X|$ be a strongly normal order, $\forall S_0 \subseteq X_0$, a cell e of $C^*_{|X|}$ such that $e \notin supp(\psi^*(S_0))$ and whose dimension is maximal is a maximal face of one cell of $supp(\psi^*(S_0))$. Moreover its image by ${\psi^*}^{-1}$ is $\alpha_{|\beta(S_0)}$-unipolar.

We are now going to prove our main result that is the α-kernel of any suborder of a strongly normal order $|X|$ is transformed by ψ^* into the support of the corresponding n-cells in $C^*_{|X|}$.

Theorem (α-kernel and support) : Let $|X| = (X, \alpha)$ be a strongly normal order and $C^*_{|X|}$ its dual abstract complex. $\forall S_0 \subseteq X_0$:

x α-free in the order $|\beta(S_0)| = (\beta(S_0), \alpha_{|S_0}) \Leftrightarrow \psi^(x) \notin supp(\psi^*(S_0))$.

Proof : *We are going to prove the theorem in two stages :*

$\Rightarrow$ *If x is $\alpha_{|\beta(S_0)}$-free in $|\beta(S_0)|$, we prove that its image by ψ^* does not belong to $supp(\psi^*(S_0))$, indeed :*

- *if x is $\alpha_{|\beta(S_0)}$-unipolar, then, by Lemma 4, $\psi^*(x) \notin supp(\psi^*(S_0))$;*
- *if x is $\alpha_{|\beta(S_0)}$-free, then, by Lemma 1, $\exists x' \in \alpha^{\square}(x)$ such that $\alpha(x) \cap S_0 = \alpha(x') \cap S_0$ and by Lemma 3 $\psi^*(x) \notin supp(\psi^*(S_0))$.*

$\Leftarrow$ *Let $e \in cl(\psi^*(S_0))$, with $e \notin supp(\psi^*(S_0))$, and $dim(e) < n$, we show that ${\psi^*}^{-1}(e)$ is $\alpha_{|\beta(S_0)}$-free*

1. if e is face of only one n-cell, e_n, $e \notin supp(\psi^*(S_0))$ because the intersection of the combinatorial closure of all n-cells having e as face is reduced to $cl(e_n)$ which is different from $cl(e)$. Moreover $\alpha({\psi^*}^{-1}(e)) \cap S_0$ contains only one element : ${\psi^*}^{-1}(e_n)$. ${\psi^*}^{-1}(e)$ is hence $\alpha_{|\beta(S_0)}$-free, by Lemma 5.

2. if e is face of at least two n-cells of $\psi^*(S_0)$, we know by Lemma 6 that $\exists e_p \in supp(\psi^*(S_0))$ such that $e < e_p$.

2-a. If e has a maximal dimension, we know by Lemma 8 that ${\psi^*}^{-1}(e)$ is $\alpha_{|\beta(S_0)}$-unipolar.

2-b. We consider then the subcomplex C^1 of $\psi^*(\beta(S_0))$ obtained from $\psi^*(\beta(S_0))$ by removing all cells considered in 1. and 2-a.. By Lemma 2, we know that C^1 is pure and strongly normal. We consider the cells of C^1 not belonging to $supp(\psi^*(S_0))$ and having a maximal dimension. Lemma 8 allows us to conclude as previously that these cells are $\alpha_{|{\psi^*}^{-1}(X^1)}$-unipolar for the order $|{\psi^*}^{-1}(X^1)|$ deduced from $\beta(S_0)$ by removing only α-unipolar and α-free elements. These cells are hence $\alpha_{|\beta(S_0)}$-free. Finally, by recursively deleting cells of maximal dimension not belonging to $supp(\psi^*(S_0))$, we remove all cells not belonging to $supp(\psi^*(S_0))$. As each of them is a cell of maximal dimension for a strongly normal subcomplex of $\psi^*(\beta(S_0))$, it is α-unipolar for a suborder of $\beta(S_0)$ obtained by removing only $\alpha_{|\beta(S_0)}$-unipolar and $\alpha_{|\beta(S_0)}$-free elements of $\beta(S_0)$. Hence each of these cells is $\alpha_{|\beta(S_0)}$-free. $\square$

We now define an *analogous of s.w.l.f. for strongly normal orders*. Let $|X|$ be a strongly normal order, a s.w.l.f. ϕ on $|X|$ is defined by : $\phi : \mathcal{P}(X_0) \times X \rightarrow \{0,1\}$

- $\phi(S_0, x) = 1$ if $x \in X_0$,
- $\phi(S_0, x) = 0$ if $x \notin \alpha$-kernel$(\beta(S_0))$,
- $\phi(S_0, x) \leq \phi(X_0, x)$;
- $\phi(S_0, x) = \phi(\alpha(x) \cap S_0, x)$;

Theorem (Order and complex with s.w.l.f.) : Let $|X|$ be a strongly normal order, the map f defined by : $\forall x \in X \forall S_0 \subseteq X_0$, $f(\psi(S_0), \psi(x)) = \phi(S_0, x)$ is a s.w.l.f. on $C^*_{|X|}$.

Proof : *Properties 1,3 and 4 of s.w.l.f. clearly hold for f. Property 2 is a consequence of theorem 1* $\square$

4 Conclusion and Perspectives

We have shown that a strongly normal order and a pure strongly normal complex can be built from one another, in such a way that the inessential elements of one match the inessential ones of the other. Both models are interesting because they are general and in particular not dimension dependent.They propose a general framework to represent the support of images and to express topology relations within. Nevertheless they are not equivalent and we list briefly the benefits that each model may take of the other. Concerning the notion of connectedness, Bertrand and al. had to consider different orders to define different connectedness, whereas the use of a s.w.l.f. allows to define several connectedness on the same order. Moreover the polyhedral complex used by Ayala *et al.* is the basis of a multilevel architecture that allow to have discrete results consistent with continuous ones. They propose for example the definition of a fundamental group that may be transferred on orders. They also prove a Seifert-Van Kampen theorem to compute fundamental groups. Ayala *et al.* have in interesting theoretical results but their work has not been exploited yet in concrete applications. Bertrand *et al.* have more practical results and propose for example a definition of simple points which allows the thinning of objects in parallel. It may be interesting to see how their method may be applied to complexes.
In future works we intend to effectively transfer the tools of one model onto the other. We are also studying if the notions of surface defined on orders and checking if it is consistent with the surface classically defined with polyhedral complexes.

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